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Quantum tunneling of vortices in two-dimensional superfluids

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We examine the problem of quantum vortex tunneling in the Gross-Pitaevskii model. The effect of gapless phonons is to produce a super-Ohmic quantum dissipation that renormalizes the effective tunneling parameters but does not destroy quantum coherence. The renormalization of the instanton frequency is computed within a specific model tunneling potential.

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I. INTRODUCTION

A quantum dissipative system typically consists of a quantum particle in an external potential, coupled to a bath of gapless excitations. In the path-integral treatment, pioneered by Feynman and Vernon, and by Caldeira and Leggett,[1–4] the bath is integrated out, resulting in a retarded self-interaction term for the particle, which, in turn, affects the instanton for the quantum tunneling process. This self-interaction term depends on (i) the form of the coupling between the particle and bath, and (ii) the spectral properties of the bath itself.

In this paper we will consider the quantum tunneling of a vortex in a two-dimensional superfluid described by the Gross-Pitaevskii model.[5] The vortex is a collective excitation described by a collective coordinate labeling its positional center. The bath consists of the phonons of the superfluid. The Caldeira-Leggett scheme is implemented most naturally within the well-known dual theory, which at a linearized level is $(2+1)$-dimensional electrodynamics, with vortices playing the role of charges, minimally coupled to a Maxwell theory whose "photon" is the phonon of the superfluid.[6–10] The correct coupling of the vortex and bosonic degrees of freedom is then a gauge coupling of the form $\mathcal{J}_\mu \mathcal{A}^\mu$—a point which is often missed in extant treatments of quantum dissipation of vortex dynamics in the literature. The vortex tunneling problem then maps onto a one-dimensional ferromagnetic Ising model with long-ranged $1/r^3$ interactions, which, from the work of Dyson,[11] is known to have no ordered phase. The absence of a phase transition in the associated Ising model means that the vortex tunneling remains coherent in the presence of phonon dissipation. This conclusion is consistent with the finite vortex tunneling amplitude that has been computed using Bogoliubov theory in Ref. 12.

Electrodynamics of vortices and phonons

There exists a formal equivalence between the two-dimensional Bose superfluid and a nonlinear version of two-dimensional electrodynamics where superfluid vortices play the role of charges, the boson particle current plays the role of the electric field, and the boson density plays the role of a (scalar) magnetic field. This equivalence has long been known, and originally was utilized in establishing the duality between the XY and Coulomb gas models of classical statistical mechanics.[6–10] Starting with a Hamiltonian density for a one-component Bose system,

$$\mathcal{H} = \frac{\hbar^2}{2m} \nabla \psi^\dagger \cdot \nabla \psi + v(\psi^\dagger \psi) - \mu \psi^\dagger \psi, \tag{1}$$

one arrives at the (Euclidean) Lagrangian density for $(2+1)$-dimensional electrodynamics,

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2} m c^2 n_0 \left\{ \mathcal{E}^2 + (\mathcal{B}-1)^2 + \frac{1}{4} (\xi \nabla \mathcal{B})^2 \right\} + 2\pi i \hbar n_0 \mathcal{J}^\mu \mathcal{A}_\mu, \tag{2}$$

where $\mathcal{E} = \hat{z} \times \boldsymbol{j} / n_0 c = -\nabla \mathcal{A}^0 - c^{-1} \partial_t \boldsymbol{\mathcal{A}}$ and $\mathcal{B} = n/n_0 = \hat{\mathbf{z}} \cdot \nabla \times \boldsymbol{\mathcal{A}}$ are dimensionless "electric" and "magnetic" fields, respectively, proportional to the boson current and particle densities, $\boldsymbol{j}$ and n.[13] The sound velocity is $c = \sqrt{n_0 v''(n_0)/m}$, where $v(n) - \mu n$ has a stable minimum at $n = n_0$, and the coherence length is $\xi = \hbar / mc$. The position 3 vector is $x^\mu = (ct, x, y)$, and the metric used to raise and lower indices is $g^{\mu\nu} = \mathrm{diag}$
$(+--)$. The vortex 3 current is given by

$$\mathcal{J}^\mu(\boldsymbol{x}, t) = \sum_i q_i \begin{Bmatrix} c \\ \dot{\boldsymbol{X}}_i \end{Bmatrix} \delta[\boldsymbol{x} - \boldsymbol{X}_i(t)], \tag{3}$$

where q_i is the integer vorticity of the ith vortex.

Integrating out the gauge field $\mathcal{A}^\mu$, one obtains the effective action for the vortices, which consists of three parts: $S_{E,\mathrm{eff}} = S^{\mathrm{I}}_{E,\mathrm{eff}} + S^{\mathrm{II}}_{E,\mathrm{eff}} + S^{\mathrm{III}}_{E,\mathrm{eff}}$. The first term corresponds to an instantaneous Coulomb interaction between vortices:

$$S^{\mathrm{I}}_{E,\mathrm{eff}} = -\frac{2\pi\hbar^2 n_0}{m} \sum_{i<j} q_i q_j \int_0^{\hbar\beta} d\tau \ln |\boldsymbol{X}_i(\tau) - \boldsymbol{X}_j(\tau)|. \tag{4}$$

The second term represents the geometric phase accrued by the vortices moving through the superfluid:

$$S^{\mathrm{II}}_{E,\mathrm{eff}} = 2\pi i n_0 \hbar \sum_i q_i \int_0^{\hbar\beta} d\tau X_i(\tau) \dot{Y}_i(\tau). \tag{5}$$

The third term reflects a retarded interaction between vortices, as well as self-interaction ("radiation reaction") due to the phonons:

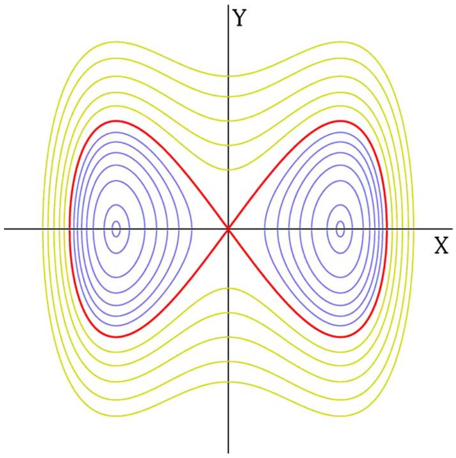

FIG. 1. (Color online) Contour plot of vortex potential $V(X,Y)$ in Eq. (8). The red curve marks the separatrix.

$$S^{\mathrm{III}}_{E,\mathrm{eff}} = \frac{2\pi^2\hbar^2 n_0}{m}\sum_{i,j} q_i q_j \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \frac{1}{\hbar\beta} \times \sum_{\nu_m}\int \frac{d^2k}{(2\pi)^2} e^{i\boldsymbol{k}\cdot[\boldsymbol{X}_i(\tau)-\boldsymbol{X}_j(\tau')]} e^{-i\nu_m(\tau-\tau')} \times \frac{\hat{\boldsymbol{k}}\times\dot{\boldsymbol{X}}_i(\tau)\cdot\hat{\boldsymbol{k}}\times\dot{\boldsymbol{X}}_j(\tau')}{\nu_m^2+\omega_{\boldsymbol{k}}^2}, \tag{6}$$

where $\nu_m = 2\pi m/\hbar\beta$ is a Matsubara frequency, and where $\omega_{\boldsymbol{k}} = ck(1+\frac{1}{4}k^2\xi^2)^{1/2}$ is the phonon dispersion. Finally, we model a vortex pinning potential by adding a fourth term to the effective action:

$$S^{\mathrm{IV}}_{E,\mathrm{eff}} = \sum_i \int_0^{\beta\hbar} d\tau\, V_i(X_i,Y_i). \tag{7}$$

Such a potential should describe, phenomenologically, the attraction of vortices to a region of potential that is repulsive for bosons. This physics lies beyond the linearized electrodynamic theory, in which vortices are point particle phase defects with no associated density variation. The density depletion due to a vortex may be computed as a nonlinear response of the Bose superfluid, as in Ref. 14.

II. VORTEX TUNNELING

We now consider the problem of quantum tunneling of a single vortex at $T=0$. The scenario is generic but one application might be the motion of a vacancy or interstitial in a vortex lattice. Due to the density depletion in the vortex core, an external potential that is repulsive to the bosons will be attractive to the vortices. For concreteness, consider a single $q=+1$ vortex, tunneling in the potential

$$V(X,Y) = \bar{V}\left\{\mathcal{U}(X) + \frac{Y^2}{2a^2}\right\}, \tag{8}$$

where $\mathcal{U}(X/a)$ is a dimensionless double-well potential with minima at $X=\pm a$, such as that in Fig. 1,

$$\mathcal{U}(X) = \frac{1}{8}C\left(\frac{X^2}{a^2}-1\right)^2, \tag{9}$$

where C is a constant. $\bar{V}$ sets the overall scale of the potential.

Ignoring for the moment the self-interaction term, the effective Lagrangian is then

$$L^0_{E,\mathrm{eff}} = 2\pi i\hbar n_0 X\dot{Y} + V(X,Y), \tag{10}$$

which corresponds to the motion of a charged particle in a magnetic field, confined to the lowest Landau level, moving in the presence of V. Quantum tunneling under such conditions was studied by Jain and Kivelson,[15] who calculated the tunneling amplitude by analytic continuation to imaginary *space*. The correctness of this approach was subsequently confirmed in detailed calculations by Fertig and Halperin,[16] who studied the problem including Landau-level mixing. Specific applications to vortex quantum tunneling have been considered by Fischer.[17]

The Landau-level structure for electrons is of course due to the finite electron mass (or effective mass), which results in a finite cyclotron energy $\hbar eB/mc$. Vortices have no intrinsic mass; however, they may inherit a polaronic mass once the phonons have been integrated out. Clearly the rest mass of the vortex is logarithmically divergent with $m_0 \simeq (\pi\xi^2 n_0)m\ln(R/\xi)$, where R is the radial size of the system and ξ is an ultraviolet cutoff (since there is no divergent energy density in the vortex core in the full nonlinear theory). But as discussed by Arovas and Freire,[14] Lorentz invariance of the effective action (minus the background field contribution) requires that the dynamical mass m_{dyn} must be identical to the rest mass m_0. In a superconductor, the two terms in the dynamical momentum $\mathbf{p}+\frac{e}{c}\mathbf{A}$ cancel at long distances from the vortex core and there is no infrared divergence; what remains is then due to the "core mass" $m_{\mathrm{core}} = \pi\xi^2 n_0 m$ of expelled bosons[18] as well as fermionic states in the core.[19]

Expanding the radiation reaction term $S^{\mathrm{III}}_{E,\mathrm{eff}}$ in powers of $\dot{\boldsymbol{X}}(\tau)$ yields a retarded interaction that can be expressed as a frequency-dependent mass $M(\omega)$, which for low frequencies takes the form[14]

$$M(\omega) \simeq m_{\mathrm{core}}\left\{\ln\left(\frac{2c}{\omega\xi}\right) + \frac{1}{2}i\pi\,\mathrm{sgn}(\omega)\right\}. \tag{11}$$

The logarithmic divergence of the real part of $M(\omega)$ is known from related work on vortex dynamics in Josephson-junction arrays.[20–22] The dissipation is the super-Ohmic. Several works have treated dissipative vortex dynamics in a Langevin-type approach, assuming Ohmic dissipation.[23–25] This naturally arises if the *position* of the vortex is coupled to an Ohmic oscillator bath, as in the standard Caldeira-Leggett approach to quantum dissipation.[3,4] In a superconductor at temperatures $k_B T \gg \Delta^2/E_F$, this is appropriate as the fermionic core states lead to Ohmic behavior. However, in our opinion, this is quite wrong for neutral superfluids since the dissipation is in fact non-Ohmic. The essential difference is that the coupling of the vortex motion to the phonon field is a gauge coupling not of the usual Caldeira-Leggett variety.

As shown in Ref. 14, the motion of a vortex in an oscillating superflow $\boldsymbol{v}_s(t)$ is described by

$$\mathbf{V}(\omega) = \frac{\boldsymbol{v}_s(\omega) + ir(\omega)\boldsymbol{v}_s(\omega)\hat{z}}{1 - r^2(\omega)}, \tag{12}$$

and

$$r(\omega) = \omega M(\omega)/2\pi\hbar n_0. \tag{13}$$

The vortex motion is therefore elliptically polarized, and at low frequencies, where $|r(\omega)| \ll 1$, the major axis is inclined at an angle $\theta_H(\omega) = \mathrm{Im}\, r(\omega) = \pi|\omega|\xi/4c$ with respect to $\hat{z} \times \boldsymbol{v}_s$. For the Ohmic case, $\theta_H(\omega)$ tends to a constant in the dc limit.

Returning to the tunneling problem, the self-interaction term may be written as

$$S_E^{\mathrm{self}} = \frac{\pi\hbar^2 n_0}{4mc^2}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' g(\tau-\tau')\dot{\boldsymbol{X}}(\tau)\cdot\dot{\boldsymbol{X}}(\tau'). \tag{14}$$

Here we have set $e^{i\boldsymbol{k}\cdot[\boldsymbol{X}(\tau)-\boldsymbol{X}(\tau')]} \approx 1$ in the exponent in Eq. (6) to capture the leading effects of the dissipative term. The Fourier transform $\hat{g}(\omega)$ of the kernel $g(\tau-\tau')$ is given by

$$\hat{g}(\omega) = \int_0^{\infty} \frac{ds}{D(s,\nu)} = \frac{1}{\sqrt{1-\nu^2}}\ln\left(\frac{1+\sqrt{1-\nu^2}}{1-\sqrt{1-\nu^2}}\right), \tag{15}$$

where $D(s,\nu) = \frac{1}{4}s^2 + s + \nu^2$, with $\nu = \omega/\omega_\Lambda$, and $\omega_\Lambda = c/\xi$ serves as an ultraviolet cutoff. We now write $g = g_< + g_>$, where $g_<$ ($g_>$) results from integrating out phonons with frequencies less (greater) than the as yet undetermined instanton frequency scale, ω_0. This leads to a cutoff $s_0 = \omega_0^2/\omega_\Lambda^2 \equiv \nu_0^2$ on the above integral, hence

$$\hat{g}_>(\omega) = \frac{1}{\sqrt{1-\nu^2}}\ln\left(\frac{1+\frac{1}{2}\nu_0^2+\sqrt{1-\nu^2}}{1+\frac{1}{2}\nu_0^2-\sqrt{1-\nu^2}}\right). \tag{16}$$

The fast phonon degrees of freedom adjust quickly to the vortex position. We can accordingly approximate $g_>(\tau) \approx g_0\delta(\tau)$ with

$$g_0 = \hat{g}_>(0) = \ln\left(1 + \frac{4}{\nu_0^2}\right). \tag{17}$$

This produces the following action:

$$S = \int_{-\infty}^{\infty} d\tau\left[\frac{1}{2}M_{\mathrm{d}}\dot{X}^2 + \frac{1}{2}M_{\mathrm{d}}\dot{Y}^2 + iM_{\mathrm{d}}\omega_{\mathrm{c}}X\dot{Y} + V(X,Y)\right] + \frac{\pi\hbar^2 n_0}{4mc^2}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' g_<(\tau-\tau')\dot{\boldsymbol{X}}(\tau)\cdot\dot{\boldsymbol{X}}(\tau'), \tag{18}$$

where $M_{\mathrm{d}} = \frac{1}{2}\pi\xi^2 n_0 g_0 m$ is the induced vortex mass, and $\omega_{\mathrm{c}} = 4\omega_\Lambda/g_0$ is the vortex cyclotron frequency. Note that $M_{\mathrm{d}}\omega_{\mathrm{c}} = 2\pi\hbar n_0$.

Since the last term in Eq. (18) arises from the slow phonons, we may treat it within the "fast flip approximation,"[26] which is to say we neglect it in computing the instanton profile. Further analytic progress can be made for the potential of Eq. (8). We first integrate out $Y(\tau)$, which results in

$$\hat{Y}(\omega) = -\frac{\omega_{\mathrm{c}}\omega\hat{X}(\omega)}{\Omega^2+\omega^2}, \tag{19}$$

where $\Omega^2 = \bar{V}/M_{\mathrm{d}}a^2$. Note that $Y(\tau)$ is purely imaginary, as it is in the Jain-Kivelson approach as well. Equation (18) may now be recast as $S = S_0 + S_{\mathrm{diss}}$ with

$$S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} M_{\mathrm{d}}\omega^2\left(1 + \frac{\omega_{\mathrm{c}}^2}{\Omega^2+\omega^2}\right)|\hat{X}(\omega)|^2 + \bar{V}\int_{-\infty}^{\infty} d\tau\, U[X(\tau)]. \tag{20}$$

Solving for the instanton of S_0 yields the frequency scale ω_0, which is the inverse of the instanton time. This must be determined self-consistently since the parameters M_{d} and ω_{c} already depend upon ω_0.

Suppose that we have solved for the instanton $X(\tau)$. Within the fast flip approximation, we can evaluate the last term in Eq. (18) by taking

$$\dot{X}(\tau) \approx 2a\sum_i \eta_i\delta(\tau-\tau_i), \tag{21}$$

which entails

$$Y(\tau) = -ia\frac{\omega_{\mathrm{c}}}{\Omega}\sum_i \eta_i e^{-\Omega|\tau-\tau_i|}, \tag{22}$$

and substituting it into the integral. Here, τ_i is the time at the center of the ith instanton and η_i alternates sign, taking the value +1 when the instanton interpolates between $X=-a$ and $X=+a$, and -1 when the direction of tunneling is reversed. The dissipative term S_{diss} contains contributions from both the X and Y components of the motion. The dominant contribution comes from the X motion with

$$\frac{1}{\hbar}S_{\mathrm{diss}}^{(X)} = \frac{\pi\hbar n_0 a^2}{mc^2}\sum_{i,j}\eta_i\eta_j g_<(\tau_i-\tau_j) \approx -\frac{\pi n_0 a^2}{4\omega_0^2\omega_\Lambda}\sum_{n,n'} g''_<[(n-n')\omega_0^{-1}]\sigma_n\sigma_{n'}, \tag{23}$$

where each $\sigma_n = \pm 1$ is an Ising spin variable. Here, we have divided the imaginary time line into intervals of width ω_0^{-1}, the characteristic instanton time scale. The nearest-neighbor interaction, which adds to the bare instanton action computed from S_0, is finite with $g_<(0) = 2\omega_\Lambda \sinh^{-1}(\omega_0/2\omega_\Lambda)$. At long-time separations, we obtain a long-ranged ferromagnetic Ising model with dimensionless interaction,

$$J_{nn'} = \frac{J_0}{|n-n'|^3}, \tag{24}$$

with $J_0 = \frac{1}{4}\pi n_0 a^2\omega_0/\omega_\Lambda$. As shown by Dyson,[11] there is no finite temperature phase transition for this model, which means that the vortex does not get trapped in either well, and the tunneling remains coherent. We find that the contribution $S_{\mathrm{diss}}^{(Y)}$ is weaker still with its long-ranged interaction decaying as $|n-n'|^{-5}$.

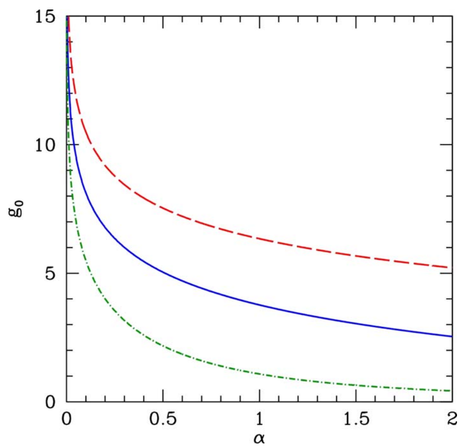

FIG. 2. (Color online) Numerical solution to Eq. (17), with anisotropy parameters $C=0.1$ (dashed red curve), $C=1.0$ (solid blue curve), and $C=10$ (dotted-dashed green curve).

Returning now to the Gaussian action S_0, we see it is described by a frequency-dependent vortex mass, which may be expressed as

$$M(\omega)=M_{\mathrm{d}}\left(1+\frac{\omega_c^2}{\Omega^2+\omega^2}\right). \tag{25}$$

In the limit $g_0=0$, where dissipation is turned off, one has $M(\omega)\to M_0=(2\pi\hbar n_0 a)^2/\bar{V}$. The mass M_0 is what one obtains for the effective one-dimensional tunneling problem in Eq. (10) after integrating out $Y(\tau)$:

$$S_0[X(\tau)]=\int d\tau\left[\frac{1}{2}M_0\dot{X}^2+\bar{V}\mathcal{U}(X)\right]. \tag{26}$$

For this one-dimensional problem, the dimensionless instanton action is

$$K=\hbar^{-1}\sqrt{2M_0\bar{V}a^2}\int_{-1}^{1}dz\sqrt{\mathcal{U}(az)-\mathcal{U}(a)}, \tag{27}$$

and the inverse instanton time is $\omega_0\approx\sqrt{-\bar{V}\mathcal{U}''(0)/M_0}$.

With dissipation present, we identify the instanton frequency via the equation

$$\omega_0^2=-\frac{\bar{V}\mathcal{U}''(0)}{M(\omega_0)}=\frac{C\bar{V}}{2a^2M(\omega_0)}. \tag{28}$$

It is now helpful to define the dimensionless parameter,

$$\alpha=\frac{\bar{V}}{mc^2}\times\frac{1}{\pi a^2 n_0}. \tag{29}$$

Expressing Eq. (28) in terms of g_0, we obtain the equation

$$\frac{4g_0}{e^{g_0}-1}=C\alpha\left(\frac{\alpha g_0^{-1}+\dfrac{2}{e^{g_0}-1}}{\alpha g_0^{-1}+\dfrac{2}{e^{g_0}-1}+8g_0^{-2}}\right). \tag{30}$$

The numerical solution to Eq. (17) is shown in Fig. 2. In the limit $\alpha\to\infty$, we obtain $g_0(\alpha)\approx 32/C\alpha^2$, hence $\nu_0=\sqrt{\frac{C}{8}}\alpha$. In this limit, the instanton frequency ω_0 is much larger than the cutoff ω_Λ, which is the effective phonon bandwidth. Accordingly, there are no fast phonon modes available to renormalize the tunneling potential, and the fast flip approximation is valid.

In the limit $\alpha\to 0$, we have $g_0=\ln(4/C\alpha)+\mathcal{O}(\ln\ln\alpha^{-1})$. The dimensionless instanton frequency ν_0 then vanishes as $\nu_0\sim\sqrt{\alpha/\ln(\alpha^{-1})}$. In this limit, dissipative effects drive the instanton frequency to zero.

Throughout our derivation we have assumed that the dimensionless parameter $\epsilon\equiv\pi\xi^2 n_0$ satisfies $\epsilon\gg 1$. Otherwise, virtual vortex-antivortex excitations can proliferate out of the vacuum.

III. CALDEIRA-LEGGETT MODEL

It is instructive to contrast our results with those of a more familiar model of quantum dissipation applied to vortex dynamics. Rather than the gauge coupling in Eq. (2), consider instead the dissipative coupling[3,4]

$$L_E^{\mathrm{diss}}=\frac{1}{2}\sum_\alpha\left[m_\alpha\dot{\boldsymbol{x}}_\alpha^2+m_\alpha\omega_\alpha^2\left(\boldsymbol{x}_\alpha-\frac{C_\alpha}{m_\alpha\omega_\alpha^2}\boldsymbol{X}\right)^2\right].$$

Integrating out the bath degrees of freedom $\{\boldsymbol{x}_\alpha\}$, we arrive at the dissipative contribution to the Euclidean action,

$$S_E^{\mathrm{diss}}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2|\hat{X}(\omega)|^2\int_0^\infty du\frac{2\hat{J}(u)}{u(u^2+\omega^2)}, \tag{31}$$

where $\hat{J}(\omega)$ is the spectral density,

$$\hat{J}(\omega)=\sum_\alpha\frac{C_\alpha^2}{4m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha). \tag{32}$$

For our vortex problem, the dissipative action is given by Eq. (14), i.e.,

$$S_E^{\mathrm{diss}}=\frac{\pi\hbar^2 n_0}{4mc^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\hat{g}(\omega)|\hat{X}(\omega)|^2. \tag{33}$$

Thus, the phonon dissipation of a single vortex may be modeled by an oscillator bath coupled to a collective coordinate, as in Eq. (31), with the spectral density

$$\hat{J}(\omega)\approx\frac{\pi\hbar^2 n_0}{4mc^2}\omega^2. \tag{34}$$

Again, this is a super-Ohmic dissipation.

IV. CONCLUSIONS

We have examined the quantum tunneling of a vortex in the Gross-Pitaevskii model, including dissipative effects due to phonon radiation in the far field of the vortex. In the absence of coupling to the dissipative phonon environment, the problem is equivalent to that of a massless charged particle in a uniform magnetic field and an external potential, i.e., the well-studied case of an electron in the lowest Landau level. Dissipative effects renormalize the instanton frequency, as computed here, but the tunneling process remains coherent.

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