Testing for Majorana Zero Modes in a $p_x + ip_y$ Superconductor at High Temperature by Tunneling Spectroscopy

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Directly observing a zero energy Majorana state in the vortex core of a chiral superconductor by tunneling spectroscopy requires energy resolution better than the spacing between core states Δ_0^2/ϵ_F. We show that, nevertheless, its existence can be decisively tested by comparing the temperature-broadened tunneling conductance of a vortex with that of an antivortex even at temperatures $T \gg \Delta_0^2/\epsilon_F$.

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Driven partially by the dream of building naturally error-resistant quantum computers, the study of topological phases of matter has become an important topic of research [1]. The simplest class of topological phases of matter that could be useful in this respect are the chiral $p_x + ip_y$ BCS paired systems [2]. There are several physical systems where $p_x + ip_y$ pairing is believed to be realized, including the A phase of superfluid ^{3}He [3] (^{3}He-A), the exotic superconductor Sr_2RuO_4 [4], and the $\nu = 5/2$ quantum Hall state [5,6]. In addition, there have been recent proposals to realize $p_x + ip_y$ pairing in cold fermion gases [7]. In these (weak) $p_x + ip_y$ systems, certain types of vortices (quasiparticles in the quantum Hall context [8]) are believed to carry zero energy Majorana fermions [8,9] which are the topologically protected degrees of freedom.

In Sr_2RuO_4 and ^{3}He-A, the vortices that carry the Majorana fermions are the so-called half-quantum vortices, which can be thought of as a vortex in the order parameter of one spin species without a vortex in the order parameter of the opposite species [10]. (Note that, in spin-polarized $p_x + ip_y$ systems, including proposed atomic gas realizations or the 5/2 state, there is no half-quantum vortex and the full-quantum vortex carries the Majorana fermion.)

Let us suppose that, in one of these systems, the relevant Majorana-fermion-carrying vortex has been observed [11]. The next important step would be to design an experiment to observe the Majorana fermion in such a vortex [12]. In the case of Sr_2RuO_4, one obvious experiment would be an energy-resolved tunneling experiment, which measures the local density of states (LDOS) [13]. An observation of a localized mode at precisely zero energy would be direct evidence of the Majorana mode. For cold atoms, an analogous experiment for observing the LDOS would be an energy-resolved local particle annihilation experiment. For the other realizations of $p_x + ip_y$ order, it is not as clear how such an experiment would be performed [14].

In principle, such tunneling experiments could provide definitive evidence for the Majorana mode. However, in practice they may be prohibitively difficult. In the vortex, there will exist subgap bound states in the core known as Caroli–de Gennes–Matricon (CdGM) states [15,16]. The spacing between these bound states is typically of order $\delta_c = \Delta_0^2/\epsilon_F$, where Δ_0 is the gap (presumably on the order of the critical temperature) and ϵ_F is the Fermi energy. Since the experimentally observed tunneling spectrum will be smeared by the temperature, this tunneling experiment would naively only have a clear signature for $T < \delta_c$. Unfortunately, such low temperatures could potentially be unattainable in any of the proposed realizations ($\delta_c \approx 7\ \mu$K in ^{3}He-A, <0.1 mK in Sr_2RuO_4). The purpose of this Letter is to demonstrate that the tunneling spectrum retains an unambiguous signature of the Majorana fermion at much higher temperatures. The signature is found by comparing the tunneling conductance peaks of a vortex with an antivortex, the direction of vorticity being defined relative to the angular momentum of the chiral order parameter.

Bogoliubov-de Gennes (BdG) theory.—We consider a two-dimensional uniform $p_x + ip_y$ superconductor of spinless fermions. The BdG excitations are given by [17]

$$\begin{pmatrix} \hat{T} - \epsilon_F & \Delta \\ \Delta^\dagger & -(\hat{T} - \epsilon_F) \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \qquad (1)$$

where $\hat{T}$ is the kinetic energy operator and ϵ_F is the Fermi energy.

We implement the BdG equation on a sphere of radius R, parametrized by the unit vector $\mathbf{\Omega} = (\theta, \phi)$. The spherical geometry has two important advantages: (i) It has no boundaries, which strongly affect the low energy spectrum. (ii) In the absence of disorder, the azimuthal angular momentum is conserved, which greatly reduces the computational difficulty of the BdG diagonalization.

The order parameter field on the sphere is taken to be of the following form [8,18]:

$$\Delta_{v\bar{v}} = \Delta_p(\mathbf{\Omega}, \mathbf{\Omega}')F_{v\bar{v}}(\bar{\mathbf{\Omega}}) = \sum_{lml'm'} \Delta_{lml'm'} Y_{-(1/2),l',m'}(\mathbf{\Omega}) Y_{-(1/2),l,-m}(\mathbf{\Omega}'),$$

$$\Delta_p(\mathbf{\Omega}, \mathbf{\Omega}') = \frac{\Delta_0}{(4\pi\xi_p^2)(l_F + \frac{1}{2})}(\alpha\beta' - \beta\alpha') \times |\alpha\alpha'^* + \beta\beta'^*|^{2(R/\xi_p)^2}, \quad (2)$$

which defines $\Delta_{lml'm'}$. Δ_0 is the pairing amplitude, the pairing range is ξ_p, and l_F is the Fermi angular momentum, given by $\epsilon_F = l_F(l_F + 1)/(2mR^2)$. The functions $\alpha = \cos(\theta/2)$ and $\beta = \sin(\theta/2)e^{-i\phi}$ are spinor functions. Y_{qlm} are monopole harmonics [19], where q, l, and m are half odd integers. The order parameter $\Delta_p(\mathbf{\Omega}, \mathbf{\Omega}')$ acquires a 2π phase when $\mathbf{\Omega}$ encircles $\mathbf{\Omega}'$, which describes $p_x + ip_y$ pairing. $|\Delta_p|$ keeps the particles within the pairing range $|\mathbf{\Omega} - \mathbf{\Omega}'| \sim \xi_p$.

The order parameter field $F_{v\bar{v}}(\bar{\mathbf{\Omega}})$ describes the vorticity of the pair center of mass $\bar{\mathbf{\Omega}} = (\mathbf{\Omega} + \mathbf{\Omega}')/2$. We choose $F_{v\bar{v}}$ to describe an antivortex on the north pole and a vortex on the south pole, with the direction of vorticity defined relative to the chirality of the $p_x + ip_y$ order parameter, depicted in Fig. 1. For the vortex pair field, we use the analytical form (without self-consistency) [20]

$$F_{v\bar{v}}(\mathbf{\Omega}) = \frac{(\sin\theta)R/\xi}{\sqrt{1 + [(\sin\theta)R/\xi]^2}} e^{i\phi} = \sum_{L=1,3,5,\ldots} f_L Y_{L1}(\mathbf{\Omega}), \quad (3)$$

which defines f_L. Y_{LM} are spherical harmonics, and $\xi = 2\epsilon_F/(\pi\Delta_0 k_F)$ is Pippard's coherence length. We take $\xi_p < \xi$ for simplicity.

The BdG equation is represented as a matrix in terms of $3j$ symbols as

$$T_{lm,l'm'} = \epsilon_F \frac{l(l+1) - \frac{1}{4}}{l_F(l_F+1) - \frac{1}{4}} \delta_{ll'}\delta_{mm'}$$

$$\Delta_{lm,l'm'} = \delta_{m',1-m}\Delta_0\sqrt{(2l+1)(2l'+1)}[D_l + (-1)^{l+l'}D_{l'}] \times \sum_L f_L \sqrt{\frac{2L+1}{16\pi}} \begin{pmatrix} l & l' & L \\ -m & m-1 & 1 \end{pmatrix} \times \begin{pmatrix} l & l' & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},$$

$$D_l \simeq \frac{l}{l_F} e^{(-l^2 + l_F^2)(\xi_p/R)^2}.$$

Diagonalizing Eq. (1) produces a set of energies E_n and corresponding eigenvectors u_n^{lm}, v_n^{lm}. By azimuthal symmetry, m is a good quantum number. The BdG wave functions on the sphere are

$$u_n(\mathbf{\Omega}) = \sum_l u_n^{lm} Y_{-(1/2),l,m}(\mathbf{\Omega}), \quad (4)$$

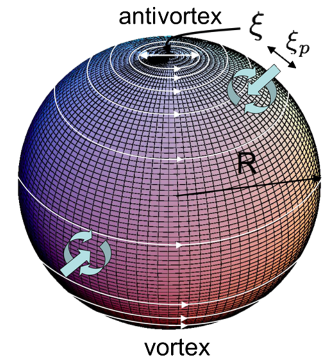

FIG. 1 (color online). A vortex pair of the $p_x + ip_y$ superconductor on the sphere, described by Eq. (2). Thin white lines represent the current flow. Wide arrows represent the pair relative angular momentum. ξ_p is the pairing range. ξ is the coherence length which determines the vortex core size.

$$v_n(\mathbf{\Omega}) = \sum_l v_n^{lm} Y^*_{-(1/2),l,-m+1}(\mathbf{\Omega}). \quad (5)$$

In Fig. 2, we depict the BdG spectrum of the vortex pair as a function of m. The continuum states above the gap $|E_n| > \Delta_0$ are extended, while the branch that approaches zero is the $p_x + ip_y$ version of the CdGM core states. Their number is of order ϵ_F/Δ_0, and their spacing is of order δ_c [16].

As seen in the inset in Fig. 2, each CdGM state is almost doubly degenerate. The splitting represents weak tunneling between the north and south pole core states. Indeed, we find that the tunnel splittings decrease exponentially with the radius of the sphere $\delta E_n \sim e^{-R/\xi}$ for $R \gg \xi$.

The lowest positive energy approaches zero as $E_0 \sim e^{-R/\xi}$. In the large sphere limit, the wave functions $u_0(\mathbf{\Omega}) \approx v_0(\mathbf{\Omega})$ are equally split between the north and south poles. The corresponding BdG fermion is constructed out of two well separated Majorana operators in

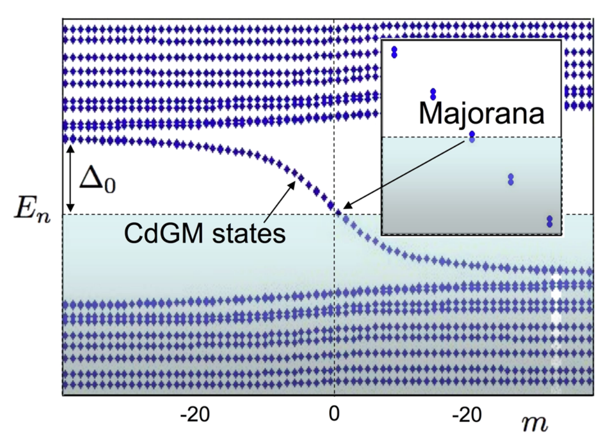

FIG. 2 (color online). BdG spectrum $E_{n(m)}$, of the vortex pair on the sphere, depicting the CdGM core states. The inset shows that their double degeneracies are split by weak tunneling between the poles. The state nearest zero energy is the Majorana mode of both the vortex and the antivortex.

the north and south poles. We have also verified that E_0 is insensitive to the addition of moderate potential disorder [21]. This agrees with previous asymptotic calculations in the plane which have shown that the Majorana excitations are ‘‘topologically protected’’ against perturbations [8,9].

The asymptotic predictions for the wave functions $u_0(r)$ for coreless vortices in the plane [7] are

$$u_0(\mathbf{x}) \sim \begin{cases} J_0(k_F r)e^{-r/\pi\xi} & \text{antivortex,} \\ J_1(k_F r)e^{i\phi}e^{-r/\pi\xi} & \text{vortex,} \end{cases} \tag{6}$$

which are valid for *both* $r \gg \xi$ and $r \ll \xi$. Numerically, we confirmed that these predictions hold even in finite core sizes.

The physical reason behind the difference in Eq. (6) is that the Majorana wave functions are sensitive to the *sum* of vorticity and relative angular momentum. J_0 is obtained only when that sum vanishes, and this is important for the experimental signature we discuss below.

Local density of states (LDOS).—At zero temperature the LDOS is defined as [13]

$$\mathcal{T}(E, r) = \sum_n |u_n(r)|^2 \delta(E - E_n) + |v_n(r)|^2 \delta(E + E_n), \tag{7}$$

where r is the distance from the vortex (or antivortex) center.

Figure 3 shows the LDOS near the cores of the vortex and the antivortex for displacements $r \ll \xi$ and energies $|E| \leq \Delta_0$. The Majorana state can be easily discerned as the zero energy peak in both vortex and antivortex cores.

The other CdGM core states also appear as oscillatory peaks, with energy spacing δ_c. The difference between the vortex and antivortex excitations is apparent: In Eq. (6) the antivortex Majorana state is peaked at $r = 0$, while the other CdGM states have nodes at $r = 0$. In contrast, the vortex Majorana state is peaked at half a Fermi wavelength away from the origin, while two CdGM states are peaked at $r = 0$. Notice that, in the vortex, the Majorana state has a significantly lower peak than in the antivortex.

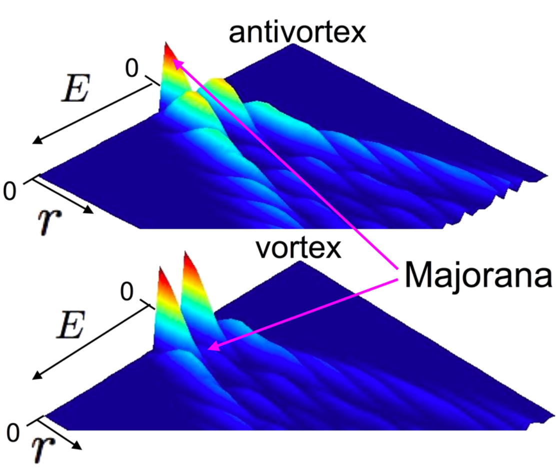

FIG. 3 (color online). Zero temperature local density of states of Eq. (7) near the vortex and the antivortex centers. The peaks belong to the CdGM states. Notice that the zero energy Majorana mode is removed from the origin in the vortex, while it is maximized at the origin in the antivortex.

In a tunneling spectroscopy experiment (e.g., Ref. [22]), the discrete LDOS spectrum is smeared by temperature broadening. The tunneling conductance [23] is defined as

$$\frac{dI}{dV}(E, r) \sim T \int dE' \left(\frac{\partial f(E - E')}{\partial E'}\right) \mathcal{T}(E', r), \tag{8}$$

where $f(E)$ is the Fermi-Dirac distribution at zero chemical potential and temperature T.

In the BCS weak coupling regime, $k_F\xi \gg 1$, and therefore δ_c could be a very small temperature scale. At moderate temperatures $\delta_c < T < \Delta_0$, the peaks of Fig. 3 are smeared on the energy axis (but not on the r axis), and therefore an asymmetry effect can be observed.

A typical tunneling conductance is depicted in Fig. 4, which shows a central peak at $r = 0$, $E = 0$, with low broad ridges dispersing away to larger r and E. We see that the central peak of the vortex is twice the height of that of the antivortex.

This effect is a direct consequence of Eq. (6). Under temperature smearing the *two* CdGM peaks at $r = 0$ of the vortex merge into one large central peak. In contrast, only a single Majorana state is responsible for the central peak of the antivortex. Since the relevant maximas in the LDOS are nearly identical, a ratio of 2 is obtained at elevated temperatures.

Our effect requires having spatial resolution in tunneling conductance better than a Fermi wavelength. If $dI/dV(r, E)$ is convoluted with an areal resolution of $(\delta r)^2 > \lambda_F^2$, the ratio between the vortex and antivortex peak heights rapidly approaches unity as $\delta r > \lambda_F$. The ratio is weakly temperature-dependent in the regime $\delta_c < T < \Delta_0$.

In real three-dimensional samples, zero bias peaks are somewhat suppressed by bulk states and surface imperfections (Ref. [22] reports a 15% enhancement above the high voltage background). Nevertheless, it is the ratio of 2 between the vortex and an antivortex enhancement which

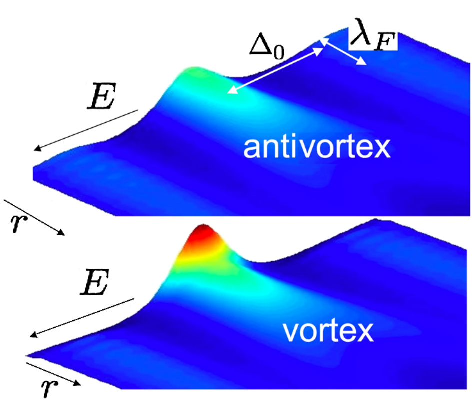

FIG. 4 (color online). Tunneling conductance of Eq. (8), in arbitrary units. λ_F is the Fermi wavelength. The temperature $T = 0.15\Delta_0$ is about 10 times larger than the CdGM level spacing.

would signal the Majorana state. To avoid changes in the background, we suggest to leave the tip at the same position while reversing the magnetic field. The field should be localized and weak enough so as not to overturn the chiral order parameter.

General cases.—Some difference between vortex and antivortex excitations is expected for any chiral symmetry breaking (CSB) superconductor. The important questions are whether this difference is observable at $T > \delta_c$ and whether it is sensitive to the existence of Majorana fermions.

For a CSB superconductor with relative angular momentum $M = 1, 2, \ldots$ ($p_x + ip_y$, $d_x + id_y, \ldots$), there is a Majorana state in the vortex core, provided the vorticity N obeys $N + M = 0, \pm 2, \pm 4, \ldots$ [9]. However, in most cases these Majorana states vanish at the origin. The only exceptions are cores of antivortices which satisfy $N = -M$. Our factor of 2 effect will be observable only for this subset of cases. Notice that $p_x + ip_y$ is the only case where the effect occurs for vorticity $|N| = 1$.

Since the core states are sensitive only to large potential gradients, moderate disorder does not destroy the Majorana states. We have explicitly confirmed this expectation numerically [21], by solving the BdG equation with a white noise potential. For disorder potential fluctuations up to order ϵ_F, the Majorana tunneling energy decays with R, and $u_0(r)$ with r, with the same exponents as the clean system, and the peak height doubling signature of the Majorana states is essentially unaffected.

Summary.—We solved the BdG spectrum of $p_x + ip_y$ vortex pair state in the spherical geometry. We showed that, even at high temperatures compared to the CdGM state spacing, a signature of the Majorana state remains when one compares the LDOS of the vortex to that of the antivortex.

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