Comparison of Tunneling Rates of Fractional Charges and Electrons across a Quantum Hall Strip

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The fundamental mechanism of current dissipation in a Hall strip is investigated by searching for the fastest process of charge relaxation between the edges. The tunneling rate of the fractional charge across a $\nu = 1/3$ Laughlin state of width Y on the cylinder is found to fit $t_{1/3} \propto \exp[-\alpha Y^2/12\lambda^2]$, where λ is the Landau length, and $\alpha \simeq 1.0$. This rate is exponentially *larger* than the electron tunneling rate, and can be interpreted by analogy to the tunneling of a vortex through a superfluid. Fractional charge tunneling dominates current relaxation. It determines the Aharonov-Bohm oscillation period, and the magnitude of quantum shot-noise. [S0031-9007(97)05065-5]

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In a superconducting strip, the elementary process of current dissipation is the transfer of a vortex from one edge to the other, or the nucleation of a vortex pair and its separation to opposite edges. In a quantum Hall strip, dissipation is given by moving a charge between the two edges. For bulk filling fractions $\nu = 1/m$ (m is an odd integer [1]), there are low lying edge excitations which carry fractional charge $Q = \nu e$. At zero temperature, and low bias current, it is natural to wonder: Which tunnels faster, fractionally charged quasiparticles, or electrons? An impurity potential which breaks translational invariance in the longitudinal direction *allows both processes.* For electrons, the tunneling rate is simply the matrix element of the potential between single electron edge states. For fractionally charged quasiparticles, however, the tunneling rates involve an overlap of correlated many electron wave functions.

Besides intellectual curiosity, there are experimental implications which motivate us to find the *relative* rates of fractional versus integer charge tunneling: (i) Luttinger liquid edge theory [2] predicts different leading powers (of current and temperature) of the longitudinal resistance for different elementary charges. (ii) The Aharonov-Bohm (AB) flux periodicity $\Delta\phi$ of current oscillations which was measured in resonant tunneling [3] depends on the elementary tunneling charge as $\Delta\phi = e\nu\phi_0/Q$ [4]. (iii) The charges which dominate the backscattering current I_B can be measured by the magnitude of the quantum shot-noise S by $S = 2QI_B$ [5]. Recent experiments in the $\nu = 1/3$ phase report excellent fits to fractional charge $Q = e/3$ [6].

Kane and Fisher (KF) [7] calculated the renormalization group flows of the tunneling coupling constants due to low lying Luttinger liquid edge excitations. They found that for $\nu < 1$, electron tunneling becomes irrelevant at low enough temperatures while fractional charge tunneling flows to strong coupling as the infrared cutoff is reduced. In KF theory, however, the tunneling rate is a free parameter, which leaves the possibility that it might be undetectable at experimental temperatures.

This Letter presents a microscopic calculation of fractional charge tunneling rates across a Hall fluid. The numerical results show that the fractional charge tunneling rate is much larger than the electron charge rate at large strip widths. Subsequently, we connect the microscopic tunneling matrix element to the interedge scattering parameter of KF theory, and discuss its experimental implications.

Our domain is the open cylinder $x \in [0, 2\pi R), -\infty < y < \infty$, with N electrons, and a radially penetrating field $B = \frac{\phi_0}{2\pi\lambda^2}$, where λ, the Landau length, is henceforth our unit of distance. This geometry can describe a quantum Hall liquid strip with two symmetric edges.

The free electron states of the lowest Landau level (LLL) are labeled by momenta $k = \gamma n$, n integer, and $\gamma \equiv 1/R$. The wave functions are

$$\psi_k(x, y) = \sqrt{\frac{\gamma}{2\pi}} \exp\left(ikx - \frac{(y-k)^2}{2}\right). \qquad (1)$$

The Laughlin state of filling fraction $\nu = \frac{1}{m}$ on the cylinder was given by Thouless [8]

$$\Psi^{1/m} = \prod_{i<j}(e^{i\gamma(x_i+iy_i)} - e^{i\gamma(x_j+iy_j)})^m \prod_i e^{-y_i^2/2}. \qquad (2)$$

It is the ground state of a suitably defined pseudopotential Hamiltonian [1,9]. The expansion of $\Psi^{1/m}$ in the LLL Fock basis is

$$\Psi_L^{1/m} = \sum_{[\mathbf{k}]} A[\mathbf{k}/\gamma] \exp\left(\sum_i k_i^2\right) |\mathbf{k}\rangle, \qquad (3)$$

where $|\mathbf{k}\rangle = |k_1, \ldots, k_N\rangle$ and $k_i \in [0, Y]$. $Y = m\gamma(N - 1)$ is defined as the *width* of the Hall liquid strip (the width of the area partially occupied by electrons depicted between the horizontal solid lines in Fig. 1.

There is an infinite family of other degenerate ground states labeled by the total momentum $P = \sum_i k_i$, which

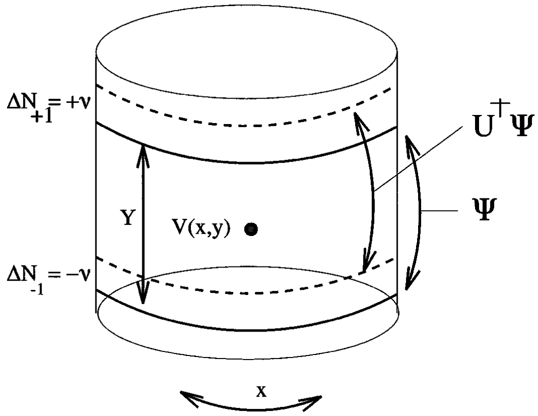

FIG. 1. Fractional charge tunneling depicted by two displaced Laughlin states of bulk density ν on the cylinder. ΔN_p are the edge charge differences, and V is an impurity potential which enables a transition between the states.

are given by uniformly shifting the momenta k_i moving the electron density up or down the cylinder. A weak ($v \to 0$) confining potential $V(y) = \frac{v}{2}(y - Y/2)^2$ selects (3) as the ground state.

The expansion coefficients A are given by [10]

$$A[\mathbf{n}] = \frac{1}{N!} \sum_{\mathbf{r}^1,\ldots,\mathbf{r}^m} (-1)^{\sum_{l=1}^m \mathcal{P}[\mathbf{r}^l]} \prod_{i=1}^N \delta\left(n_i - \sum_{l=1}^m r_i^l\right), \quad (4)$$

where $\mathbf{r}^l$ is a permutation of the set $0, 1, \ldots, N - 1$, and $\mathcal{P}$ is the parity of a permutation.

The coefficients A have a complicated structure [11], but it is useful to note that the components with $A[\mathbf{k}/\gamma] \neq 0$ can be derived from a single parent Tao-Thouless (TT) state [12]

$$|\mathbf{k}^{TT}\rangle = |0, m\gamma, 2m\gamma, \ldots, Y\rangle . \quad (5)$$

For this state $A[\mathbf{k}^{TT}/\gamma] = 1$. All other $|\mathbf{k}\rangle$ components are given by successively squeezing pairs of momenta toward each other. Rezayi and Haldane have shown [9] that in the regime $1 \ll Y \ll N$ the occupation number is constant for k far from the edges, i.e., $n_k = \langle c_k^\dagger c_k \rangle = 1/m$ for $1 < k < Y - 1$.

An impurity potential in the LLL Fock representation is

$$\mathcal{V} = \sum_{kk'} V_{k,k'} c_k^\dagger c_{k'} , \quad (6)$$

where $c_k^\dagger$ creates an electron in state ϕ_k. The ground state to ground state tunneling rate of charge qe between the edges, to leading order in $\mathcal{V}$, is

$$t_q = \langle \Psi | \mathcal{V} U^{mq} | \Psi \rangle , \quad (7)$$

where U is the unitary phase operator which translates all the single particle momenta by one interval

$$U^\dagger c_k^\dagger U = c_{k+\gamma}^\dagger . \quad (8)$$

$U^\dagger$ moves a fractional charge $1/m$ from the $p = -1$ to the $p = +1$ edge (see Fig. 1), and thus increases the total momentum of the Hall state by $P \to P + Q$, where $Q = N\gamma$.

For a weak impurity potential, the tunneling rate of a fractional charge is thus given by

$$t_{1/m} = \langle \Psi | \mathcal{V} U | \Psi \rangle = \sum_k V_{k,k+Q} M_{k,k+Q}$$
$$M_{k,k+Q} = \frac{1}{Z} \sum_{\mathbf{k},\mathbf{k}'} A[\mathbf{k}/\gamma] A[\mathbf{k}'/\gamma] e^{\frac{1}{2}(\mathbf{k}^2 + \mathbf{k}'^2)} \times \prod_{n=1}^N \delta^N[\mathbf{k} + \mathbf{Q}(n), \mathbf{k}' - \gamma \mathbf{1}], \quad (9)$$

where $Z = \langle \Psi \mid \Psi \rangle$, $Q_i(n) = Q\delta_{in}$, and $\mathbf{1}_i = 1$. $M_{k,k+Q}$ reflects the many-body overlap of the relatively displaced Laughlin states. Its weighted sum $M(Q) = V^{-1} \sum_k V_{k,k+Q} M_{k,k+Q}$ was computed numerically for local impurity potentials $V\delta(x)$ and $V\delta(x)\delta(y - Y/2)$. The calculation was carried out for $\nu = 1/3$ states with five up to eight electrons. As shown in Fig. 2 at large widths we find the asymptotic decay

$$|M(Q)| \propto \exp\left(-\frac{\alpha}{2} Q^2\right), \quad (10)$$

where $\alpha \approx 1.0$, and independent of the number of electrons. Combining (10) with $V_{0,Q} \propto \exp(-\frac{1}{4}Q^2)$ yields the tunneling rate's asymptotic dependence on width

$$t_{1/3} \sim \exp(-\alpha Y^2/12) . \quad (11)$$

In comparison, a unit charge tunneling rate, which is proportional to the potential matrix element, is

$$t_1 = \langle \Psi | \mathcal{V} U^3 | \Psi \rangle = n_0^2 V(0, Y) . \quad (12)$$

For a localized potential of the form $V\delta(s)\delta(y - Y/2)$,

$$t_1 \sim \gamma^2 \exp(-Y^2/4) , \quad (13)$$

where $n_0 \approx \gamma^\beta$ is appropriate for a density profile which vanishes as a power law $n_k \approx k^\beta$ at the edge. [The numerical results for n_k of the Laughlin state (3) up to eight particles is $\beta = 1.0$.] Thus, *the tunneling exponent is 3 times larger for quasiparticles than for electrons.*

The result (11) could be understood using the superfluid description of the fractional Hall phase, which can be derived by the Chern-Simons Ginzburg-Landau functional [13]. At the mean field level, the ground state is a Bose superfluid of density $\rho_s = \frac{1}{m} B/\phi_0$. The dissipation of current involves tunneling of vortices between opposite edges, where a vortex of unit circulation carries a fractional electric charge of e/m. Ignoring auxiliary gauge field fluctuations, and interactions at the core length scale, the vortex dynamics are governed by a Magnus force $e\phi_0\rho_s \mathbf{v} \times \hat{\mathbf{z}}$. Thus they are quantized as particles with charge e in the lowest Landau level of an effective field $\tilde{B} = \phi_0\rho_s$, and Landau length $\tilde{\lambda} = \sqrt{m}\,\lambda$, with wave functions given by (1). For $m = 3$, the matrix element of $\mathcal{V}$ between two vortex wave functions at the edges readily recovers (11), with $\alpha = 1$.

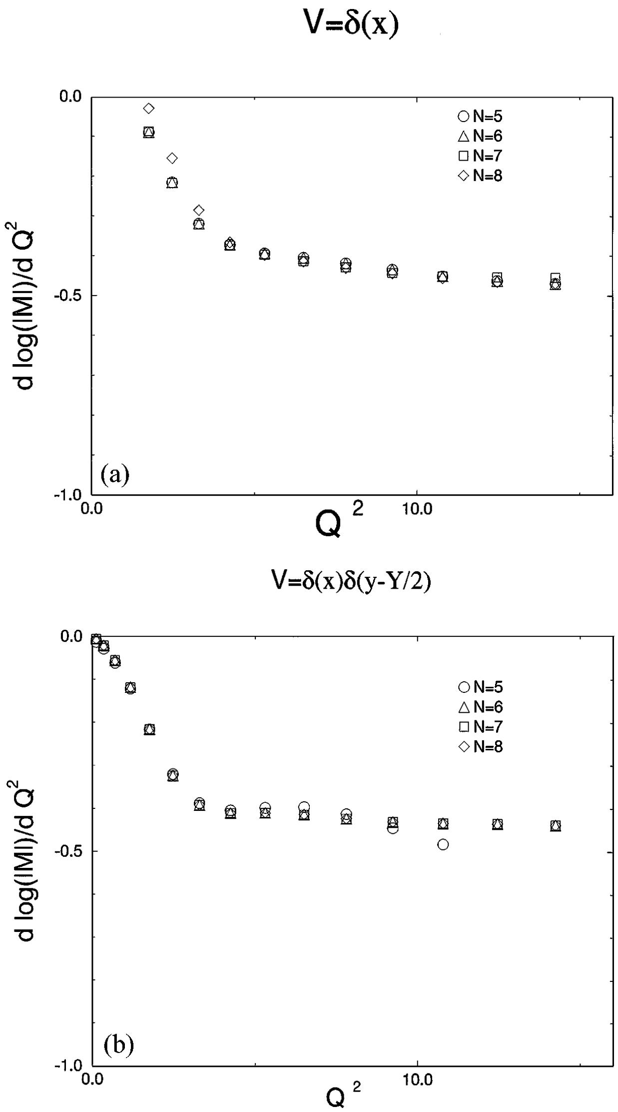

FIG. 2. Numerical evaluation of the asymptotic decay of the many-body factor $M(Q)$; see Eq. (10) for the $\nu = 1/3$ Laughlin states. (a) and (b) show similar dependence for two different localized impurity potentials.

How do tunneling matrix elements couple to edge excitations? A half-strip density operator is defined as follows:

$$\rho_p(q) = \int_{Y/2}^{Y_p} dy \int_0^{2\pi R} dx\, e^{iqx} \rho(x,y)$$
$$\approx \sum_k \theta_{p,k}\theta_{p,k+q} c_{k+q}^\dagger c_k + \mathcal{O}(q^2)\,, \qquad (14)$$

where $p = \pm 1$, $Y_p = (1+p)Y/2$ are the two edge y coordinates, and

$$\theta_{pk} = \begin{cases} 1 & p(k - Y/2) > 0\,, \\ 0 & p(k - Y/2) < 0\,. \end{cases} \qquad (15)$$

The last approximation in (14) applies to the long wavelength regime $q \ll 1$. The commutation relations of $\rho_p(q)$ are

$$[\rho_p(q), \rho_{p'}(q')] = \delta_{pp'}\delta_{q,-q'} \sum_k \theta_{pk}\theta_{p,k+q}(n_{p,k} - n_{p,k+q}) + \delta_{pp'}\{q \neq -q'\}\,. \qquad (16)$$

Since excitations in the bulk have an energy gap Δ_B, the low energy sector includes only particle-hole excitations near the edges, i.e., $c_k^\dagger d_{k+q}\Psi$ with $k \approx Y_p$, and energies $\omega_q = vq$, where v is the gradient of the confining potential. $\{q \neq q'\}$ terms in (16) create excitations deep in the bulk which introduce corrections suppressed by factors of ω_q/Δ_B and q/Y. Also, in this sector $n_{p,k}$ is approximately diagonal

$$n_{p,k} \simeq \begin{cases} \nu & p(Y_p - k) \gg 1\,, \\ 0 & (Y_p - k) \gg 1\,. \end{cases} \qquad (17)$$

Thus, Wen's Kac-Moody algebra of edge bosons [2] is recovered:

$$[\rho_p(q), \rho_{p'}(q')] \simeq \delta_{pp'}\delta_{q,-q'}\gamma^{-1}\nu q\,. \qquad (18)$$

The edge charge operator is $N_p = \sum_k \theta_{pk} n_{p,k}$, which is conjugate to the edge phase operators U_p

$$[N_p, U_p^\dagger] = p n_{p,Y/2} U_p^\dagger \simeq p\nu U_p^\dagger\,. \qquad (19)$$

The total phase operator (8) is $U^\dagger = U_1^\dagger U_{-1}$. The edge quasiparticle creation operator is constructed following Haldane [14]

$$\phi_p = p\gamma\left(xN_p/2 + i \sum_{q \neq 0} \theta(-pq) \frac{e^{-iqx}}{q} \rho_p(q) \right),$$
$$\psi_p^\dagger(x) = \mathcal{A} e^{iY_p x} e^{i\phi_p^\dagger(x)} U_p^\dagger e^{i\phi_p(x)}, \qquad (20)$$

where $\mathcal{A}$ is an undetermined normalization constant. $\psi_p^\dagger(x)$ creates a localized edge excitation of extra charge ν as evidenced by the commutator with $\rho_p(x) \equiv \sum_q e^{iqx}\rho_p(q)$:

$$[\rho_p(x), \psi_{p'}^\dagger(x')] = \nu\delta_{pp'}\delta(x - x')\psi_{p'}^\dagger(x)\,. \qquad (21)$$

The impurity potential operator in the low energy sector simply transfers a localized fractional charge between the edges. It must therefore be proportional to the normal ordered operator

$$\mathcal{V}(x) = :\psi_p^\dagger(x)\psi_{-p}(x): + \text{H.c.}$$
$$= \mathcal{A}^2 e^{iYx} e^{i\sum_p p\phi_p^\dagger(x)} U^\dagger e^{i\sum_p p\phi_p(x)} + \text{H.c.} \qquad (22)$$

The normalization $\mathcal{A}^2$ is precisely the bare fractional charge tunneling parameter amplitude in KF theory [7]. It can now be determined by sandwiching both sides of Eq. (22) between the relatively displaced ground states leading to

$$\mathcal{A}^2 = \langle\Psi|\mathcal{V}U|\Psi\rangle = t_{1/m}\,. \qquad (23)$$

We make the following two comments: (i) Tao and Haldane [15] have shown that in the absence of an impurity potential, the quantum Hall ground state of $\nu = 1/m$ on the *torus* has an m fold degeneracy. A time dependent AB flux threading the torus moves the ground state between the m different states of this manifold, and the Hall conductance is precisely $\sigma_{xy} = \frac{1}{m}(e^2/h)$. An impurity potential couples between degenerate ground states, as it does on the cylinder, opening a minigap Δ between the ground state and the first excited state [16]. The quantum Hall effect can be observed provided the flux does not vary extremely slowly [15], i.e., $\Delta/\hbar \ll \phi_0/V_x$, where V_x is the induced electromotive force.

(ii) For the infinite plane geometry, the tunneling exponent between two *localized* quasiparticle states centered on delta function impurities at $\mathbf{r}_1, \mathbf{r}_2$ is [17]

$$S(\mathbf{r}_1 - \mathbf{r}_2) \propto \exp\left(-\frac{B}{4m\phi_0}|\mathbf{r}_1 - \mathbf{r}_2|^2\right). \qquad (24)$$

The scaling of the tunneling action with B/m was argued to be a general property of low elementary excitations with fractional charge $1/m$. Generalizing this idea further, Jain, Kivelson, and Trivedi [18] have formulated a "law of corresponding states" which relate the dissipative response of quantum Hall liquids at different filling fractions using a conjecture that they scale with $\frac{e}{m}B$. Although the results reported here are consistent with this law, they are limited to *wide* Hall strips in the presence of *weak* impurity potentials. They depend on the particular correlations of the Laughlin state. *The Tao-Thouless state* [12], for example, has the same bulk density and total momentum as Laughlin's state, *but its impurity matrix element for transfer of fractional charge between the edges is zero*.

KF have shown that edge excitations enhance the fractional charge and suppress the unit charge contributions to the backscattering current [7]. Thus the renormalization group flow enhances the bare tunneling ratio, which strengthens the experimental relevance of KF theory.

Current oscillations in the presence of a slowly time dependent AB flux measures the dominant current relaxation mechanism. The lowest level crossing between adiabatic energy curves occurs at $\phi_{\mathrm{AB}} = \phi_0/2$ between ground states which are relatively shifted by charge e/m at their edges [4]. The impurity potential opens there a minigap of size $t_{1/m}$. A minigap t_1 opens at higher energies between energy curves with minima separated by $m\phi_0$. We do not expect to observe transfer of charge e reflected by a periodicity of $m\phi_0$ in the current oscillations, even for nonadiabatic changes of flux, because $t_{1/m} \gg t_1$. At the same time, resonant tunneling oscillations have directly measured the discrete fractional charge tunneling in and out of an antidot from the edges [4].

Finally, the quasiparticle charge which dominates the backscattering current between the edges can be measured by quantum shot-noise at zero temperature and bias [5]. Recent experimental reports of measuring fractional charge in quantum shot-noise of fractional quantum Hall systems [6] are consistent with the expectation that fractional charges tunnel faster than electrons.

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