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Landau-Zener Tunneling for Dephasing Lindblad Evolutions

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Abstract: We consider a family of time-dependent dephasing Lindblad generators which model the monitoring of the instantaneous Hamiltonian of a system by a Markovian bath. In this family the time dependence of the dephasing operators is (essentially) governed by the instantaneous Hamiltonian. The evolution in the adiabatic limit admits a geometric interpretation and can be solved by quadrature. As an application we derive an analog of the Landau-Zener tunneling formula for this family.

Lindblad generators describe the quantum evolutions of finite systems coupled to a memoryless bath [1,2]. They give a useful phenomenological description of thermalization, decoherence, and measurement [3–5]. We consider a family of time-dependent dephasing Lindblad generators, first introduced in [6], which models the continuous monitoring of the instantaneous energy. This is the case, for example, in the Zeno effect [7]. The family can be defined for arbitrary dephasing rate, however, its physical interpretation in the strong dephasing regime requires some care [8].

The family of dephasing Lindbladians that we consider is defined in such a way that any instantaneous stationary states of the Hamiltonian are also instantaneous stationary states for the Lindbladian. This makes the family special and non-generic. (Generic Lindbladians have a unique stationary state.) In the adiabatic limit, the evolution generated by this family can be solved by quadrature and admits a geometric interpretation. The qualitative features of the dynamics differ from the corresponding dynamics of generic Lindbladians [9–11] reflecting the special character of the family.

In 1932 Landau [12] and independently Zener [13], Majorana [14], and Stückelberg [15] found an explicit formula for the tunneling in a *generic* near crossing undergoing unitary, adiabatic evolution. Here we describe the corresponding result for the non-unitary case associated with a dephasing Lindbladian. The solution appears to be the simplest generalization of the Landau-Zener problem which is still exactly soluble.

The influence of dissipation and noise on the tunneling of a two-level system has been extensively studied in the physics literature [16–20]. The results presented here differ in one or both of the following aspects: First, in the framework: We *assume* a Lindbladian, and do not attempt to derive an effective dynamics from a (unitary) model of a bath or a (unitary) model of stochastic noise source. Second: The Lindbladians are, as stated, of the dephasing type. Our results contribute to the mathematical physics of Lindblad operators. This approach has the virtue that the adiabatic dynamics can be solved by quadrature and does not rely on the assumption of weak dephasing which one normally needs to make when modelling decoherence with a unitary bath or noise.

In the limit of weak dephasing, the tunneling formula we derive can be compared with results of [19] for Zener tunneling due to a dephasing noise. The two formulas have the same functional form up to an overall constant which is left undetermined in [19].

Since tunneling is dominated by the near crossing dynamics, the universal aspects of near crossing in a two-level system are captured by a Hamiltonian that depends *linearly* on time. By an appropriate choice of basis and of the zero of energy the relevant dynamics is governed by the Hamiltonian

$$H(s, g_0) = \frac{1}{2}\begin{pmatrix} s & g_0 \\ g_0 & -s \end{pmatrix}, \quad (s = \varepsilon t), \tag{1}$$

where $\varepsilon > 0$ is the adiabatic parameter and $g_0 > 0$ is the minimal gap. The tunneling probability T is the probability of a state, which originates asymptotically on one eigenvalue branch, to end up in the other at late times. The formula Landau and Zener found[1] for this Hamiltonian is:

$$T = e^{-\pi g_0^2/2\hbar\varepsilon}. \tag{2}$$

The singularity of the limit $\hbar\varepsilon \to 0$ reflects the singularity of the adiabatic and semiclassical limits, and their coincidence in this case.

The universal aspects of tunneling for near crossing in an *open* system described by a dephasing Lindbladian is described as follows. The adiabatic evolution of the density matrix ρ is governed by

$$\hbar\varepsilon\dot{\rho} = \mathcal{L}_s(\rho), \quad (\varepsilon > 0), \tag{3}$$

where the slowly varying parameter $s = \varepsilon t$, having the physical dimension of an energy, is viewed as the slow clock. $\mathcal{L}_s$ is the changing Lindblad operator

$$\mathcal{L}(\rho) = -i[H, \rho] - \hbar\gamma(P_-\rho P_+ + P_+\rho P_-); \tag{4}$$

H is the Hamiltonian, which for a generic near crossing is given in Eq. (1); $P_\pm = |\pm\rangle\langle\pm|$ are the two spectral projections of H; finally, $\gamma > 0$ is the dephasing rate.[2] $\gamma = 0$ is the case considered by Landau and Zener. In both cases transitions between the ground and the excited state only occur because the generator of the dynamics depends on s. The tunneling probability

$$T = \mathrm{tr}(\rho P_+)(\infty), \quad (\rho(-\infty) = P_-(-\infty)) \tag{5}$$

is the error in fidelity of the ground state.

[1] Landau, who used semiclassical methods did not actually attempt to calculate the multiplicative overall factor in front of the exponential in Eq. (2). Fortunately, this factor happens to be unity. Zener solved the differential equation exactly in terms of Weber functions and derived Eq. (2) exactly. Zener was aware of Landau's solution but for some reason, incorrectly, believed that Landau missed a factor of 2π in the exponent.

[2] γ^{-1} is the dephasing time commonly denoted by T_2.

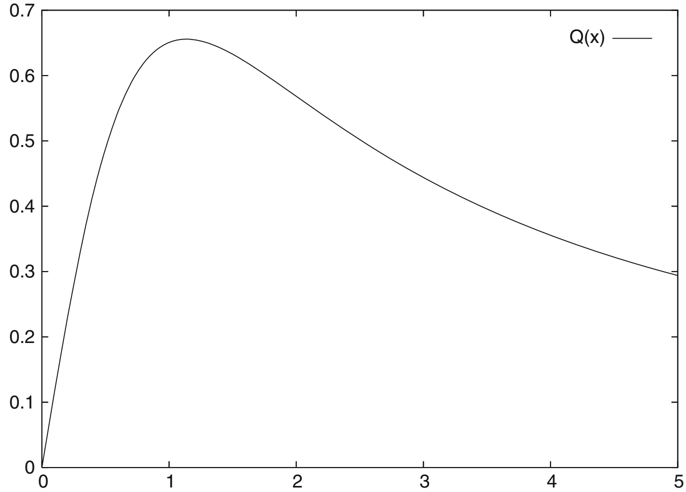

Fig. 1. The function $Q(x)$. The argument is the ratio of dephasing rate to the minimal gap. The function has a maximum at $x = 1.13693$

The adiabatic tunneling formula with dephasing, which we shall derive below, is[3]

$$T = \frac{\varepsilon\hbar}{2g_0^2}\, Q\left(\frac{\hbar\gamma}{g_0}\right) + O(\varepsilon^2), \tag{6}$$

where Q is the algebraic function (shown in the figure)

$$Q(x) = \frac{\pi}{2}\,\frac{x(2+\sqrt{1+x^2})}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)^2}. \tag{7}$$

A few remarks about this result are in order:

- The adiabatic limit means that $\sqrt{\hbar\varepsilon}$ is the smallest energy scale in the problem and in particular, $\varepsilon \ll \hbar\gamma^2$. When this fails, the error terms in Eq. (6) need not be small compared to the leading term.
- When the dephasing is weak, $\hbar\gamma \ll g_0$, Eq. (6) reduces to

$$T \approx \frac{3\pi}{16} \cdot \frac{\varepsilon\gamma\hbar^2}{g_0^3}. \tag{8}$$

This term has the same form as one of the tunneling terms found by Shimshoni and Stern [19] in a (different) model where the two-level system is dephased by noise. The method they use cannot give the overall constant $3\pi/16$ [21], nor does it allow investigating the full range of $\hbar\gamma/g_0$.

[3] Since the problem has two dimensionless parameters: $\hbar\varepsilon/(\hbar^2\gamma^2 + g_0^2)$ and $\hbar\gamma/g_0$ it is not obvious what is the correct dimensionless expression corresponding to $O(\varepsilon^2)$. The correct interpretation of $O(\varepsilon)$ turns out to be the product $O\left(\frac{\hbar\varepsilon}{\hbar^2\gamma^2+g_0^2}\,\frac{\hbar\gamma}{g_0}\right)$ which can be verified by Eq. (10).

- When $\hbar\gamma \gg g_0$ Eq. (6) reduces to

$$T \approx \frac{\pi\varepsilon}{4g_0\gamma}. \tag{9}$$

This may be understood as a manifestation of the quantum Zeno effect [7] by the following interpretation (cf. [4, Sect. 5]): Imagine that the energy H is measured with probability $\gamma\delta t$ after a short waiting time δt. If a measurement happens, its outcome shall not be recorded. In the process the state is changed from ρ first to $\tilde\rho = e^{-iH\delta t/\hbar}\rho e^{iH\delta t/\hbar}$ and then to

$$\begin{aligned}(1-\gamma\delta t)\tilde\rho + \gamma\delta t \sum_{i=\pm} P_i\tilde\rho P_i \\ = \rho - \frac{i}{\hbar}[H,\rho]\delta t + \gamma\left(\sum_{i=\pm} P_i\rho P_i - \rho\right)\delta t + O((\delta t)^2).\end{aligned}$$

In the limit $\delta t \to 0$ the resulting dynamics is generated by Eq. (4). Hence the dephasing term in the Lindblad generator can be viewed as a continuous monitoring of the state of the system at rate γ. Transitions between the states $P_\pm$, which the changing Hamiltonian term potentially induces, are suppressed at high measurement rates, as stated in Eq. (9) and in line with the Zeno effect.

Equation (6) follows from, and is a special case of, a more general and basic formula for the tunneling when the adiabatic evolution takes place on a finite interval of (slow) time $[s_0, s_1]$ and one also allows $\gamma(s)$ to be time-dependent

$$T = 2\varepsilon\hbar^2 \int_{s_0}^{s_1} \gamma(s)\frac{\mathrm{tr}(P_+\dot P_-^2 P_+)}{g^2(s)+\hbar^2\gamma^2(s)}\,ds + O(\varepsilon^2). \tag{10}$$

Here $g(s)$ is the instantaneous gap in $H(s)$,

$$g^2(s) = s^2 + g_0^2. \tag{11}$$

The positivity of the integrand in Eq. (10) when $\gamma > 0$ makes the tunneling irreversible. This changes the characteristics of the ε dependence of T from exponentially small in Eq. (2) to linear in Eq. (6). The Landau-Zener formula, Eq. (2), is buried in the error terms of Eq. (10).

Equation (10) reduces the tunneling problem to integration. In the case where γ is constant and s runs from $-\infty$ to ∞ the numerator Eq. (10) is simply

$$\mathrm{tr}(P_+\dot P_-^2 P_+) = \frac{g_0^2}{4g^4(s)}. \tag{12}$$

Elementary algebra then leads to Eq. (6) with

$$Q(x) = x\int_{-\infty}^{\infty}(t^2+1)^{-2}(t^2+1+x^2)^{-1}dt. \tag{13}$$

The integral can be evaluated explicitly to give Eq. (7).

The key idea behind the derivation of the adiabatic tunneling formula, Eq. (10), is a geometric view of spectral projections as *adiabatic invariants*. The evolution of *observables* is governed by the adjoint of the Lindblad generator, $\mathcal{L}^*$, (this is the Heisenberg picture), where the adjoint refers to the Hilbert-Schmidt inner product. In particular, the adjoint of the dephasing Lindblad operator of Eq. (4) acting on the observable A is given by (from now on we set $\hbar = 1$)

$$\mathcal{L}^*(A) = i[H, A] - \gamma(P_- A P_+ + P_+ A P_-). \tag{14}$$

It differs from Eq. (4) by the replacement of i by $-i$. As we shall now see an instantaneously stationary observable $A(s) \in \mathrm{Ker}(\mathcal{L}_s^*)$ that has no motion in $\mathrm{Ker}(\mathcal{L}_s)$ is an *adiabatic invariant*. More precisely,

Theorem 1. *Let $A(s)$ be an observable which lies in the instantaneous kernel of $\mathcal{L}_s^*$, i.e.*

$$\mathcal{L}_s^*\,(A(s)) = 0 \tag{15}$$

and suppose that, in addition, the linear equation

$$\dot{A}(s) = \mathcal{L}_s^*(X(s)) \tag{16}$$

admits a solution $X(s)$. Then one has

$$\mathrm{tr}(A(s)\rho_\varepsilon(s))\,\Big|_{s_0}^{s_1} = \varepsilon\,\mathrm{tr}(X(s)\rho_\varepsilon(s))\Big|_{s_0}^{s_1} - \varepsilon\int_{s_0}^{s_1} \mathrm{tr}(\dot{X}(s)\rho_\varepsilon(s))\,ds, \tag{17}$$

where $\rho_\varepsilon(s)$ is a solution of the adiabatic Lindblad evolution. $A(s)$ is an adiabatic invariant in the sense that its expectation is conserved up to a small error, $O(\varepsilon)$, given by the right hand side of Eq. (17) whereas the change in a generic observable is $O(\varepsilon^{-1})$ and in the Lindblad generator is $O(1)$.

The identity, Eq. (17), readily follows from

$$\begin{aligned}
\frac{d}{ds}\,\mathrm{tr}(A(s)\rho_\varepsilon(s)) &= \mathrm{tr}(\dot{A}(s)\rho_\varepsilon(s)) + \mathrm{tr}(A(s)\dot{\rho}_\varepsilon(s))\\
&= \mathrm{tr}(\mathcal{L}_s^*(X(s))\rho_\varepsilon(s)) + \varepsilon^{-1}\,\mathrm{tr}(A(s)\mathcal{L}_s(\rho_\varepsilon(s)))\\
&= \mathrm{tr}(X(s)\mathcal{L}_s(\rho_\varepsilon(s))) + \varepsilon^{-1}\,\mathrm{tr}(\mathcal{L}_s^*(A(s))\rho_\varepsilon(s))\\
&= \varepsilon\,\mathrm{tr}(X(s)\dot{\rho}_\varepsilon(s))
\end{aligned} \tag{18}$$

and integration by parts.

Equation (16) may be interpreted as a condition that $A(s)$ undergoes parallel transport: The equation has a solution provided $\dot{A}(s) \in \mathrm{Range}\,(\mathcal{L}_s^*)$ which is the case if $A(s)$ has no motion in $\mathrm{Ker}\,(\mathcal{L}_s)$.

It is straightforward to verify that the instantaneous spectral projections $P_j(s)$ of a dephasing Lindblad generator are adiabatic invariants in the sense of the theorem. Evidently, $\mathcal{L}_s^*\,(P_+(s)) = 0$. Moreover, Eq. (16) is solved by

$$X(s) = -i\sum_{k\neq j}\frac{P_k\dot{P}_+P_j}{e_k - e_j + i\gamma} \tag{19}$$

with $e_\pm$ the two eigenvalues of H. To see this note first that $X(s)$ is purely off-diagonal[4] by construction and so is $\dot{P}_+$, namely

$$\dot{P}_+ = P_-\dot{P}_+P_+ + P_+\dot{P}_+P_-. \tag{20}$$

[4] The orthogonal complement to the kernel is spanned by the operators $|\mp\rangle\,\langle\pm|$.

This follows from $P_+ = P_+^2$, which implies $\dot{P}_+ = \dot{P}_+ P_+ + P_+ \dot{P}_+$ and in turn $P_\pm \dot{P}_+ P_\pm = 0$. The equality of the off-diagonal components of Eq. (16) follows from

$$\mathcal{L}^*(P_k A P_j) = i(e_k - e_j + i\gamma) P_k A P_j, \quad (k, j = \pm, k \neq j). \tag{21}$$

The probability of leaking out of the instantaneous ground state is given by Eq. (17) with $A(s) = P_+(s)$. Equation (10) then follows by appealing to the adiabatic theorem [9–11] which allows to replace the instantaneous state by the instantaneous projection on the right hand side of Eq. (17),

$$\rho_\varepsilon(s) = P_-(s) + O(\varepsilon), \tag{22}$$

uniformly in s_0, s. The rest is simple algebra.

For the convenience of the reader we include a proof of Eq. (22). Let $U_\varepsilon(s, s_0)$ be the propagator for the differential equation (3), whence $\rho(s) = U_\varepsilon(s, s_0) P_-(s_0)$. We recall that the solution of its inhomogeneous variant, $\dot{x} = \varepsilon^{-1} \mathcal{L}_s(x) + y$, is given by the Duhamel formula

$$x(s) = U_\varepsilon(s, s_0) x(s_0) + \int_{s_0}^{s} U_\varepsilon(s, s') y(s') ds'. \tag{23}$$

The remainder to be estimated, $r(s) = \rho(s) - P_-(s)$, satisfies $\dot{r} = \varepsilon^{-1} \mathcal{L}_s(r) - \dot{P}_-$, because of $\mathcal{L}_s(P_-) = 0$. Before applying Eq. (23), we observe that the equation $\dot{P}_-(s) = \mathcal{L}_s(X(s))$ admits Eq. (19) as a solution upon replacing i, $\dot{P}_+$ by $-i$, $\dot{P}_-$. The differential equation thus becomes

$$(r - \varepsilon X)^{\cdot} = \varepsilon^{-1} \mathcal{L}_s(r - \varepsilon X) - \varepsilon \dot{X},$$

resulting in

$$r(s) - \varepsilon X(s) = U_\varepsilon(s, s_0)(r(s_0) - \varepsilon X(s_0)) - \varepsilon \int_{s_0}^{s} U_\varepsilon(s, s') \dot{X}(s') ds'.$$

Since $\mathcal{L}_s$ is dissipative, i.e. $\mathrm{tr}(\rho \mathcal{L}_s(\rho)) \leq 0$, we obtain $\|U_\varepsilon(s, s_0)\| \leq 1$, $(s \geq s_0)$. Together with $r(s_0) = 0$ we conclude that $r(s) = O(\varepsilon)$, as claimed. The uniformity follows from decay: $X(s) = O(s^{-3})$, $\dot{X}(s) = O(s^{-4})$, $(s \to \pm\infty)$.

In conclusion: We have introduced a class of adiabatically changing dephasing Lindblad operators which allowed us to calculate the tunneling in a generic two-level crossing and extend the Landau-Zener tunneling to dephasing Lindbladians with arbitrary dephasing rate. Dephasing makes the tunneling irreversible and so fundamentally different from tunneling in the unitary setting. This irreversibility is responsible for the difference in the functional form of the tunneling formulas.

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