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Is the number of photons a classical invariant?

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Abstract. We describe an apparent puzzle in classical electrodynamics and its resolution. It is concerned with the Lorentz invariance of the classical analogue of the number of photons.

1. Introduction

Photons are quantum objects and *a priori* have no business in classical electrodynamics. So, what can one possibly mean by the question 'Is the number of photons a classical invariant?'

Consider a box filled with monochromatic radiation of frequency ω. If U denotes the total electromagnetic energy in the box, then, the right-hand side of

$$\hbar N = \frac{U}{\omega} \tag{1}$$

is a purely classical quantity. The left-hand side gives the interpretation and quantization of this quantity, namely, that it counts the number of photons, N, in units of $\hbar$. What then is the classical significance of U/ω?

In quantum mechanics the number of photons is quantized. As such, it must be Lorentz invariant, for under Lorentz transformations that are close to the identity, it can only change by a little, and since it is quantized it cannot change at all. This implies that the number of photons must be a Lorentz invariant, even under Lorentz transformations that are far from the identity. With this hindsight, and since Lorentz invariance is a classical concept, one learns that the classical significance of U/ω is its Lorentz invariance.

Since neither the energy nor the frequency are Lorentz invariants, the Lorentz invariance of the ratio is not manifest, and as we shall see is actually rather subtle. If, indeed, the Lorentz invariance of the ratio holds by a direct classical argument, without recourse to the quantization argument above, one can rediscover, and to some extent also motivate the existence of photons on purely classical grounds. This approach has its limitations, of course. One still needs quantum mechanics to understand quantization, and $\hbar$ to actually count photons.

After the preprint of this paper was posted on the Los Alamos electronic archive, Professor Andrew Zangwill drew our attention to a paper by Zeldovich [5] who considered this problem in 1966. In section 7 of this paper, we shall discuss Zeldovich's paper in some detail.

2. The puzzle

Here is what appears to be a reasonable calculation of how equation (1) Lorentz transforms. In this calculation U/ω turns out not to be Lorentz invariant.

Consider a linearly polarized, plane monochromatic wave of frequency ω travelling in the $\hat{\boldsymbol{x}}$ direction. The electric and magnetic fields are

$$\boldsymbol{E} = E_0\,\cos(kx - \omega t)\,\hat{\boldsymbol{y}} \qquad \boldsymbol{B} = E_0\,\cos(kx - \omega t)\,\hat{\boldsymbol{z}}. \tag{2}$$

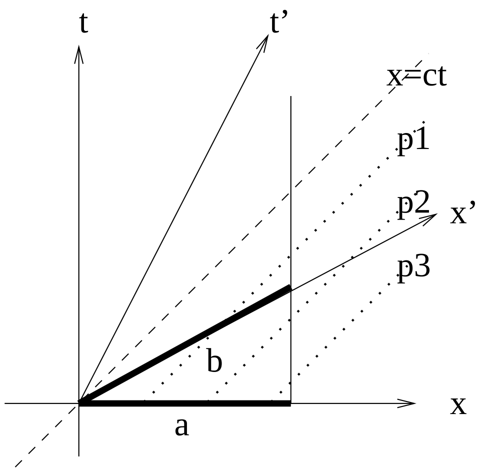

Figure 1. Space–time diagram. Each photon is represented by a dotted line (denoted by p1–p3). The bold lines 'a' and 'b' represent the box as viewed at $t = 0$, $t' = 0$ from the two frames S and S'. The number of intersections between the photon world lines and the box gives the total photons inside the box. It can be seen that p2 and p3 are not counted in S', and therefore there will be more photons counted in S.

The electromagnetic energy density is

$$\frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = \frac{E_0^2}{4\pi}\,\cos^2(kx - \omega t). \tag{3}$$

Consider a *fictitious* rectangular box of proper length L, aligned with the x axis, whose cross section is A. Suppose that the length of the box is much larger than the wavelength of the radiation. The total energy in the box is

$$U = \frac{AE_0^2}{4\pi}\int_0^L \mathrm{d}x\,\cos^2(kx - \omega t) \approx \frac{A\,L\,E_0^2}{8\pi}. \tag{4}$$

The number of photons in this box, according to (1), appears to be

$$\hbar N = \frac{U}{\omega} = \frac{E_0^2 A L}{8\pi\omega}. \tag{5}$$

Now, let us compute the number of photons, N', in the same box, but as viewed in a frame, S', moving with velocity v along the x axis. In S', the electric field amplitude is [4]

$$E_0' = \frac{E_y - (v/c)B_z}{\sqrt{1-(v/c)^2}} = \frac{E_0 - (v/c)E_0}{\sqrt{1-(v/c)^2}} = E_0\sqrt{\frac{1-v/c}{1+v/c}}. \tag{6}$$

The length of the box experiences Lorentz contraction and is now

$$L' = L\sqrt{1-(v/c)^2}. \tag{7}$$

The electromagnetic energy in the box in the moving frame is therefore

$$U' \approx \frac{(E_0')^2}{8\pi}AL' = \frac{E_0^2}{8\pi}\frac{1-v/c}{1+v/c}AL\sqrt{1-(v/c)^2}. \tag{8}$$

Now ω is transformed according to the Doppler formula [2]

$$\omega' = \omega\sqrt{\frac{1 - v/c}{1 + v/c}}. \tag{9}$$

Hence the number of photons in the moving box appears to be

$$\hbar N' = \frac{U'}{\omega'} \approx \frac{E_0^2 AL}{8\pi\omega}(1 - v/c) \approx \hbar N(1 - v/c) \tag{10}$$

which is manifestly not Lorentz invariant.

Figure 1 gives a geometric description of this result and illustrates in a direct way why different numbers of photons seem to appear in different frames.

3. What went wrong?

What, if anything, went wrong? One easy way out is to say that photons can only be correctly discussed in a quantum context. To correctly compute the number of photons one has to construct quantum fields, creation and annihilation operators, and compute the number of photons in the framework of quantum field theory. It is, of course, correct that a deeper understanding of photons requires quantum fields. However, it seems unlikely that this is the only resolution of a simple paradox. In any case, this is hardly a satisfactory resolution of it.

The origin of the paradox is not computational or quantum mechanical, but conceptual. It all has to do with what is the correct energy U to put in (1). Let us analyse this in some detail.

Equation (1) must be viewed as a formula that gives the number of photons in a field configuration at a given instant. A field configuration is, of course, extended in space. The field configuration associated with a plane wave is problematic because the total electromagnetic energy is infinite, and so is the total number of photons. The energy in a box is finite, however. Yet the box we picked is a virtual box: a box that lets light escape and enter. So what we learn is that one cannot take a part of a field configuration and chop it more or less arbitrarily and still hope that equation (1) will correctly count the number of photons. The equation comes with the proviso that the energy is the total electromagnetic energy of a field configuration. To make a field configuration with finite energy (and well-defined frequency) one can confine the electromagnetic field in an ideal, but still real box. This means a box with reflecting (that is the real part) and lossless (that is the ideal part) walls. The field configuration we have picked does not possess these properties.

A second way to resolve the paradox is to think about equation (1) differently, namely, to think of U as the energy absorbed by a photodetector. In this case, the energy U is associated with the energy flux swept by a photodetector while it is operating, see figure 2. The relevant box is now not a box in space but a box in time. The advantage of a detector is that one can also apply equation (1) to field configurations, like plane waves, with infinite energy. Since simultaneity is not a Lorentz-invariant concept, extended objects are a pain in special relativity, and a source of many paradoxes. Therefore, a good photodetector must be a small, and ideally, point-like object.

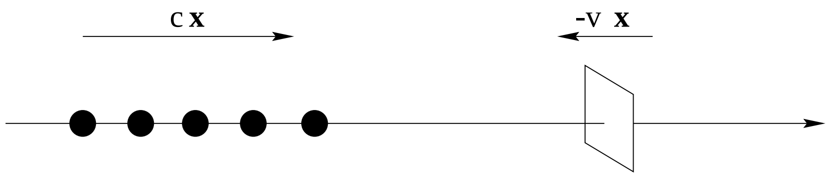

Figure 2. A second way to resolve the paradox. The square plate represents the photodetector, and the dots represent photons.

4. Photons in a box

Photons confined to a box correspond classically to a standing wave. A standing wave is a superposition of two monochromatic waves of equal frequency and amplitude, travelling in opposite directions.

Let $N_{\rightarrow}$ and $N_{\leftarrow}$ denote the number of right- and left-travelling waves, respectively. In the box's rest frame, these numbers are equal, and we will denote them by $N/2$. In the moving frame, the numbers transform according to (10):

$$\begin{aligned} N'_{\rightarrow} &= \tfrac{1}{2}N(1 - v/c) \\ N'_{\leftarrow} &= \tfrac{1}{2}N(1 + v/c). \end{aligned} \tag{11}$$

Happily, we find $N = N'$ and it is therefore invariant. So, although the number of right and left movers are not Lorentz invariant, their sum is. This is good news, because there are no additional quantum numbers in this problem besides the total number of photons.

Although this calculation gives the desired result, it is cheating: generally, electromagnetic energies do not add linearly. However, in this case the total energy can be decomposed into two contributions due to the left- and right-travelling waves. Let $\boldsymbol{E}_{\rightarrow} = \hat{\boldsymbol{y}}E_{\rightarrow}(x, t)$ and $\boldsymbol{B}_{\rightarrow} = \hat{\boldsymbol{z}}E_{\rightarrow}(x, t)$ denote the electric and magnetic fields of the right-going wave, respectively. Analogously, the fields of the left-going wave are $\boldsymbol{E}_{\leftarrow} = \hat{\boldsymbol{y}}E_{\leftarrow}(x, t)$ and $\boldsymbol{B}_{\leftarrow} = -\hat{\boldsymbol{z}}E_{\leftarrow}(x, t)$. The sign of $\boldsymbol{B}_{\leftarrow}$ is negative because the direction of motion is reversed. The energy density is

$$\begin{aligned} &\frac{1}{8\pi}\big(\boldsymbol{E}_{\rightarrow} + E_{\leftarrow}\big)^2 + \frac{1}{8\pi}\big(\boldsymbol{B}_{\rightarrow} + B_{\leftarrow}\big)^2 \\ &\qquad = \frac{1}{8\pi}\big(E_{\rightarrow}(x, t) + E_{\leftarrow}(x, t)\big)^2 + \frac{1}{8\pi}\big(E_{\rightarrow}(x, t) - E_{\leftarrow}(x, t)\big)^2 \\ &\qquad = \frac{1}{8\pi}\big(2E^2_{\rightarrow}(x, t) + 2E^2_{\leftarrow}(x, t)\big). \end{aligned} \tag{12}$$

We see that the cross terms cancel, and the energies of the two waves indeed add linearly. Note that this result is true regardless of the reference frame, since we did not assume any relation between $E_{\rightarrow}(x, t)$ and $E_{\leftarrow}(x, t)$.

Another way of solving the problem of additivity is shown in figure 3.

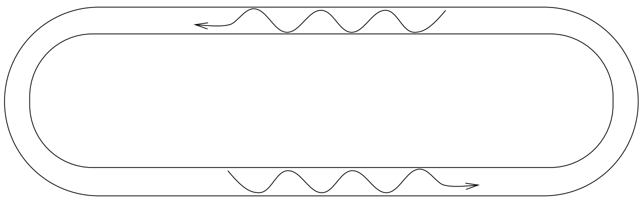

Figure 3. Photons in a closed optical fibre. Here, unlike in the box, photons going in opposite directions do not interfere, and the energies of the right and left movers are clearly additive.

5. Photodetector

A different approach to counting photons in a Lorentz-invariant way is to replace the box by a photodetector. Consider a monochromatic plane wave passing through a thin photon detector whose surface is perpendicular to the x axis, as can be seen in figure 2. We will find the number of photons passing through the detector during a given proper time t, assuming that the photons are point particles.

In the rest frame of the detector, the total energy received by the detector during time τ is

$$U = \frac{E_0^2}{8\pi} Ac\tau. \tag{13}$$

where A is the surface area of the detector. This yields

$$\hbar N = \frac{E_0^2 Ac\tau}{8\pi\omega} \tag{14}$$

for the number of photons detected.

In a moving frame the field intensity and frequency transform according to (6) and (9), respectively. The measurement time experiences time dilation

$$t' = \frac{\tau}{\sqrt{1-(v/c)^2}}. \tag{15}$$

What volume will the detector sweep during t'? The detector moves towards the photons a distance of vt', while each photon, treated as a point particle, travels towards the detector a distance of ct'. Therefore, the last photon to meet the detector at time t' is exactly a distance $vt' + ct'$ from the detector at $t = 0$. The volume swept by the detector is $A(v+c)t'$. Now we can find N':

$$\begin{aligned}
\hbar N' &= \frac{(E_0')^2}{8\pi}\, A(c+v)\, t' \frac{1}{\omega'} \\
&= \frac{E_0^2}{8\pi}\, \frac{1-v/c}{1+v/c}\, A(c+v)\, \frac{\tau}{\sqrt{1-(v/c)^2}}\, \frac{1}{\omega}\sqrt{\frac{1+v/c}{1-v/c}} \\
&= A\frac{E_0^2 c\tau}{8\pi\omega} = \hbar N.
\end{aligned} \tag{16}$$

The number of photons seen in the two frames is the same.

6. Classical invariants and the Ehrenfest principle

The point of view which we have taken here, namely that of examining the significance of classical quantities associated with (discrete) quantum numbers, goes back to Ehrenfest and the early days of quantum mechanics [3]. Ehrenfest stressed the relation of quantum numbers with classical adiabatic invariants.

Let us recall how this applies to the classical harmonic oscillator. The ratio of the energy to the frequency of an oscillator is a classical quantity whose importance in quantum mechanics comes from the fact that it is a function of the quantum number:

$$\hbar\left(n + \frac{1}{2}\right) = \frac{U}{\omega}. \tag{17}$$

The ratio of energy to frequency is the classical adiabatic invariant for the harmonic oscillator [1]. The Ehrenfest adiabatic principle can also be applied to the quantization of angular momentum, and the quantization of energy levels in the hydrogen atom. Ehrenfest was very specific in identifying adiabatic invariants with quantum numbers.

It is not clear how to apply the Ehrenfest adiabatic principle to the number of photons. However, the Lorentz invariance of the number of photons suggests that one may take a broader interpretation of the Ehrenfest principle where quantum numbers are associated with a class of classical invariants, which includes adiabatic and Lorentz invariants as special cases.

7. The Zeldovich formula

The question 'Is the number of photons a classical invariant?' has been asked, and answered, by Zeldovich in 1966 [5]. He pointed out that the number of photons is both an adiabatic and a Lorentz invariant. However, although he made the claim of Lorentz invariance, he did not *show* this.

Zeldovich wrote an interesting expression for the classical invariant, which is a generalization of equation (1) to the case where the field is not monochromatic. The main purpose of this section is, however, to show that the Zeldovich formula indeed describes a Lorentz invariant, at least for plane waves that are not monochromatic.

Zeldovich's formula for the number of photons is:

$$\hbar N = \frac{1}{8\pi} \int \mathrm{d}^3 k \, \frac{|\hat{\boldsymbol{E}}(\boldsymbol{k}, t)|^2 + |\hat{\boldsymbol{B}}(\boldsymbol{k}, t)|^2}{c|\boldsymbol{k}|} \tag{18}$$

which is a natural generalization of equation (1) to the polychromatic case. Here $\hat{\boldsymbol{E}}(\boldsymbol{k}, t)$ and $\hat{\boldsymbol{B}}(\boldsymbol{k}, t)$ are the Fourier transforms of the electric and magnetic fields:

$$\begin{aligned} \hat{\boldsymbol{E}}(\boldsymbol{k}, t) &= \frac{1}{(2\pi)^{3/2}} \int \boldsymbol{E}(\boldsymbol{x}, t) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}^3 x \\ \hat{\boldsymbol{B}}(\boldsymbol{k}, t) &= \frac{1}{(2\pi)^{3/2}} \int \boldsymbol{B}(\boldsymbol{x}, t) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}^3 x. \end{aligned} \tag{19}$$

It would be interesting to have an elementary demonstration of the Lorentz invariance of equation (18). Here, instead, we shall be content with the Lorentz invariance in the special case of a plane wave; i.e. a field configuration that is independent of the y and z coordinates. For a plane wave the number of photons is, of course, infinite, and an interesting finite quantity is the number of photons per unit area in the y–z plane. Since this area does not contract under Lorentz boosts in the $\hat{\boldsymbol{x}}$ directions, the corresponding invariant is

$$\hbar n = \frac{1}{8\pi} \int \mathrm{d}k \, \frac{|\hat{\boldsymbol{E}}(k, t)|^2 + |\hat{\boldsymbol{B}}(k, t)|^2}{c|k|}. \tag{20}$$

In the Lorentz frame S a plane wave is made of both right and left movers:

$$\begin{aligned} &\boldsymbol{E}(x, t)(E_{\rightarrow}(x - ct) + E_{\leftarrow}(x + ct))\, \hat{\boldsymbol{y}} \\ &\boldsymbol{B}(x, t)(E_{\rightarrow}(x - ct) - E_{\leftarrow}(x + ct))\, \hat{\boldsymbol{z}}. \end{aligned} \tag{21}$$

Using the standard transformation law for the fields [4], one finds that the fields in the frame S', as functions of its (primed) coordinates,

$$x = \frac{x' + (v/c)t'}{\sqrt{1 - (v/c)^2}} \qquad t = \frac{t' + (v/c)x'}{\sqrt{1 - (v/c)^2}} \tag{22}$$

are

$$\begin{aligned} \boldsymbol{E}'(x', t') &= \left(b\, E_{\rightarrow}(b(x' - ct')) + \frac{1}{b}\, E_{\leftarrow}\left(\frac{x' + ct'}{b}\right)\right) \hat{\boldsymbol{y}}' \\ \boldsymbol{B}'(x', t') &= \left(b\, E_{\rightarrow}(b(x' - ct')) - \frac{1}{b}\, E_{\leftarrow}\left(\frac{x' + ct'}{b}\right)\right) \hat{\boldsymbol{z}}' \end{aligned} \tag{23}$$

where

$$b \equiv \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

We can now compute $\hat{\boldsymbol{E}}'(k', t')$ and $\hat{\boldsymbol{B}}'(k', t')$:

$$\begin{aligned}\hat{\boldsymbol{E}}'(k', t') &= \hat{\boldsymbol{y}}' \frac{1}{(2\pi)^{3/2}} \int \left(bE_{\rightarrow}(bx') + \frac{1}{b} E_{\leftarrow}\left(\frac{x'}{b}\right)\right) \mathrm{e}^{-\mathrm{i}k'x'} \mathrm{d}x' \\ &= \hat{\boldsymbol{y}}' \left(\hat{E}_{\rightarrow}\left(\frac{k'}{b}\right) + \hat{E}_{\leftarrow}(k'b)\right)\end{aligned} \tag{24}$$

and similarly

$$\hat{\boldsymbol{B}}'(k', t') = \hat{\boldsymbol{z}}' \left(\hat{E}_{\rightarrow}\left(\frac{k'}{b}\right) - \hat{E}_{\leftarrow}(k'b)\right). \tag{25}$$

The number of photons per unit area in the frame S' is

$$\begin{aligned}\hbar n' &= \frac{2}{8\pi} \int \left(\left| \hat{E}_{\rightarrow}\left(\frac{k'}{b}\right)\right|^2 + \left| \hat{E}_{\rightarrow}(k'b)\right|^2 \right) \frac{\mathrm{d}k'}{c|k'|} \\ &= \frac{2}{8\pi} \int \left(|\hat{E}_{\rightarrow}(k)|^2 + |\hat{E}_{\rightarrow}(k)|^2 \right) \frac{\mathrm{d}k}{c|k|} = \hbar n\end{aligned} \tag{26}$$

where the factor 2 comes from the contribution of the electric and magnetic fields, and the mixed terms drop. This establishes Lorentz invariance.

8. Epilogue

This is an account of a simple paradox and its resolution. It is remarkable that in spite of the quantum nature of photons, one can correctly compute their number in a box as if they were mere golf balls in a bucket. However, precisely because photons are, at the same time, associated with an extended field configuration, this calculation is also subtle, and can lead to wrong results if one is not careful.

The account given here grew out of teacher–student interaction in the spring semester class of classical electrodynamics at the Technion. Puzzles are effective means of teaching and learning, especially when the teacher does not already know their resolution.

Acknowledgments

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Note added in proof. Dr J J Labarthe has kindly pointed out that this problem was considered first by A Einstein in section 8 of his celebrated article [6] on relativity. A translation of this paper can be found in the appendix of [7].

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