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Quantum response of dephasing open systems

J E Avron[1], M Fraas[1,4], G M Graf[2] and O Kenneth[1,3]
[1] Department of Physics, Technion, 32000 Haifa, Israel
[2] Theoretische Physik, ETH Zurich, CH-8093 Zürich, Switzerland
[3] School of Physics and Astronomy, Tel-Aviv University, Tel Aviv 69978, Israel
E-mail: martin.fraas@gmail.com

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Abstract. We develop a theory of adiabatic response for open systems governed by Lindblad evolutions. The theory determines the dependence of the response coefficients on the dephasing rates and allows for residual dissipation *even when the ground state is protected by a spectral gap*. We give the quantum response a geometric interpretation in terms of Hilbert space projections: for a two-level system and, more generally, for systems with a suitable functional form of the dephasing, the dissipative and non-dissipative parts of the response are linked to a metric and to a symplectic form. The metric is the Fubini–Study metric and the symplectic form is the adiabatic curvature. When the metric and symplectic structures are *compatible*, the non-dissipative part of the inverse matrix of response coefficients turns out to be *immune to dephasing*. We give three examples of physical systems whose quantum states induce compatible metric and symplectic structures on control space: qubit, coherent states and a model of the integer quantum Hall effect.

Two frameworks that provide insight and understanding of the transport coefficients are Kubo's theory of linear response [1, 2] and the theory of adiabatic response [3–5]. Both have a quantum version and a classical version, agree when there is overlap and endow the non-dissipative transport coefficients with geometric meaning of the adiabatic (Berry's) curvature [6].

A notable success of this approach to the 'geometrization of quantum response' has been its application to the integer quantum Hall effect where one observes quantized resistances of the form h/ne^2 with n being an integer. The accuracy and robustness of the quantization are understood as reflecting the nontrivial topology of the quantum state characterized by a topological invariant, the Chern number, which is the n measured as the Hall resistance [7].

[4] Author to whom any correspondence should be addressed.

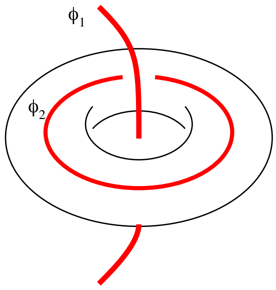

Figure 1. A torus in coordinate space threaded by two loops (red) carrying fluxes ϕ_1 and ϕ_2 viewed as a model of the quantum Hall effect. The (strong) magnetic field (not shown) is perpendicular to the surface of the torus. The control space $\mathcal{M}$ is the space of fluxes. By the Aharonov–Bohm periodicity, $\mathcal{M}$ is a torus as well.

'Geometrization of transport' also lies at the heart of the topological classification of states of matter [8–10] and the considerable current interest in topological insulators [11].

Our aim here is to carry over the program of 'geometrization of quantum response' to open quantum systems, more precisely, to extend the theory of adiabatic quantum response from the Hamiltonian setting to the Lindbladian setting [12, 13]. The extension gives geometric meaning to both the non-dissipative and dissipative response coefficients and allows us to examine how they are affected by decoherence. It allows us to address the stability of topological quantum numbers, such as Chern numbers, against dephasing.

In the adiabatic response, the Hamiltonian, $H(\phi)$, is viewed as a function of the control parameters ϕ that drive the system [3, 4, 15]. The space of controls shall be denoted by $\mathcal{M}$. We focus on observables that are gradients of H over $\mathcal{M}$, namely

$$F_\mu = \frac{\partial H}{\partial \phi^\mu}. \tag{1}$$

For example, in the system shown in figure 1, the controls are magnetic fluxes, that, by the Aharonov–Bohm effect, may be thought of as angular variables so that the control space $\mathcal{M}$ is the torus $\mathbb{T}^2$. F_μ is the loop current in the μ loop as one can see by the principle of virtual work. For other notions of currents in the Lindbladian context, see [16, 17].

Our main object of interest is the matrix of response coefficients f defined through the *linear and instantaneous* terms in $\dot\phi$ in the (adiabatic) expansion of the response

$$\mathrm{Tr}(\rho_t F_\mu) = f_{\mu\nu}\, \dot\phi^\nu(t) + \cdots. \tag{2}$$

Summation convention over repeated indices is implied. All of the other terms, which are not proportional to $\dot\phi$, do not concern us. In the example of figure 1, $\dot\phi^\mu$ is the emf on the ν loop and f is then the conductance matrix relating the loop currents to emfs.

The quantum state ρ_t in equation (2) arises from the ground state as the solution at time t of the equation of motion

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}(\rho)$$

governed by the adiabatically changing *Lindbladian*,

$$\mathcal{L}(\rho) = -\mathrm{i}[H, \rho] + \sum_\alpha ([\Gamma_\alpha \rho, \Gamma_\alpha^*] + [\Gamma_\alpha, \rho\Gamma_\alpha^*]). \tag{3}$$

Here, $\Gamma_\alpha = 0$ corresponds to a unitary evolution.

In adiabatic transport [3–5], one is interested in the situation where the control parameters ϕ move adiabatically along a path in control space $\mathcal{M}$. It turns out that adiabatic methods used to study the unitary case $\Gamma_\alpha = 0$ can be extended to also study the open systems described by Lindbladians, provided that the instantaneous stationary states move continuously with the controls. The response of a stationary state, $\mathcal{L}(\rho(\phi)) = 0$, of the Lindbladian $\mathcal{L}$ to a driving is given by (see appendix B)

$$f_{\mu\nu} = \mathrm{Tr}(F_\mu \mathcal{L}^{-1}(\partial_\nu \rho)). \tag{4}$$

$f_{\mu\nu}$ has a geometric interpretation when the stationary state ρ is a (spectral) projection. A particular choice of Lindbladians that achieve this is $\Gamma_\alpha = \Gamma_\alpha(H)$ for some function of H. We call this family of changing Lindbladians, where $\Gamma_\alpha(H)$ are slaved to the *instantaneous* Hamiltonian, *dephasing Lindbladians*. As long as the Hamiltonian and hence the Lindbladian are fixed, the evolution conserves energy, while entropy increases. Once the system is adiabatically driven, energy dissipates proportionally to the symmetric part of f, as seen from $\mathrm{Tr}(\dot\rho_t H) = 0$ and equation (2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}(\rho_t H) = \mathrm{Tr}(\rho_t \partial_\mu H)\dot\phi^\mu = f_{\{\mu,\nu\}}\dot\phi^\mu\dot\phi^\nu + \cdots. \tag{5}$$

The result is in line with the transition rate between energy eigenstates [18], which is also quadratic in $\dot\phi$. In appendix A, we give an example showing how dephasing Lindbladians naturally emerge in certain stochastic unitary evolutions.

The slaving of the decoherence terms to the instantaneous Hamiltonian has the consequence that the instantaneous stationary states of the Hamiltonian and the Lindbladian coincide. Moreover, when the controls ϕ vary adiabatically, one expects spectral projections of H to evolve so that they remain close to the corresponding instantaneous spectral projections $P(\phi)$. The evolution of spectral projections can be described in geometric terms. The geometric quantum response is concerned with the relation between the response matrix f and the geometry of $P(\phi)$ on control space.

Before we describe this relation, it is convenient to review the geometry on the control space associated with a single spectral bundle $\phi \mapsto P(\phi)$. Consider the (operator-valued) 1-form giving the natural adiabatic connection (here $P_\perp = 1 - P$),

$$A = A_\mu\, \mathrm{d}\phi^\mu, \qquad A_\mu = P_\perp \partial_\mu P \tag{6}$$

from which one can construct the second rank tensor on control space

$$2\,\mathrm{Tr}(A \otimes A^*) = (g_{\mu\nu}(\phi) - \mathrm{i}\,\omega_{\mu\nu}(\phi))\mathrm{d}\phi^\mu \otimes \mathrm{d}\phi^\nu, \tag{7}$$

where $\otimes$ is the product of forms. The symmetric part is the natural notion of infinitesimal distance for projections, the Fubini–Study metric [19],

$$g_{\mu\nu} = \mathrm{Tr}\, P_\perp\{\partial_\nu P,\ \partial_\mu P\} = \mathrm{Tr}(\partial_\nu P)\,(\partial_\mu P). \tag{8}$$

The antisymmetric part gives the adiabatic (or Berry's) curvature [6],

$$\omega_{\mu\nu} = \mathrm{i}\,\mathrm{Tr}(P_\perp[\partial_\mu P,\, \partial_\nu P]) = -\mathrm{i}\,\mathrm{Tr}(P[\partial_\mu P,\, \partial_\nu P]). \tag{9}$$

The basic properties of g and ω are direct consequences of equations (8) and (9) and are summarized in the following statement.

Proposition 1. *$g \geqslant 0$ defines a metric on the space of controls $\mathcal{M}$. It is even under time-reversal, i.e. $g(P) = g(\theta P \theta^{-1})$ for θ antiunitary, as it is under electron–hole exchange, $g(P) = g(P_\perp)$. ω is a closed 2-form, $d\omega = 0$, endowing $\mathcal{M}$ with a symplectic structure, if non-degenerate. It is antisymmetric under time reversal and hence $\omega = 0$ if $H(\phi)$ is time reversal invariant. ω is antisymmetric under electron–hole exchange, $\omega(P) = -\omega(P_\perp)$.*

Geometry of quantum response. For the sake of simplicity, we consider the case when H is a finite dimensional matrix and $\Gamma_\alpha(H)$ are real functions of H. The (instantaneous) eigenvalues of H shall be denoted by ε_j, and the corresponding eigenstates by $|j\rangle$. The ground state shall be denoted by $j = 0$. The eigenvalues of $\mathcal{L}$ (corresponding to eigenstates $|j\rangle\langle k|$) shall be denoted by λ_{jk}. By inspection, $\mathrm{Im}(\lambda_{jk}) = -\varepsilon_j + \varepsilon_k$, while $\mathrm{Re}(\lambda_{jk}) \leqslant 0$, i.e. all eigenvalues lie in the (closed) left half-plane. If two states are degenerate $\varepsilon_j = \varepsilon_k$, then clearly $\lambda_{jk} = 0$.

It will be convenient to introduce the notion of dimensionless dephasing rates $\gamma_{jk} \geqslant 0$ associated with the pair of non-degenerate eigenstates

$$\gamma_{jk} = -\frac{\mathrm{Re}(\lambda_{jk})}{|\varepsilon_j - \varepsilon_k|} \geqslant 0. \tag{10}$$

For a pair of degenerate states, where both the numerator and the denominator vanish, the dephasing rate is defined to be zero.

We are now at a position where we can state our first main result. In appendix B, we describe a formula relating the response matrix f to the spectral projections and their gradients for general dephasing Lindbladians. The formula admits a particularly simple geometric interpretation provided that we make a specific choice for $\Gamma(H)$. We shall first state the result in this special case and then comment on the general case.

Theorem 1. *The matrix of transport coefficients f, equation (2), associated with the adiabatic evolution characterized by a single dephasing rate, equation (10), $\gamma = \gamma_{j0} \geqslant 0$ to all other energy levels j, is given by*

$$f = \frac{\gamma}{1+\gamma^2}\, g + \frac{1}{1+\gamma^2}\, \omega. \tag{11}$$

g is the Fubini–Study metric of the ground state bundle and ω its adiabatic curvature.

Let us make the following observations:

1. The condition of a single dephasing rate is automatically satisfied in any two-level system. It is also satisfied if H has two degenerate eigenvalues. When H has three or more distinct eigenvalues, the condition is non-trivial and can be interpreted as a condition on the functional form of $\Gamma(H)$, e.g. $\Gamma(H) = \sqrt{\gamma(H - \varepsilon_0)}$ with ε_0 being the lowest eigenvalue of H.
2. The dissipative response is associated with the symmetric part of f by equation (5) and is fully determined by the metric g. It vanishes in the limit $\gamma = 0$ as it must—there is no dissipation in adiabatic unitary evolution with a gap condition. In contrast, in open dephasing systems, a gap condition does not provide protection from dissipation.
3. The non-dissipative response is associated with the antisymmetric part of f and is fully determined by the adiabatic curvature and the dephasing rate. (An extension of the Berry phase to systems with decoherence in the form of a complex phase is studied in [14].)

4. The theorem gives a geometric interpretation to both the dissipative and non-dissipative parts of the adiabatic quantum response.
5. For weakly dephasing systems, $0 \leqslant \gamma \ll 1$, the dissipative response depends linearly on γ, while the non-dissipative response depends quadratically on γ. Dephasing affects both the dissipative and non-dissipative response coefficients.
6. Chern numbers are integers obtained by integrating the adiabatic curvature over a closed two-dimensional (2D) control space. The theorem says that the relation between the control space average of transport coefficients and Chern numbers is a function of the dephasing γ. Hence, topological quantum numbers are not robust against dephasing.
7. The theorem generalizes to the multi-dephasing rate case, where γ_{j0} of equation (10) is j dependent, at the price of replacing $P_\perp \partial_\mu P$ by a weighted sum of $P_j \partial_\mu P$. The Fubini–Study metric is then replaced by a metric that does not have a standard name.
8. The linear response formula, equation (2), gives the leading term in the adiabatic expansion of the response provided that the spectrum of $H(\phi)$ is independent of ϕ. In the general case when the eigenvalues are ϕ dependent, the expansion equation (2) has additional terms at low orders: one that is not small, but depends on $\phi(t)$ only, and one that is of the same order as $\dot{\phi}(t)$, but is given by a quadratic expression in $\dot{\phi}(t')$ integrated over $t' \leqslant t$. However, as these terms are not proportional to the instantaneous driving $\dot{\phi}(t)$, we shall not consider them here (cf [4]).

Control space with compatible metric and symplectic structures. Our second main result concerns the special class of Hamiltonians when the metric g and the symplectic structure ω are *compatible*. In the case of a single pair of controls, this is expressed by $\det g = \det \omega$. In general and in terms of a basis in which both g and ω are 2×2-block diagonal, the two structures are compatible if they are so inside each block. Equivalently, we say that g and ω are compatible if

$$\omega^{-1} g + g^{-1} \omega = 0. \tag{12}$$

Equation (12) implies the equality of the determinants.

By proposition 1, compatibility is possible *only when time reversal is broken*. Below, we shall give three natural physical examples that give rise to compatible metric and symplectic structure. The three examples correspond to the three prototypical control spaces $\mathcal{M}$: the sphere, the plane and the torus, respectively.

Theorem 2. *Suppose that g and ω are compatible. Then the inverse of the matrix of quantum response of theorem 1 has the form*

$$f^{-1} = \gamma g^{-1} + \omega^{-1}. \tag{13}$$

The claim is easily verified by multiplying equation (11) by equation (13) and using equation (12).

Remarkably, the non-dissipative response associated with the antisymmetric part of f^{-1} is *independent* of γ and so immune to dephasing. It is determined by the adiabatic curvature alone, just like in the case of unitary evolution.

Compatibility test. Given P, one is interested in simple tests that tell us whether g and ω are compatible without explicitly computing the matrices g and ω.

Theorem 3. *The following are equivalent.*

(i) g, ω *defined on* $\mathcal{M}$ *are compatible in the sense of equation (12).*

(ii) $\mathcal{M}$ *has local holomorphic and antiholomorphic coordinates* z^j, $\bar{z}^j$ *making it a complex manifold and the map* $P(\phi)$ *satisfies* $P_\perp \bar{\partial}_j P = 0$.

(iii) The image of the map $P : \mathcal{M} \to \mathrm{Gr}(n+1, r; \mathbb{C})$ *is a complex submanifold of the complex Grassmannian manifold* $\mathrm{Gr}(n+1, r; \mathbb{C})$. *(Here* $r = \mathrm{rank}(P)$ *and* $n+1$ *is the dimension of the Hilbert space.)*

Moreover, when P *is a 1D projection (i.e.* $r = 1$*), test (ii) is equivalent to the claim that* P *may be expressed as* $P = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ *where* $|\psi\rangle = |\psi\rangle_z$ *is holomorphic, i.e.* $\bar{\partial}_j |\psi\rangle = 0$.

We show how test (ii) implies compatibility (i) when $\mathcal{M}$ is 2D. We write locally $z = \phi^1 + \tau\phi^2$, where $\tau = \tau_1 + \mathrm{i}\tau_2$ with $\tau_2 \neq 0$. Then $\bar{\partial} = \frac{1}{2i\tau_2}(\tau\partial_1 - \partial_2)$, and assertion (ii) is equivalent to the assertion that

$$\tau A_1 - A_2 = 0. \tag{14}$$

Substituting this identity into equation (7) yields

$$g_{22} = |\tau|^2 g_{11}, \quad g_{12} = \tau_1 g_{11}, \quad \omega_{12} = \tau_2 g_{11}. \tag{15}$$

The equality of the determinants now follows by inspection.

For $P = |0\rangle\langle 0|$, test (ii) is equivalent to $\partial_{\bar{z}} |0\rangle \propto |0\rangle$ and hence the possible $\bar{z}$ dependence of $|0\rangle_z$ is merely through a normalization/phase factor.

Complex structure–Kähler manifolds. The compatibility condition is equivalent to the statement that the operator $J = \omega^{-1} g$ acting on $T_\phi\mathcal{M}$, the tangent space at ϕ, satisfies $J^2 = -1$. It can therefore be used to turn the real vector space $T_\phi\mathcal{M}$ into a complex vector space by defining multiplication by the scalar $\alpha + \mathrm{i}\beta \in \mathbb{C}$ to be given by the action of the operator $\alpha + \beta J$.

The compatibility condition equation (12) automatically holds for the Fubini–Study metric and the adiabatic curvature when the control space is the whole projective space $\mathbb{PC}^n$ and the projections P parametrize themselves. This is related to the fact that $\mathbb{PC}^n$ is a Kähler manifold. The same also holds for the Grassmanian manifold. Our control space $\mathcal{M}$ may be viewed (through the map $\phi \mapsto P(\phi)$) as immersed in $\mathbb{PC}^n$ when $\mathrm{rank}(P) = 1$ or in the Grassmanian when $\mathrm{rank}(P) > 1$. Indeed, we defined $g^{\mathcal{M}}$ and $\omega^{\mathcal{M}}$ on $\mathcal{M}$ using this identification by restricting $g^{\mathbb{PC}^n}$, $\omega^{\mathbb{PC}^n}$ to its immersion. $g^{\mathcal{M}}$, $\omega^{\mathcal{M}}$ inherit the compatibility from $g^{\mathbb{PC}^n}$, $\omega^{\mathbb{PC}^n}$ if and only if $\mathcal{M}$ is a *complex* submanifold of $\mathbb{PC}^n$ (or of $\mathrm{Gr}(n+1, r)$), in which case $\mathcal{M}$ inherits the Kähler property, too. This leads to theorem 3.

We conclude by giving three examples of natural controlled physical systems whose ground state bundle has a metric compatible with the curvature.

The Qubit family. The controlled Hamiltonian of a qubit is

$$H = \hat{\phi} \cdot \vec{\sigma}, \qquad \hat{\phi} \in \mathbb{S}^2. \tag{16}$$

The control space $\mathcal{M} = \mathbb{S}^2$ is the unit sphere. The ground state projections are given by

$$2P = 1 - \hat{\phi} \cdot \vec{\sigma}. \tag{17}$$

The associated Fubini–Study metric and the symplectic form are readily computed from equations (8) and (9),

$$2g_{ij} = \partial_i\hat{\phi} \cdot \partial_j\hat{\phi}, \qquad 2\omega_{ij} = -\hat{\phi} \cdot \partial_i\hat{\phi} \times \partial_j\hat{\phi}. \tag{18}$$

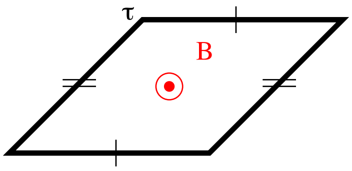

Figure 2. A flat torus with skewness τ is obtained by identifying the parallel ends of the skewed parallelogram. A constant magnetic field with an integer number, B, of flux quanta penetrates the torus.

Both give consistent areas (half the standard area) on $\mathbb{S}^2 = \mathcal{M}$.

Coherent states. The Hamiltonian of a phase-space controlled oscillator is

$$H(\zeta,\mu) = \tfrac{1}{2}(p-\mu)^2 + \tfrac{1}{2}(x-\zeta)^2, \tag{19}$$

where $(\zeta,\mu)\in\mathbb{R}^2 = \mathcal{M}$, the Euclidean plane. The manifold of ground states is the coherent states. The controls are boosts and shifts hence

$$A_1 = P_\perp \partial_\zeta P = -\mathrm{i}P_\perp[p, P] = -\mathrm{i}P_\perp(p-\mu)P. \tag{20}$$

Similarly,

$$A_2 = P_\perp \partial_\mu P = \mathrm{i}P_\perp[x, P] = \mathrm{i}P_\perp(x-\zeta)P. \tag{21}$$

Since $a = (x-\zeta) + \mathrm{i}(p-\mu)$ annihilates the ground state of the shifted oscillator, equation (14) holds with $\tau = i$

$$P_\perp a P = 0. \tag{22}$$

Landau Hamiltonian. As a third example of a control space with Kähler structure, consider the Landau Hamiltonian associated with a torus of unit area and skewness $\tau = \tau_1 + \mathrm{i}\tau_2$, $\tau_2 > 0$ (see figure 2). The torus is threaded by Aharonov–Bohm magnetic fluxes (ϕ_1, ϕ_2) (see figure 1) and is penetrated by constant magnetic field $2\pi B$, with B being an integer (see figure 2). The corresponding Hamiltonian is [20],

$$H(\tau,\phi) = \frac{1}{\tau_2^2} D^* D, \tag{23}$$

where $D = \mathrm{i}(\tau\partial_x - \partial_y) - 2\pi\tau(By+\phi)$ and $\phi = \phi_1 - \phi_2/\tau$. We impose the usual magnetic translation boundary conditions [21],

$$\psi(x,y) = \psi(x+1,y) = \mathrm{e}^{2\pi\mathrm{i}Bx}\,\psi(x,y+1). \tag{24}$$

The (single particle) ground state is B-fold degenerate with energy $E = 0$ independent of τ and ϕ. For simplicity, we consider the case of a single particle and $B = 1$. The lowest Landau level is then related to theta functions [22],

$$\psi(x,y) = (2/\tau_2)^{1/4}\,\mathrm{e}^{-\pi|\tau|^2(\mathrm{Im}\,\phi)^2/\tau_2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}nx}\,\mathrm{e}^{\mathrm{i}\pi\tau(y+n+\phi)^2}. \tag{25}$$

Since, except for the normalization, ψ depends only on the 'holomorphic coordinate' ϕ, and not on the 'anti-holomorphic' coordinate $\bar{\phi}$, it follows from theorem 3 that g and ω are compatible. Alternatively, the analyticity of ψ is also seen from that of D; note that the boundary conditions are independent of ϕ.

Conclusions

This paper studies the response of a driven open quantum system governed by adiabatically evolving Lindbladians. When the Lindblad operator is *dephasing*, coherence and phase information is degraded but the energy of the system is still conserved. In this case, we find that the response admits a geometric interpretation induced by the behavior of the instantaneous Hilbert space projections.

We focus on the response associated with observables that can be derived by the principle of virtual work from the Hamiltonian of the system. For these observables we find that, in contrast with the case of unitary evolution, there is residual dissipation: a spectral gap does not protect against dissipation in open systems in general.

Our first main result, theorem 1, concerns the special case of dephasing two-level systems and, more generally, certain dephasing systems characterized by a single rate γ. In this case, the response matrix admits a simple geometric interpretation: the non-dissipative transport is proportional to the adiabatic curvature ω and the dissipative response to the Fubini–Study metric g. Since the proportionality factor relating non-dissipative transport to the adiabatic curvature is a function of the dephasing rate γ, it follows that the robustness of Chern numbers *does not* translate into a robustness of the transport coefficients against dephasing.

Our second main result, theorem 2, is concerned with a geometric mechanism that provides protection against dephasing for certain non-dissipative response coefficients. The mechanism leading to such a protection is the *compatibility* of the adiabatic curvature with the metric. In this case, the relation between transport coefficients and Chern numbers is *independent* of the dephasing rate.

Our third main result concerns the test of compatibility. In particular, we show that when the instantaneous stationary states are holomorphic functions of the driving parameters, the Fubini–Study metric is compatible with the adiabatic curvature.

We conclude with three fundamental examples of compatible physical Hamiltonians associated with the three types of control spaces. The first example of a compatible system is the qubit Hamiltonian whose control space is the sphere, the second example of a compatible system is the Harmonic oscillator whose control space is the plane and the third example is a model of the quantum Hall effect on a torus.

Acknowledgments

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Appendix A

Here we describe an example illustrating how the adiabatic dephasing Lindbladian, with a slaved dephasing term, naturally arises from stochastic unitary evolutions. The example is a stochastic variant of Berry's paradigm of the notion of adiabatic curvature, namely, a spin-1/2 in a magnetic field [6]. The evolution equation is,

$$\dot{\rho} = -\mathrm{i}[\vec{B}\cdot\vec{\sigma}, \rho], \tag{A.1}$$

where $\vec{B} \in \mathbb{R}^3$ is a time dependent magnetic field and $\vec{\sigma}$ the vector of Pauli matrices. The case considered by Berry is when $\vec{B}$ changes its orientation adiabatically say with fixed magnitude. We want now to consider the stochastic version of this model where the *magnitude* of $\vec{B}$ is a stochastic variable, while its orientation is changing smoothly (adiabatically) in time. Formally, this corresponds to replacing B in the evolution equation by

$$\vec{B} \to W_t \vec{B}_0,$$

where W_t is (scalar, biased) white noise. The canonical interpretation of equation (A.1) as a stochastic differential equation goes through the Ito calculus [23]. To do so, it is convenient to expresses white noise in terms of the corresponding Brownian motion

$$\mathrm{d}b_t := W_t \mathrm{d}t.$$

The rules of Ito calculus say that $\mathrm{d}\rho$ has to be expanded to first order in $\mathrm{d}t$ and to second order in $\mathrm{d}b$. This gives the stochastic evolution equation,

$$\mathrm{d}\rho = -\mathrm{i}[H_0, \rho]\mathrm{d}b - \tfrac{1}{2}(\mathrm{d}b)^2[H_0, [H_0, \rho]], \qquad H_0 = \vec{\sigma} \cdot \vec{B}_0,$$

where B_0 is the smooth (nonstochastic) function of time. In particular, it follows that the (noise average) state $\rho_a = \mathbb{E}(\rho)$ satisfies the adiabatic Lindblad equation,

$$\dot{\rho}_a = \mathcal{L}(\rho_a) = -\mathrm{i}\mu[H_0, \rho_a] - \tfrac{1}{2}D[H_0, [H_0, \rho_a]],$$

where μ is the bias of the white noise $\mu = \mathbb{E}(W_t)$ and D its variance $\mathbb{E}(W_t W_s) = D\delta(t-s)$. If $D \neq 0$, this gives a dephasing evolution where the dephasing is slaved to the time dependence of the Hamiltonian. (A general framework for deriving the Lindbladian for general stochastic evolutions is described in e.g. [24].)

Appendix B

Here we outline the proof of theorem 1 by evaluating the terms proportional to $\dot{\phi}$ in equation (2).

Let P_j denote the spectral projections for H. Since $\mathcal{L}(P_j) = 0$, the spectral projections are instantaneous stationary states. Let $P = P_0$ denote the projection on the ground state. We also denote $E_{jk} = |j\rangle\langle k|$. This is an eigenvector of the Lindbladian with eigenvalue λ_{jk}, i.e. $\mathcal{L}(E_{jk}) = \lambda_{jk} E_{jk}$.

By the adiabatic theorem, the state adheres to the spectral projection, $\rho(t) = P(\phi(t)) + O(\dot{\phi})$. The first-order correction $\delta\rho$ to the state satisfies

$$\mathcal{L}(\delta\rho) = \dot{P}, \tag{B.1}$$

as can be seen from the substitution $\rho = P + \delta\rho$ into the Lindblad equation (3) and using $\mathcal{L}(P) = 0$. The correction can be decomposed as $\delta\rho = \delta_\perp\rho + \delta_\parallel\rho$ into parts $\delta_\perp\rho \in \mathrm{Range}\,\mathcal{L}$ and $\delta_\parallel\rho \in \mathrm{Ker}\,\mathcal{L}$, which are orthogonal with respect to the inner product defined by the trace. Note that $\mathcal{L}$ considered as a map on Range $\mathcal{L}$ is invertible and that $\dot{P} \in \mathrm{Range}\,\mathcal{L}$. Thus, equation (B.1) implies $\delta_\perp\rho = \mathcal{L}^{-1}(\dot{P})$, where the inverse $\mathcal{L}^{-1}$ is well defined. In fact, since the eigenstates of $\mathcal{L}$ are E_{jk}, one may readily write

$$\delta_\perp\rho = \mathcal{L}^{-1}(\dot{P}) = \sum_{j\neq k} \frac{\langle j|\,\dot{P}\,|k\rangle}{\lambda_{jk}} E_{jk}. \tag{B.2}$$

Strictly speaking, here we restricted ourselves to the case of simple eigenvalues; more generally, $\langle j|\,\dot{P}\,|k\rangle = 0$ between degenerate eigenstates, whence the appropriate reading of the sum (B.2) is by omitting such pairs.

The complementary part $\delta_\parallel \rho$ may be determined as well (cf [4, 18]) and happens to depend on history, but it will not be needed.

Now, ρ carries two contributions to the response equation (2), of which the leading one, $\mathrm{Tr}(PF_\mu)$, equals $\partial_\mu \mathrm{Tr}(PH) = \partial_\mu \varepsilon_0$. This term is not proportional to $\dot{\phi}$ and does not concern us (note that it vanishes when the spectrum is independent of ϕ, cf observation 8 after the theorem). As for the first order correction $\delta\rho$, two contributions arise in turn through $F_\mu = \frac{\partial H}{\partial \phi^\mu} = \partial_\mu \sum \varepsilon_j P_j$. The first, $\sum (\partial_\mu \varepsilon_j) P_j$, lies in Ker $\mathcal{L}$ and matches $\delta_\parallel \rho$. This term does not concern us either and again vanishes when $\partial_\mu \varepsilon_j$ does.

The other part $\sum \varepsilon_j \partial_\mu P_j$ lies in range $\mathcal{L}$ and gives the requisite linear response term of the expectation value $\langle F_\mu\rangle = \mathrm{Tr}(\rho\, \partial_\mu H)$,

$$\sum_i \varepsilon_i \mathrm{Tr}((\partial_\mu P_i)\, \delta_\perp \rho) = \sum_{i\neq 0} (\varepsilon_i - \varepsilon_0) \mathrm{Tr}((\partial_\mu P_i)\, \delta_\perp \rho), \tag{B.3}$$

where we used $\partial_\mu \sum_i P_i = 0$. Equation (B.3) can now be written as $\sum_i (\varepsilon_i - \varepsilon_0)\mathcal{A}_i$, where

$$\mathcal{A}_i = \sum_{j\neq k} \langle k|\, \partial_\mu P_i \,|j\rangle \, \frac{\langle j|\, \dot{P}\, |k\rangle}{\lambda_{jk}}. \tag{B.4}$$

Using

$$\langle j|\dot{P}|k\rangle = \langle j|P_j \dot{P} P_k|k\rangle = (\delta_{k,0} + \delta_{j,0})\langle j|\dot{P}|k\rangle, \tag{B.5}$$

it follows that the double sum in equation (B.4) reduces to the single sum

$$\mathcal{A}_i = \sum_{j\neq 0} \Big(\frac{\langle 0|\, \partial_\mu P_i \,|j\rangle \; \langle j|\, \dot{P}\, |0\rangle}{\lambda_{j0}} + \mathrm{c.c.} \Big). \tag{B.6}$$

Since $\overline{\lambda_{kj}} = \lambda_{jk}$, we find that $\mathcal{A}_i$ is manifestly real as it must be. Using the fact that (recall that $i \neq 0$)

$$\langle 0|\, \partial_\mu P_i \,|j\rangle = \langle 0|\, P \partial_\mu P_i \,|j\rangle = -\, \langle 0|\, (\partial_\mu P) P_i \,|j\rangle = -\delta_{ij}\, \langle 0|\, \partial_\mu P \,|j\rangle, \tag{B.7}$$

we finally find that

$$\begin{aligned} \sum_{j\neq 0} (\varepsilon_j - \varepsilon_0)\mathcal{A}_j &= -\sum_{j\neq 0} \frac{\varepsilon_j - \varepsilon_0}{\lambda_{j0}} \mathrm{Tr}(P(\partial_\mu P) P_j \dot{P}) + \mathrm{c.c.} \\ &= \sum_{j\neq 0} \frac{1}{i + \gamma_{j0}} \mathrm{Tr}((\partial_\mu P) P_j \dot{P}) + \mathrm{c.c.}, \end{aligned} \tag{B.8}$$

where $\gamma_{j0} \geqslant 0$ is the dimensionless characterization of the spectral data of equation (10).

Simplification occurs, for γ_{j0} is independent of j. This is, of course, automatically the case for a two-level system where j takes one value $j = 1$. (A similar simplification occurs when there is one dominant $1/\gamma_{j0}$.) The sum over j can now be carried out explicitly,

$$\langle F_\mu \rangle = \sum (\varepsilon_j - \varepsilon_0)\mathcal{A}_j = \frac{\gamma - i}{1+\gamma^2} \mathrm{Tr}((\partial_\mu P) P_\perp \dot{P}) + \mathrm{c.c.} \tag{B.9}$$

Writing $\dot{P} = \sum_\nu (\partial_\nu P)\, \dot{\phi}$, we obtain the expression in the theorem.

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