Adiabatic theorem without a gap condition: Two-level system coupled to quantized radiation field

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We prove an adiabatic theorem for the ground state of the Dicke model [Phys. Rev. **93**, 99 (1954)] in a slowly rotating magnetic field and show that for weak electron-photon coupling, the adiabatic time scale is close to the time scale of the corresponding two-level system without the quantized radiation field. There is a correction to this time scale, which is the Lamb shift of the model. The photon field affects the rate of approach to the adiabatic limit through a logarithmic correction originating from an infrared singularity characteristic of QED. [S1050-2947(98)02112-X]

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I. INTRODUCTION

In this work we investigate the relation between adiabatic theorems for models that, like QED, allow for the creation and annihilation of photons and the corresponding quantum mechanical models where the electron is decoupled from the photon field. We study this problem in the context of a specific and essentially soluble model: the Dicke model [1]. The corresponding quantum-mechanical model is a two-level system, such as a spin in an adiabatically rotating magnetic field, which is a basic paradigm of adiabatic theory [2].

In the usual quantum adiabatic theorem [3,4] the gap between eigenvalues plays an important role: It fixes the adiabatic time scale and determines the rate at which the adiabatic limit is approached. There is no such gap in the corresponding QED models, so the nature of the adiabatic theorem in the two cases has qualitatively different features. For example, there is no gap in the spin-boson [10] and Dicke models (for weak coupling), both of which describe a two-level system in a radiation field.

The first problem we address is whether there is an adiabatic theorem for the ground state in a radiation field. Assuming a positive answer, the second question is, What property of the QED model plays the role of the gap in the adiabatic theorem? Another way of phrasing this question is how does the adiabatic time scale of the two-level system compare to that of the QED model? Are the two close in the limit of a small fine-structure constant α and if so, how close? The third question compares the rate of approach to the adiabatic limit in the two models.

Consider a two level system, such as a spin or a twofold Zeeman split atomic level, in an external magnetic field pointing in the z direction. When radiation effects are neglected, the corresponding Hamiltonian is

$$H = m\sigma_z, \quad m = \mu B. \tag{1}$$

The corresponding Dicke model is

$$H_D = H\otimes \mathbf{1} + \alpha^{-1}\mathbf{1}\otimes E + \sqrt{\alpha}\sigma_+ \otimes a^\dagger(f) + \sqrt{\alpha}\sigma_- \otimes a(f), \tag{2}$$

where

$$E = \int |k| a^\dagger(k) a(k) d^d k \tag{3}$$

and[1]

$$f(k) = \sqrt{\frac{2\pi}{|k|}} \langle \psi_1 | \{ e^{-i\mathbf{k}\cdot\mathbf{x}}, p \} | \psi_2 \rangle, \tag{4}$$

with $\psi_j(x)$ the atomic wave functions of the two-level system. Polarization indices are omitted since the helicity of the photon does not play an interesting role in the questions we study. We use atomic units where $e = \hbar = 1$, so $\alpha = 1/c = 1/137$ is small. μ, the magnetic moment, is also of order α in these units.

Following Berry [2], we consider the case where the magnetic field changes its direction adiabatically and has fixed magnitude. The time-dependent Hamiltonian for the two level system is

$$H(s) = \mu B(s)\cdot\sigma = U(s) H U^*(s), \tag{5}$$

with $U(s) \in \mathrm{SU}(2)$ the appropriate rotation. The corresponding adiabatic Dicke model has the time-dependent Hamiltonian

$$H_D(s) = [U(s)\otimes \mathbf{1}] H_D [U^*(s)\otimes \mathbf{1}]. \tag{6}$$

Our aim is to compare the evolution of the ground state of H_D with the instantaneous ground state of $H_D(s)$.

Adiabatic theorems for quantum systems coupled to a field have been studied in [5,6]. In [5] Narnhofer and Thirring give a characterization of extremal Kubo-Martin-Schwinger states by adiabatic invariance. When applicable, this result shows, in particular, that the ground state is adiabatically invariant. The characterization depends on the condition of asymptotic Abelianess, which does not hold for the models we consider. In [6] Davies and Spohn give a derivation of linear response theory for a system coupled to a bath in the adiabatic limit. The notion of adiabaticity in this work

[1] $\{\,,\}$ stands for the anticommutator.

is such that the coupling between the field and the quantum system vanishes in the adiabatic limit. This is not a standard notion of adiabaticity.

Let us now describe our results. First, we show that there is an adiabatic theorem for the ground state of the Dicke model, even though the model has no spectral gap to protect the ground state. Second, we show that the distance to a nearby resonance in the Dicke model plays the role of a gap. Third, we show that the adiabatic time scale for the Dicke model and the two-level system agree in the limit of small α. The (inverse) of two time scales differs by the Lamb shift of the Dicke model. Finally, we show that the approach to the adiabatic limit in the two models is different: While in the two-level system the approach to the adiabatic limit is with an error $O(1/\tau)$, the approach to the limit in the Dicke model is with an error of $O(\sqrt{\ln \tau/\tau})$. The logarithm comes from an infrared divergence characteristic of QED.

Although the results we derive here are for a rather special model, we suggest that something similar happens also for more realistic models. The success of the quantum adiabatic theorem in numerous applications that depend on a correct prediction of the adiabatic time scale is evidence that at least the time scale aspect of our results may carry over to more realistic models. It would be interesting to know if this is indeed the case for the spin-boson model [7–13]. The spin-boson model is a more realistic QED version of a two-level system that, unlike the Dicke model, is not explicitly soluble. However, as much progress in the spectral analysis of the spin-boson problem has been made recently, the problem we pose here may be a reasonable challenge.

II. THE ADIABATIC THEOREM AND A COMMUTATOR EQUATION

In this section we explain what we mean by ''adiabatic theorem'' and give a condition for an adiabatic theorem to hold. This condition is that the commutator equation, Eq. (10) below, has solutions X,Y, which are bounded operators.[2] We also introduce notation and terminology, and collect known facts that we need. To simplify the presentation, we shall stay away from making optimal assertions.

We consider Hamiltonians that are bounded from below and choose the origin of the energy axis so that the spectrum begins at zero. Let $H(s)\geqslant 0$ be a family of such self-adjoint Hamiltonians. The unitary evolution generated by the Hamiltonian $U_\tau(s)$ is the solution of the initial-value problem

$$i\dot{U}_\tau(s)=\tau H(s)U_\tau(s),\quad U_\tau(0)=1,\quad s\in[0,1]. \tag{7}$$

τ is the adiabatic time scale and we are concerned with the limit of large τ. The physical time is $t=\tau s\in[0,\tau]$. Since τ is large $H(s)=H(t/\tau)$ varies adiabatically. We assume that all operators are defined on some fixed dense domain in the Hilbert space.

The (instantaneous) ground state is in the range of the kernel of $H(s)$ and we assume that the kernel is smooth and one dimensional. Let $P(s)\neq 0$ be the projection on the kernel of $H(s)$, i.e., $H(s)P(s)=0$, $\dim P=\mathrm{Tr}\,P=1$. By smoothness we mean that $\dot{P}(s)$ is a bounded operator.

The adiabatic theorems we consider are concerned with the large time behavior of the evolution of the ground state where $t=O(\tau)$ or, equivalently, $s=O(1)$. The smoothness of the kernel implies that there is a natural candidate for an adiabatic theorem for the ground state, which is independent of whether or not $H(s)$ has a gap in its spectrum, namely, if $\psi(0)\in \mathrm{Range}P(0)$ at time $s=0$, then it evolves in time so that $\psi_\tau(s)=U_\tau(s)\psi(0)$ lies in $\mathrm{Range}P(s)$ at time s in the adiabatic limit $\tau\to\infty$.

To formulate the adiabatic theorem with error estimates we need to get hold of *adiabatic phases* [2]. To do that we introduce the adiabatic evolution of Kato [4]: Let $U_A(s)$ be the solution of the evolution equation

$$\dot{U}_A(s)=[\dot{P}(s),P(s)]U_A(s),\quad U_A(0)=1,\quad s\in[0,1]. \tag{8}$$

It is known that

$$U_A(s)P(0)=P(s)U_A(s), \tag{9}$$

that is, $U_A(s)$ maps $\mathrm{Range}P(0)$ onto $\mathrm{Range}P(s)$. We can now formulate the basic adiabatic theorem.

Theorem 1. Let $H(s)P(s)=0$ for all $0\leqslant s\leqslant 1$, with P the differentiable projection on the ground state and $\|\dot{P}(s)\|\leqslant D$. Suppose that the commutator equation

$$[\dot{P}(s),P(s)]=[H(s),X(s)]+Y(s) \tag{10}$$

has operator valued solutions $X(s)$ and $Y(s)$ so that for $\varepsilon\searrow 0$

$$\|X(s)\|+\|\dot{X}(s)\|\leqslant C\times\begin{cases}\varepsilon^{-\nu}\\ |\ln\varepsilon|,\end{cases}\quad \|Y(s)\|\leqslant\hat{C}\varepsilon^{\mu}, \tag{11}$$

with $\mu,\nu\geqslant 0$. Then

$$\|[U_\tau(s)-U_A(s)]P(0)\|\leqslant\tilde{C}\times\begin{cases}\tau^{-\mu/(\nu+\mu)}\\ \dfrac{\ln\tau}{\tau}\end{cases} \tag{12}$$

for $s\in[0,1]$

We make the following remarks.

(i) In the case where there is a gap in the spectrum, one can always find $X(s)$ bounded so that $\nu=0$ and $Y=0$; see [14]. X and therefore $\tilde{C}$ are of the order of $(\mathrm{gap})^{-1}$. This gives an error of $1/\tau$ and generalizes the adiabatic theorem of Born, Fock, and Kato for discrete spectra to more complicated spectra provided there is a gap.

(ii) The theorem states that the physical evolution clings to the instantaneous spectral subspace. In particular, if P is one dimensional, it states that the physical evolution of the ground state remains close to the instantaneous ground state.

(iii) Here and throughout we are concerned only with the adiabatic theorem to lowest order. If s is chosen outside the support of $\dot{P}$ then much stronger results can be obtained. See, e.g., [15].

(iv) The adiabatic time scale τ_0 set by this theorem is $\tau_0=O((2+D)C)$.

[2]For X we require that its derivative is bounded.

Proof. Let $W(s)=U_A^\dagger(s)U_\tau(s)$, with $W(0)=1$. From the equation of motion and the commutator equation, (10),

$$\begin{aligned}P(0)\dot{W}(s)&=-P(0)U_A^\dagger(s)\{i\tau H(s)+[\dot{P}(s),P(s)]\}U_\tau(s)\\&=-U_A^\dagger(s)P(s)\{i\tau H(s)+[\dot{P}(s),P(s)]\}U_\tau(s)\\&=-U_A^\dagger(s)P(s)[\dot{P}(s),P(s)]U_\tau(s)\\&=-U_A^\dagger(s)P(s)\{[H(s),X(s)]+Y(s)\}U_\tau(s)\\&=-U_A^\dagger(s)P(s)\{-X(s)H(s)+Y(s)\}U_\tau(s)\\&=\frac{i}{\tau}P(0)U_A^\dagger(s)X(s)\dot{U}_\tau(s)\\&\quad-P(0)U_A^\dagger(s)Y(s)U_\tau(s).\end{aligned}\tag{13}$$

To get rid of derivatives of U_τ, which are large by the equation of motion, we rewrite the first term on the right-hand side [up to the $P(0)$ on the left] as

$$\begin{aligned}U_A^\dagger(s)X(s)\dot{U}_\tau(s)&=[U_A^\dagger(s)\dot{X}(s)U_\tau(s)]\\&\quad-U_A^\dagger(s)\dot{X}(s)U_\tau(s)-\dot{U}_A^\dagger(s)X(s)U_\tau(s)\\&=[U_A^\dagger(s)\dot{X}(s)U_\tau(s)]-U_A^\dagger(s)\dot{X}(s)U_\tau(s)\\&\quad+U_A^\dagger(s)[\dot{P}(s),P(s)]X(s)U_\tau(s).\end{aligned}\tag{14}$$

From this it follows, by integrating, that for $s\in[0,1]$

$$\begin{aligned}\|[U_\tau(s)-U_A(s)]P(0)\|&=\|P(0)[U_\tau^\dagger(s)-U_A^\dagger(s)]\|\\&=\|P(0)[1-W(s)]\|\leqslant\hat{C}\varepsilon^\mu+\frac{(2+D)C}{\tau}\times\begin{cases}\varepsilon^{-\nu}\\|\ln\varepsilon|.\end{cases}\end{aligned}\tag{15}$$

Choosing $\varepsilon=\tau^{-1/(\mu+\nu)}$ gives

$$\|[U_\tau(s)-U_A(s)]P(0)\|\leqslant\tilde{C}\times\begin{cases}\tau^{-\mu/(\nu+\mu)}\\\dfrac{\ln\tau}{\tau}.\end{cases}\tag{16}$$

This concludes the proof of the theorem.

It is convenient to rewrite this solvability condition in a way that one needs to solve for a fixed X and Y rather than functions $X(s)$ and $Y(s)$. This is accomplished as follows.

Corollary 1. Let $P(s)$ be the family

$$P(s)=V(s)PV^\dagger(s),\quad V(s)=\exp(is\sigma).\tag{17}$$

It is enough to solve for the commutator equation

$$iK=[H,X]+Y,\quad K=\{\sigma,P\}-2P\sigma P,\tag{18}$$

for fixed X and Y, so that for $\varepsilon\searrow 0$

$$\|X\|\leqslant C\times\begin{cases}\varepsilon^{-\nu}\\|\ln\varepsilon|,\end{cases}\quad\|Y\|\leqslant\hat{C}\varepsilon^\mu,\tag{19}$$

with $\mu,\nu\geqslant 0$ and $\|\dot{P}(s)\|\leqslant D$. $X(s)$ and $Y(s)$ are then determined by the obvious unitary conjugation.

Proof. Since $P(s)=V(s)PV^\dagger(s)$, we have

$$\dot{P}(s)=iV(s)[\sigma,P]V^\dagger(s)\tag{20}$$

and

$$\begin{aligned}[\dot{P}(s),P(s)]&=iV(s)[[\sigma,P],P]V^\dagger(s)\\&=iV(s)(\{\sigma,P\}-2P\sigma P)V^\dagger(s).\end{aligned}\tag{21}$$

■

III. AN ADIABATIC THEOREM FOR A THRESHOLD STATE: THE FRIEDRICHS MODEL

As preparation for the analysis of the Dicke model, we prove an adiabatic theorem for the Friedrichs model that has a bound state at the threshold of the continuum. There is an inherent difficulty in the situation of a bound state at threshold in general and in the Friedrichs model [16–18] in particular, namely, that a bound state at threshold is not a stable situation. Under a small deformation of the Hamiltonian, the ground state will, generically, split away from the absolutely continuous spectrum and a gap develops. Since our aim is to study families related by a unitary operator, this problem does not appear. That is, we consider the family $H_F(s)=V(s)H_FV^\dagger(s)$, where H_F has a bound state at threshold and $V(s)$ is a smooth family of unitary operators.

The Friedrichs model

We shall consider a family of Hamiltonians, closely related to the standard Friedrichs model [16], parametrized by the scaled time s, a real number $d>0$ that plays the role of dimension, and a function f that describes the deformation of the family. Since we are interested only in the low-energy behavior of the family we shall introduce an ''ultraviolet cutoff'' to avoid inessential difficulties.

The Hilbert space of the Friedrichs model (with an ultraviolet cutoff) is $\mathcal{H}=\mathbb{C}\oplus L^2([0,1],k^{d-1}dk)$. A vector $\psi\in\mathcal{H}$ is normalized by

$$\psi=\begin{pmatrix}\beta\\f(k)\end{pmatrix}\|\psi\|^2=|\beta|^2+\int_0^1|f(k)|^2k^{d-1}dk,\quad\beta\in\mathbb{C}.\tag{22}$$

We choose a special, and trivial, case of a diagonal Hamiltonian whose action on a vector ψ is

$$H_F\psi=\begin{pmatrix}0&0\\0&k\end{pmatrix}\begin{pmatrix}\beta\\f(k)\end{pmatrix}=\begin{pmatrix}0\\kf(k)\end{pmatrix}.\tag{23}$$

H has a ground state at zero energy with projection

$$P=\begin{pmatrix}1&0\\0&0\end{pmatrix}.\tag{24}$$

The rest of the spectrum is the unit interval $[0,1]$ and is absolutely continuous. The density of states in this model is proportional to E^{d-1}.

We construct the family $H(s)$ by conjugating H with a family of unitary operators

$$V_f(s)=\exp[is\sigma(f)],\quad \sigma(f)=\begin{pmatrix}0 & \langle f|\\ |f\rangle & 0\end{pmatrix},\tag{25}$$

where f is a vector in $L^2([0,1],k^{d-1}dk)$.

Theorem 2. Let $H_F(s;d,f)$ be the family of Friedrichs models with a ground state at threshold for all s,

$$H_F(s;d,f)=V_f(s)H_FV_f^\dagger(s).\tag{26}$$

Suppose that

$$g(k)=ik^{-1}f(k)\in L^2([0,1],k^{d-1}dk),$$

$$V_f(s)=\exp[is\sigma(f)].\tag{27}$$

Then the quantum evolution of the ground state of $H_F(s;d,f)$ is adiabatic and its deviation from the instantaneous ground state is at most $O(1/\tau)$.

We make the following remarks.

(i) Note that if the conditions in the theorem hold in dimension d_0, then they hold in all dimensions $d\geqslant d_0$. The physical interpretation of that is that the density of states at low energies decreases with d. So even though there is spectrum near zero, there is only very little of it.

(ii) If g is not in L^2 there may still be an adiabatic theorem with slower falloff in τ by accommodating $Y\neq 0$. An example will be discussed in Sec. IV.

(iii) The Friedrichs model is vanilla: H_F has no interesting energy scale to fix the adiabatic time scale. The scale is set by the perturbation alone: $\tau_0=O((1+\|f\|^2)\|g\|)$. This is quite unlike the case in the usual adiabatic theorem and unlike what we shall show for the Dicke model.

Proof. In this case K of Corollary 1 is $K=\sigma(f)$. With $g\in L^2$, $\sigma(g)$ is a bounded (in fact, finite rank) operator and an easy calculation gives

$$[H_F,\sigma(g)]=\begin{pmatrix}0 & \langle -kg|\\ |kg\rangle & 0\end{pmatrix}=i\sigma(f).\tag{28}$$

Hence

$$X=\sigma(g),\quad Y=0\tag{29}$$

solve the commutator equation (18) with a bounded $X(s)$ and $Y(s)=0$. ∎

IV. ADIABATIC THEOREM FOR THE DICKE MODEL

In this section we describe an adiabatic theorem for the Dicke model [1] that states that the adiabatic rotation of a two-level system evolves the ground state so that it adheres to the instantaneous ground state and the time scale, at least in three dimensions, is essentially the time scale fixed by quantum mechanics without photons. The rate of approach to the adiabatic limit is different from that of a two level system and has a logarithmic correction in three dimensions. This section also collects known facts about the Dicke model that we need.

A. The Dicke model

The spin-boson Hamiltonian is the canonical QED version of a two-level system [7,12,13]. The Dicke model is a simplified version of the spin-boson Hamiltonian in the rotating-wave approximation. The rotating-wave approximation can indeed be motivated by the single-mode Dicke model. In the multimode case we consider the rotating-wave approximation as describing which terms in the spin-boson Hamiltonian are kept and which are not.

The model describes a two-level system coupled to a massless boson field in d dimensions. The Hamiltonian is

$$H_D(m,d,f,\alpha)=m(1-P)\otimes\mathbf{1}+\alpha^{-1}\mathbf{1}\otimes E+\sqrt{\alpha}\sigma_+\otimes a^\dagger(f)+\sqrt{\alpha}\sigma_-\otimes a(f),\tag{30}$$

acting on the Hilbert space $\mathbb{C}^2\otimes\mathcal{F}$ with $\mathcal{F}$ being the symmetric Fock space over $L^2(R^d,d^dk)$. Here

$$P=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \sigma_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

$$\sigma_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad E=\int |k|a^\dagger(k)a(k)d^dk.\tag{31}$$

$m>0$ is the gap in the quantum Hamiltonian (without photons). $a(f)$ and $a^\dagger(f)$ are the usual creation and annihilation operators on $\mathcal{F}$ obeying the canonical commutation relations

$$[a(f),a^\dagger(g)]=\langle f|g\rangle.\tag{32}$$

We denote by $|0\rangle$ the field vacuum and by Ω the projection on the vacuum.

It may be worthwhile to explain where the various powers of α in H come from. For the radiation field the α^{-1} comes from $\hbar\omega=\hbar c|k|$, which explains why the field energy comes with a large coupling constant. The $\sqrt{\alpha}$ has one inverse power of c from minimal coupling, $(e/2mc)(pA+Ap)$. Half a power of α comes from the standard formula for the vector potential

$$A(x):=\int d^3k\sqrt{\frac{2\pi c}{|k|}}[e^{-i\mathbf{k}\cdot\mathbf{x}}a^\dagger(k)+e^{i\mathbf{k}\cdot\mathbf{x}}a(k)].\tag{33}$$

Compare, e.g., [19].

With reasonable atomic eigenfunctions $f(k)$, Eq. (4) has fast decay at infinity and the model is ultraviolet regular. In the infrared limit $f(k)$ behaves like

$$f(k)\to -i\sqrt{\frac{2\pi}{|k|}}\int[\psi_1^*(x)(\nabla\psi_2)(x)-(\nabla\psi_1)^*(x)\psi_2(x)]d^dx.\tag{34}$$

In particular we see that for small k

$$f(k)=K\sqrt{\frac{1}{|k|}}.\tag{35}$$

The square root singularity is a characteristic infrared divergence of QED and it has consequences for the adiabatic theorem as we shall see. Note that with f having a square root singularity the model makes sense (as an operator) provided $d>1$; otherwise $a^\dagger(f)$ is ill defined since f is not in L^2.

An important parameter in the model is

$$\mathcal{E}=\left\langle f\left|\frac{1}{|k|}\right|f\right\rangle. \tag{36}$$

Bearing in mind the square root singularity of f, we see that

$$\mathcal{E}\sim\int\frac{d^dk}{|k|^2} \tag{37}$$

is finite for all $d>2$.

B. Spectral properties

What makes the Dicke model simple is that it has a constant of motion [7]. If we let $N=\int a^\dagger(k)a(k)d^dk$ be the photon number operator, then $\mathcal{N}$ commutes with H_D where

$$\mathcal{N}=\begin{pmatrix}N & 0\\ 0 & N+1\end{pmatrix}=\mathbf{1}\otimes N+P\otimes\mathbf{1}. \tag{38}$$

The spectrum of $\mathcal{N}$ is the non-negative integers. The spectral properties of $H_D(m,d,f,\alpha)$ can be studied by restricting to subspaces of $\mathcal{N}$.

For $\mathcal{N}=0$ the kernel of $\mathcal{N}$ is one dimensional and is associate with the projection

$$P=\begin{pmatrix}\Omega & 0\\ 0 & 0\end{pmatrix}. \tag{39}$$

Ω is the projection on the field vacuum. It is easy to see that $PH_D(m,d,f,\alpha)P=0$, so the model always has a state at zero energy. This state may or may not be the ground state. It is the ground state if $\alpha^2\mathcal{E}<m$ [7].

For $\mathcal{N}=1$ the space is basically $\mathcal{H}$ of the Friedrichs model. The correspondence of vectors in the two spaces is

$$\begin{pmatrix}a^\dagger(g)\\ \beta\end{pmatrix}|0\rangle\leftrightarrow\begin{pmatrix}g\\ \beta\end{pmatrix}. \tag{40}$$

The Hamiltonian action in the Friedrichs model language is

$$H_D(m,d,f,\alpha)\leftrightarrow\begin{pmatrix}\dfrac{|k|}{\alpha} & |\sqrt{\alpha}f\rangle\\ \langle\sqrt{\alpha}f| & m\end{pmatrix}. \tag{41}$$

It is a known fact about the Friedrichs model [16,17] that provided

$$\alpha^2\mathcal{E}<m \tag{42}$$

the model has no bound state and the spectrum is $[0,\infty)$ and is absolutely continuous. Since f has a square root singularity at the origin (and has fast decay at infinity), this condition holds for $d\geqslant 3$ if α (or f) is not too large. In three dimensions, provided the level spacing $m\gg\alpha^2$ in atomic units (about 10^{-3} eV), the inequality holds. In two dimensions the left-hand side is logarithmically divergent and the spectrum in the $\mathcal{N}=1$ sector has a bound state at negative energy. This state lies below the bound state of the $\mathcal{N}=0$ sector. We do not consider this situation and henceforth stick to $d\geqslant 3$.

For $\mathcal{N}\geqslant 2$ it is known [7] that the bottom of the spectrum in all these sectors is at zero if Eq. (42) holds.

C. Adiabatic rotations

Suppose that the two-level system of the Dicke model describes, e.g., two Zeeman split energy levels of an atom in constant external magnetic field B pointing in the z direction. Rotations about the z axis do not change the orientation of the magnetic field, commute with $\mathcal{N}$, and are uninteresting. Rotations about the x axis change the orientation of the magnetic field and are implemented by

$$V(s)=\exp(is\sigma_x)\otimes\mathbf{1}. \tag{43}$$

Such rotations do not commute with $\mathcal{N}$. Indeed,

$$[\mathcal{N},\sigma]=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\otimes\mathbf{1}=J\otimes\mathbf{1},\quad \sigma=\sigma_x\otimes\mathbf{1}. \tag{44}$$

D. The adiabatic theorem

Theorem 3. Let $H_D(s;m,d,f,\alpha)=V(s)H_D(m,d,f,\alpha)V^\dagger(s)$, $s\in[0,1]$, be the family of time-dependent Dicke models with f square integrable, with square root singularity at $k=0$; $m>\alpha^2\langle f|\,1/|k|\,|f\rangle$; $d\geqslant 3$; and $V(s)=\exp(is\sigma)$ as in Eq. (43). Then U_A, the adiabatic evolution associated with the ground state of $H_D(s;m,d,f,\alpha)$, and U_τ, the Schrödinger evolution, are close in the sense that

$$\|[U_A(s)-U_\tau(s)]P(0)|\leqslant C\times\begin{cases}\dfrac{1}{\tau} & \text{if } d>3\\ \dfrac{\sqrt{\ln\tau}}{\tau} & \text{if } d=3.\end{cases} \tag{45}$$

The time scale is determined by $m-\alpha^2\mathcal{E}$ and coincides with the gap without photons m up to a correction by the Lamb shift $\alpha^2\mathcal{E}$.

Proof. From Corollary 1 we find $K=\sigma\otimes\Omega$. We will first show that a solution of the commutator equation (18) for $d>0$ is

$$X=\frac{iX_1-X_2(g)}{m-\alpha^2\mathcal{E}},\quad Y=0, \tag{46}$$

where

$$X_1=J\otimes\Omega,\quad X_2(g)=P\otimes[a^\dagger(g)\Omega+\text{H.c.}],$$

$$g=i\alpha^{3/2}\frac{f}{|k|}. \tag{47}$$

Note that the gap of the two-level system m is renormalized to $m+i\alpha\langle f|g\rangle$, which is just the Lamb shift (see the Appendix). This is a small correction, of order α^2.

A useful formula we shall need is

$$Ea^\dagger(g)\Omega=a^\dagger(|k|g)\Omega. \tag{48}$$

Let us compute the commutators of X_1, X_2 with H:

$$[H,X_1]=\left[\begin{pmatrix} \frac{E}{\alpha} & \sqrt{\alpha}a^\dagger(f) \\ \sqrt{\alpha}a(f) & m+\frac{E}{\alpha}\end{pmatrix},\begin{pmatrix} 0 & -\Omega \\ \Omega & 0\end{pmatrix}\right]=m\sigma\otimes\Omega+\sqrt{\alpha}P\otimes[a^\dagger(f)\Omega+\Omega a(f)]. \tag{49}$$

For the second commutator

$$[H,X_2]=\left[\begin{pmatrix} \frac{E}{\alpha} & \sqrt{\alpha}a^\dagger(f) \\ \sqrt{\alpha}a(f) & m+\frac{E}{\alpha}\end{pmatrix},\begin{pmatrix} a^\dagger(g)\Omega+\text{H.c.} & 0 \\ 0 & 0\end{pmatrix}\right]$$

$$=\frac{1}{\alpha}P\otimes[a^\dagger(|k|g)\Omega-\Omega a(|k|g)]+\sqrt{\alpha}\begin{pmatrix} 0 & -\langle g|f\rangle \\ \langle f|g\rangle & 0\end{pmatrix}\otimes\Omega. \tag{50}$$

So if we take g of Eq. (47) then

$$[H,iX_1-X_2]=i(m-\alpha^2\mathcal{E})\sigma\otimes\Omega. \tag{51}$$

We see that we can formally solve for the commutator equation (18) provided $\mathcal{E}$ is finite.

This is not, however, the only condition. X is a bounded operator in the Hilbert space provided $g\in L^2$ otherwise $a^\dagger(g)$ is ill defined:

$$\int\frac{|f|^2}{|k|^2}d^dk\sim\int\frac{1}{|k|^3}d^dk<\infty. \tag{52}$$

The integral is finite if $d\geqslant 4$, but is logarithmically divergent if $d=3$. For $d=3$ we need to squeeze X back to the bounded operators. We do that by allowing for $Y\neq 0$.

Let χ_ε be the characteristic function of a ball of radius ε and $\chi_\varepsilon^c=1-\chi_\varepsilon$ and let $g_\varepsilon^c=\chi_\varepsilon^c g$ and $g_\varepsilon=\chi_\varepsilon g$. Let us take $X_2(g_\varepsilon^c)$, which is well defined; its norm is $O(\alpha^{3/2}\sqrt{|\ln\varepsilon|})$. For X we take, as before,

$$X=\frac{iX_1-X_2(g_\varepsilon^c)}{m+i\sqrt{\alpha}\langle f|g_\varepsilon^c\rangle}. \tag{53}$$

From this

$$\|X\|=O\left(\frac{1+\alpha^{3/2}|\ln\varepsilon|^{1/2}}{|m-\alpha^2\mathcal{E}|}\right). \tag{54}$$

For Y we take

$$(m+i\sqrt{\alpha}\langle f|g_\varepsilon^c\rangle)Y=[H,X_2(g)-X_2(g_\varepsilon^c)]=[H,X_2(g_\varepsilon)]$$

$$=\frac{1}{\alpha}P\otimes[a^\dagger(|k|g_\varepsilon)\Omega-\Omega a(|k|g_\varepsilon)]$$

$$+\sqrt{\alpha}\begin{pmatrix} 0 & -\langle g_\varepsilon|f\rangle \\ \langle f|g_\varepsilon\rangle & 0\end{pmatrix}\otimes\Omega \tag{55}$$

and we used the computation of the commutator (50). With f having a square root singularity,

$$\|Y\|=O\left(\frac{\sqrt{\alpha}\varepsilon+\alpha^2\varepsilon}{|m-\alpha^2\mathcal{E}|}\right). \tag{56}$$

This puts us in the frame of Theorem 1; except for the minor modification, the logarithm appears with a square root. Chasing the square root establishes the main result. ■

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APPENDIX: RESONANCE AND LAMB SHIFT OF THE DICKE MODEL

The $\mathcal{N}=1$ sector of the Dicke model has a resonance that serves to define the Lamb shift. The resonance is a solution of the analytically extended eigenvalue equation, (A1); see [16,18,13]. The real part of the shift is, by definition, the Lamb shift of the model, while the imaginary shift is the lifetime. For $d\geqslant 3$, the Lamb shift is dominant and the lifetime is a higher order in α. For the application to the adiabatic theorem we need only the dominant contribution, i.e., only the Lamb shift. Computing the Lamb shift is easy. Computing the lifetime is harder. For the sake of completeness we compute both, even though we only need one.

The eigenvalue equation is

$$E-m=\alpha^2G(\alpha E), \tag{A1}$$

where $G(e)$ is defined as the analytic continuation from the upper half plane of

$$G(e)=\int_{\mathbb{R}^d}\frac{|f|^2}{e-|k|}d^dk,\quad \text{Im}\,e\geqslant 0. \tag{A2}$$

By taking the imaginary part, it is easy to see that Eq. (A1) has no solution in the upper half plane. To solve the equation in the lower half plane one needs an explicit expression, at least for small α, and e near αm, of this analytic continua-

tion. Then we can solve Eq. (A1) by iteration and to lowest order we have

$$E_r \approx m + \alpha^2 G(\alpha m). \tag{A3}$$

Clearly $G(\alpha m) \to -\mathcal{E}$ in the limit $\alpha \to 0$, so to leading order

$$E_r \approx m - \alpha^2 \mathcal{E}. \tag{A4}$$

To this order, one does not see the imaginary part of the resonance energy. $\alpha^2\mathcal{E}$ is, by definition [19], the Lamb shift of the model. It may be worthwhile to point out that the Lamb shift for the hydrogen atom [20] is actually of *higher* order, namely, $\alpha^3 \ln(\alpha^{-1})$. Since the Lamb shift of hydrogen also involves an ultraviolet regularization while the present model is ultraviolet regular, it is not surprising that the order of the two is different.

Estimating the lifetime is, as we noted, irrelevant to the adiabatic theorem. So a reader will lose little by skipping the rest of this appendix. However, for the benefit of the reader who is interested in how the computation of the lifetime goes, it is given below.

We shall show below that for $d \geqslant 3$ and $|e - \alpha m| < \alpha m$, the analytic continuation of $G(e)$ to the lower half plane and to the next relevant order in α is given by

$$G(e) = -\mathcal{E} - i\pi K \Omega^d e^{d-2}, \quad \mathrm{Im}\, e \leqslant 0, \tag{A5}$$

where K is as in Eq. (35) and Ω^d is the surface area of the unit ball in d dimensions. From Eq. (A3) and taking into account Eq. (35), we get for the Lamb shift and the lifetime

$$\begin{aligned} E_r &\approx m - \alpha^2 \mathcal{E} - i\alpha^2 \pi K \Omega^d (m\alpha)^{d-2} \\ &= m - \alpha^2 \mathcal{E} - i\alpha^2 \pi \Omega^d (m\alpha)^{d-1} |f(\alpha m)|^2. \end{aligned} \tag{A6}$$

The lifetime is higher order in α than the Lamb shift and is of order α^d. For $d = 3$ this is indeed the order of the lifetime of atomic levels that decay by dipole transition. For small α the Lamb shift dominates the lifetime, both in the Dicke model and in hydrogen.

It remains to show that the analytic continuation of $G(e)$ to the lower half plane in a neighborhood of $m\alpha$ is indeed given by Eq. (A5). This can be done as follows. Let B_r be a ball of radius $r = 2m\alpha$. Then, in the upper half plane

$$G(e) = \left(\int_{B_r} + \int_{B_r^c} \right) \frac{|f|^2}{e - |k|} d^d k = G_r(e) + G_r^c(e). \tag{A7}$$

Clearly, $G_r^c(e)$ extends analytically to a half circle in the lower half plane $|e - \alpha m| < \alpha m$. In the limit of $\alpha \to 0$, by continuity,

$$G_r^c(0) \to -\mathcal{E}. \tag{A8}$$

This is the dominant piece and it is real.

Consider the analytic continuation of $G_r(e)$ for $|e - \alpha m| \leqslant \alpha m$. Since, for a small argument, $f(k)$ is given by Eq. (35), one has (in the upper half plane)

$$G_r(e) = K\Omega^d \int_0^{2m\alpha} \frac{k^{d-2}}{e - k} dk = K\Omega^d \int_\gamma \frac{k^{d-2}}{e - k} dk, \tag{A9}$$

where γ is the obvious semicircle in the complex k plane and Ω^d the surface area of the unit ball in d dimensions. The right-hand side is analytic in e in the lower half plane provided $|e - \alpha m| < \alpha m$ and so gives the requisite analytic continuation. Since e is small and of order α, to leading order we have

$$\begin{aligned} G_r(e) &= K\Omega^d \int_\gamma \frac{(k - e + e)^{d-2}}{e - k} dk \\ &= -K\Omega^d \sum_{j=0}^{d-2} \binom{d-2}{j} e^{d-j-2} \int_\gamma (k - e)^{j-1} dk \\ &\approx -K\Omega^d e^{d-2} \int_\gamma \frac{dk}{k - e} \\ &= -K\Omega^d e^{d-2} \left[i\pi + \ln\left(\frac{2\alpha m}{e} \right) + O(\alpha \ln \alpha) \right] \end{aligned} \tag{A10}$$

and the error term in the approximation that we did not compute is real and being subdominant to $\mathcal{E}$ is irrelevant.

[1] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[2] M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984); in *Geometric Phases in Physics*, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989).
[3] M. Born and V. Fock, Z. Phys. **51**, 165 (1928).
[4] T. Kato, J. Phys. Soc. Jpn. **5**, 435 (1958).
[5] H. Narnhofer and W. Thirring, Phys. Rev. A **26**, 3646 (1982).
[6] E. B. Davies and H. Spohn, J. Stat. Phys. **19**, 511 (1978).
[7] H. Heubner and H. Spohn, Ann. Inst. Henri Poincaré: Phys. Theor. **62**, 289 (1995).
[8] V. Bach, J. Fröhlich, and I. M. Sigal, Lett. Math. Phys. **34**, 183 (1995).
[9] V. Bach, J. Fröhlich, and I. M. Sigal, Adv. Math. **137**, 205 (1998).
[10] V. Bach, J. Fröhlich, and I. M. Sigal, Adv. Math. **137**, 299 (1998).
[11] V. Bach, J. Fröhlich, I. M. Sigal, and A. Sofer (unpublished).
[12] J. Dereziński and C. Gérard (unpublished).
[13] R. Minlos and H. Spohn, Am. Math. Soc. Trans. **177**, 159 (1996).
[14] J. E. Avron, R. Seiler, and L. G. Yaffe, Commun. Math. Phys. **110**, 33 (1987); **153**, 649(E) (1993).
[15] M. Klein and R. Seiler, Commun. Math. Phys. **128**, 141 (1990).
[16] K. O. Friedrichs, Commun. Pure Appl. Math. **1**, 361 (1948).
[17] K. O. Friedrichs, *Perturbations of Spectra in Hilbert Space* (AMS, Providence, RI, 1965).
[18] J. S. Howland, J. Math. Anal. Appl. **50**, 415 (1975).
[19] C. Leonardi, F. Persico, and G. Vetri, Riv. Nuovo Cimento **9**, 1 (1986).
[20] H. Bethe, Phys. Rev. **72**, 339 (1947).