THE LAGRANGIAN OF A TOP

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Abstract. The solution to problem 1 in Moed B, Fall 2002

1. Conjugate momenta

$$
\begin{aligned}
p_\theta &= I_1\,\dot\theta \\
p_\phi &= (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot\phi + I_3 \cos\theta\,\dot\psi \\
p_\psi &= I_3(\dot\phi\,\cos\theta + \dot\psi)
\end{aligned}
\tag{1.1}
$$

2. Constants of motion

ϕ and ψ are cyclic coordinates so their conjugate momenta are constants of motion.

The Lagrangian is time independent so the energy is a constant of motion.

3. Euler-Lagrange

$$
\begin{aligned}
\dot p_\theta &= (I_1 - I_3)\dot\phi^2 \sin\theta\cos\theta - I_3\,\dot\phi\dot\psi\sin\theta - K\sin\theta \\
\dot p_\phi &= 0 \\
\dot p_\psi &= 0
\end{aligned}
\tag{3.1}
$$

4. Special solution

From the constancy of p_ϕ it follows that θ is a constant of motion if $\dot\phi \neq 0$. If $\dot\phi = 0$ this follows from the constancy of p_ϕ provided $\dot\psi \neq 0$.

Assuming that at least one of $\dot\phi$, $\dot\psi \neq 0$, we get from the remaining equation of motion

$$0 = (I_1 - I_3)\dot\phi^2 \sin\theta\cos\theta - I_3\,\dot\phi\dot\psi\sin\theta - K\sin\theta$$

which implies that either $\sin\theta = 0$ or θ is a solution of

$$(I_1 - I_3)\dot\phi^2\cos\theta - I_3\,\dot\phi\dot\psi = K$$

It remains to consider the case that $\dot\phi = \dot\psi = 0$. The equation for θ reduces to

$$\dot p_\theta = -K\sin\theta$$

which is the equation of motion of the physical pendulum.

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