TWO MASSES ON A HOOP

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1 The Lagrangian

The kinetic energy is

$$T = \frac{mR^2\dot{\varphi_1}^2}{2} + \frac{mR^2\dot{\varphi_2}^2}{2}$$

From the Cosine Theorem, the potential energy is

$$U = \frac{k}{2}(\sqrt{2R^2 - 2R^2\cos(\varphi_2 - \varphi_1)} - \ell_0)^2 = \frac{k}{2}(2R\sin(\frac{\varphi_2 - \varphi_1}{2}) - \ell_0)^2$$

So the Lagrangian is

$$L = T - U = \frac{mR^2\dot{\varphi_1}^2}{2} + \frac{mR^2\dot{\varphi_2}^2}{2} - \frac{k}{2}(2R\sin(\frac{\varphi_2 - \varphi_1}{2}) - \ell_0)^2$$

The E.L. equations are:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\varphi}_i}) = \frac{\partial L}{\partial \varphi_i} \qquad (i = 1, 2)$$

$$mR^2\ddot{\varphi}_1 = kR(2R\sin(\frac{\varphi_2 - \varphi_1}{2}) - \ell_0)\cos(\frac{\varphi_2 - \varphi_1}{2})$$

$$mR^2\ddot{\varphi}_2 = -kR(2R\sin(\frac{\varphi_2 - \varphi_1}{2}) - \ell_0)\cos(\frac{\varphi_2 - \varphi_1}{2})$$

2 Equilibrium

$\varphi_1 = const$ and $\varphi_2 = const$ are solutions of the equations iff

$$\sin(\frac{\varphi_2 - \varphi_1}{2}) = \frac{\ell_0}{2R}$$

or

$$\varphi_2 - \varphi_1 = \pi$$

The first equation is for stable equilibrium (The spring is not streched) and the second is for unstable equilibrium (The spring is streched to its maximum).

3 Small Oscillations

When $\ell_0 = \sqrt{2}R$

$$\sin(\frac{\varphi_2^0 - \varphi_1^0}{2}) = \frac{\sqrt{2}}{2}$$

φ_i^0 is the angle in stable equilibrium ($\varphi_2^0 - \varphi_1^0 = \frac{\pi}{2}$) and ϕ_i is the small deviation from it $(i = 1, 2)$.

$$\sin(\frac{\varphi_2^0 + \phi_2 - \varphi_1^0 - \phi_1}{2}) = \sin(\frac{\varphi_2^0 - \varphi_1^0}{2}) + \frac{1}{2}\cos(\frac{\varphi_2^0 - \varphi_1^0}{2})(\phi_2 - \phi_1) + O(\phi_1^2, \phi_2^2) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}(\phi_2 - \phi_1) + O(\phi_1^2, \phi_2^2)$$

So the Lagrangian in this approximation is

$$L = \frac{mR^2\dot{\phi}_1^{\,2}}{2} + \frac{mR^2\dot{\phi}_2^{\,2}}{2} - \frac{kR^2}{4}(\phi_2 - \phi_1)^2$$

4 Normal modes and frequencies

From the last form of the Lagrangian

$$K = \frac{kR^2}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$M = mR^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(K - \omega^2 M) = 0 \Rightarrow (\frac{k}{2} - \omega^2 m)^2 - (\frac{k}{2})^2 = 0$$

The solutions for the frequencies are:

• $\omega_1 = 0$ with the normal mode $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (The two masses move together in the same direction)

• $\omega_2 = \sqrt{\frac{k}{m}}$ with the normal mode $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (The two masses move in opposite directions with the same amplitude and frequency $\sqrt{\frac{k}{m}}$)