COURSE 1

ADIABATIC QUANTUM TRANSPORT

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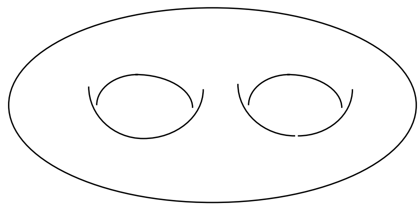

Fig. 0.1 : A Pretzel with 2 handles

Overview

Non-dissipative relatives of the ordinary (dissipative) conductance, which appears in the classical Ohm's law,

Current =Conductance × Voltage,

have geometric significance. The Hall conductance, [47, 51], is an example. Much of what I shall say will be devoted to giving precise sense to [49, 50, ???]

Conductance = Adiabatic Curvature.

For open systems, the transport of charge is related also to *comparing dimensions of Hilbert spaces*, [10, 14], leading to a second geometric relation:

Conductance = Dimension.

In the classical theory of surfaces, the Gaussian curvature of the surface M is related to a topological invariant, the Euler characteristic $\chi(M)$:

$$\int_M Curvature = 2\pi\chi(M) = 4\pi\,(1-h).$$

h is the number of handles (the sphere has no handles). An analog, relates the transport coefficients with topological invariants, known as Chern numbers [20]. Points where energy level cross play a role analogous to handles.

From a mathematical point of view what characterizes the approach presented here is the study of geometric objects associated with *families* of Schrödinger operators whose parameters are *Aharonov-Bohm fluxes.* From a physical point of view this is a study of transport and its relation to the sensitivity of quantum systems, as measured by Berry's phase, [16], to variation in Aharonov-Bohm fluxes.

The first ten chapters set the stage and provide background material about geometry, gauge fields and quantum mechanics. The last three chapters describe the applications to transport.

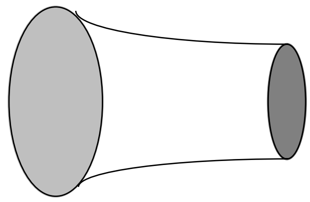

Fig. 1.1: A horn in Euclidean three space
has negative Gaussian curvature

1. Manifolds and their Calculus

1.1. Manifolds

The Aharonov-Bohm effect tells us that interesting quantum phenomena occur in multiply connected systems. A class of model surfaces we shall consider are Riemanian manifolds. Riemanian says that the manifold M comes with a metric g_M. Compact manifolds model *closed* quantum systems and noncompact manifolds model *open* quantum systems. On a compact manifold transport is associated with "currents going around holes" and so forces us to mind multiply connected systems. On a non-compact manifold charges can also be lost to infinity.

The examples listed below describe various models of two dimensional surfaces. Some of these serve as conventional models for two dimensional electron gases, some are more esoteric.

a. The Poincaré-Lobachevski upper half plane [11, ???]: $\boldsymbol{H} = \{z|\ \mathrm{Im}\, z > 0\}$, has metric $(\mathrm{d}s)^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$. Geodesics are semi-circles, whose centers lie on the x-axis. The Gaussian curvature equals -1, and fixes a length scale. $\boldsymbol{H}$ looks, locally, like a horn in 3-space, Fig. 1.1, where every point is a saddle. Like the plane and the sphere, it is a symmetric space where every point looks like every other point. Part of the interest in this model comes from its relation to chaotic dynamics, in particular the geodesic flow has exponentially diverging trajectories [11, 34]. Möbius transformations,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z' = \gamma(z) = \frac{az+b}{cz+d}, \quad \det\gamma = 1; \quad a,b,c,d \in \boldsymbol{R},$$

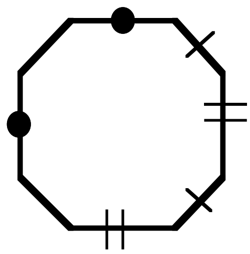

Fig. 1.2: The 2-handle torus of Fig. 0.1
represented as an octagon with pairs of edges identified.

play a role analogous to the Euclidean motions of the Euclidean plane. They take geodesics to geodesics, and the upper half plane to itself. Two useful formulae are:

$$\mathrm{d}z' = \frac{\mathrm{d}z}{(cz+d)^2}, \quad y' = \frac{y}{|cz+d|^2.} \tag{1.1}$$

b. Multi-handle tori: Riemann surfaces, (with two handles or more), can be represented by polygons in the Poincaré plane with edges identified [27, 33], see Fig. 1.2.

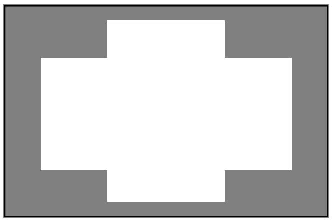

Fig. 1.3: A punctured torus with a long boundary

c. Punctured Tori: Punctured tori, such as the one shown in Fig. 1.3, have the topology one normally associates with an idealized quantum Hall effect [44], shown schematically in Fig. 1.4. When an emf drives the system in one loop, a Hall current flows in the other loop.

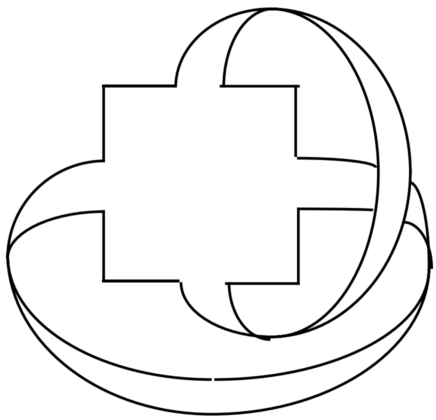

Fig. 1.4: A configuration associated with the Hall effect

d. Graphs: Graphs represent quantum wires, tight binding models, and arrays of Josephson junctions [1, 4, 7,30].

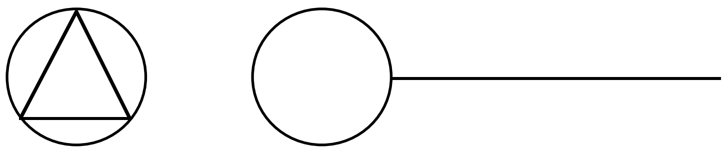

Fig. 1.5: Graphs, compact and non compact

The rest of this chapter is an introduction to the calculus of forms, homology and cohomology [3, 25, 28].

1.2. Forms

Let $x \in M$, with local coordinates $(x^1, \ldots, x^m)$, $m = \dim M$. 0-form are functions; 1-form are, locally,

$$\omega^1(x) = \sum_{j=1}^{m} \omega_j(x)\, \mathrm{d}x^j$$

and 2-form are given, locally, by

$$\omega^2(x) = \sum_{j<k}^{n} \omega_{jk}(x)\, \mathrm{d}x^j \wedge \mathrm{d}x^k, \quad \mathrm{d}x^j \wedge \mathrm{d}x^k = -\mathrm{d}x^k \wedge \mathrm{d}x^j.$$

For example, vector potentials (=gauge fields) $\boldsymbol{A}$ are 1-forms. Magnetic fields are 2-forms.

Forms can be added and multiplied:

$$\omega^k \wedge \omega^j = (-)^{jk} \omega^j \wedge \omega^k.$$

1.3. The Exterior derivative

The exterior derivative is a linear map from k-forms to (k+1)-forms:

$$\mathrm{d}\omega^k = \mathrm{d}\left(\sum \omega^k_J \; \mathrm{d}x^J\right) = \sum_{J,j} \left(\partial_j \omega^k_J\right) \; \mathrm{d}x^j \wedge \mathrm{d}x^J.$$

J is a multi-index.

Poincaré's lemma:

$$\mathrm{d}^2 = 0.$$

Proof:

$$\mathrm{d}^2 = \left(\partial_j \mathrm{d}x^j\right) \wedge \left(\partial_k \mathrm{d}x^k\right) = \left(\partial_{jk}\right) \; \mathrm{d}x^j \wedge \mathrm{d}x^k = 0.$$

In three space d is *grad*, *curl* and *div* when it operates on $0, 1, 2$ forms. Hence *curl grad* = *div curl* = 0, are versions of Poincaré's lemma.

1.4. Riemanian Metric

The metric $(\mathrm{d}s)^2 = \sum g_{jk} \, \mathrm{d}x^j \otimes \mathrm{d}x^k$, gives a volume form:

$$\mathrm{d} \; \mathrm{Vol} = \sqrt{g} \; \mathrm{d}x^1 \wedge \ldots \; \mathrm{d}x^m; \quad g = \det\{g_{jk}\} > 0.$$

The metric of an m-dimensional manifold is *conformal* if $g_{jk}(x) = \sqrt[m]{g}\delta_{jk}$.

1.5. Scalar products

The scalar product of p-forms is given by:

$$\mathrm{d}x^j \cdot \mathrm{d}x^k = g^{jk}, \quad \left(\mathrm{d}x^1 \wedge \ldots \mathrm{d}x^p\right) \cdot \left(\mathrm{d}y^1 \wedge \ldots \mathrm{d}y^p\right) \; = \det\,(\mathrm{d}x^j \cdot \mathrm{d}y^k).$$

1.6. Hodge $*$

Hodge $*$ is a linear operator on forms, so that for an m-dimensional manifold

$$\begin{aligned} * : \; k - forms \longrightarrow (m-k) forms, \\ \omega^k \wedge \omega^{m-k} = \left(*\omega^k \cdot \omega^{m-k}\right) \mathrm{d} \; Vol. \end{aligned}$$

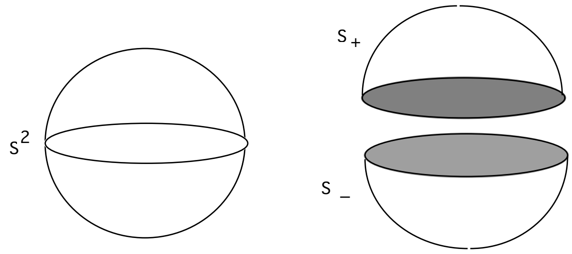

Fig. 1.6: The sphere and its hemispheres

For example, a two dimensional surface with a conformal metric has

$$\begin{aligned} *(\mathrm{d}x) = \mathrm{d}y,\ *(\mathrm{d}y) = -\mathrm{d}x,\ *(1) = \mathrm{d}\,Vol; \\ *(\mathrm{d}z) \;=\; -i\,\mathrm{d}z,\ *\,(\mathrm{d}\bar{z}) \;=\; i\,\mathrm{d}\bar{z}. \end{aligned} \tag{1.2}$$

1.7. Cohomology

ω^k is *closed* if $\mathrm{d}\omega^k = 0$ and is *exact* if $\omega^k = \mathrm{d}f$. Every exact form is closed. A closed form is exact only locally, in a disc like domain. Sensible people study electrostatics in Euclidean three space, where the assertion that $curl\ E = 0$ implies $E = grad\ V$ is indeed valid. In spaces with complicated topology it is not.

Example: The Maxwell equation $\mathrm{d}\boldsymbol{B} = 0$, $(div\ \vec{B} = 0)$, makes sense on any manifold (and is automatic if the manifold is two dimensional). On R^3 it implies $B = \mathrm{d}A$, $(B = div\ A)$, but in Euclidean space with the origin removed, A is only locally defined (e.g. on hemispheres). When a monopole sits at the origin, a Dirac string of singularities prevents defining the vector potential globally.

Cohomology groups are defined by:

$$H^k(M) = \{closed\ k - forms\}/\{exact\ k - forms\}.$$

The quotient means that f and g are identified if $f - g$ is exact. The closed/exact forms often live in infinite dimensional function space. The cohomology, in contrast, is finite dimensional.

For example the circle has no exact zero forms, and the closed 0-forms are constant. This makes $H^0(S^1) \simeq \boldsymbol{R}$. All 1-forms $f(\theta)d\theta$ are automatically closed. The exact ones $(\mathrm{d}\ g)(\theta)$, have zero averages. Thus $H^1(S^1) \simeq \boldsymbol{R}$.

1.8. Harmonic forms

a is harmonic if it is closed and co-closed:

$$\mathrm{d}a = *\mathrm{d} * a = 0.$$

For example, on R^2, (with conformal metric) if f is holomorphic then $a = \mathrm{d}f$ is harmonic by Eq. (1.1). On $\boldsymbol{T}^2$ periodicity selects a two dimensional subspace:

$$a_1 = \alpha\,\mathrm{d}x + \beta\,\mathrm{d}y, \quad \alpha,\, \beta \in \boldsymbol{R}.$$

It is a basic result of Hodge theory that for closed Riemann surfaces any closed form ω can be uniquely decomposed into a harmonic and exact parts: $\omega = h + \mathrm{d}\chi$.

1.9. Integration

k-forms can be integrated on k-chains:

$$\int_{c_k} w^k.$$

k-chains are k-dimensional submanifolds: A point is a 0 chain; A line a 1-chain; A surface is a 2-chain, etc. The pairing between chains and forms to give a real number make the space of chains and forms dual to to each other.

Stokes Theorem

$$\int_c \mathrm{d}\ \omega^{k-1} = \int_{\partial c} w^{k-1}.$$

∂c is the oriented boundary of c.

1.10. Homology

c is closed if $\partial c = 0$, and exact if $c = \partial \tilde{c}$.

Poincaré's lemma

$$\partial^2 = 0.$$

This is known as "the boundary of the boundary is zero".

The Homology groups are defined by:

$$H_k(M) = \{closed\ k - chains\}/\{exact\ k - chains\}.$$

The quotient means that one identify c with c' if $c - c' = \partial C$.

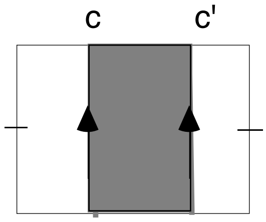

Fig. 1.7: c and c' are closed and homologous 1-chains on the torus $c - c'$ is the boundary of the hatched area.

For example the homology groups of the two torus are:

$$H_0(T^2) \simeq H_2(T^2) \simeq \boldsymbol{R}, \quad H_1(T^2) \simeq \boldsymbol{R} \oplus \boldsymbol{R}.$$

1.11. Periods

The pairing of closed k-forms with closed k-chains provides a way to normalize the basis for the cohomology group, e.g. by setting the basic periods to be 0 and 1.

2. Aharonov-Bohm Fluxes

2.1. Summary

When Aharonov-Bohm gauge fields act on a system which has several holes the Aharonov-Bohm fluxes make a vector $\phi = (\phi_1, \ldots, \phi_n)$. There are some elementary questions one has to mind when considering the vector space of fluxes. For example, does it have a distinguished origin? A distinguished basis? and a distinguished metric? The answer is no, in general, for the first two questions and yes for the second. We shall chose the metric so that when the flux is time dependent the length squared of $\dot{\phi}$ is related to the electromagnetic energy.

2.2. Fluxes

Imagine a quantum system threaded by coils that run through the holes. The coils carry currents that can be controlled independently and give rise

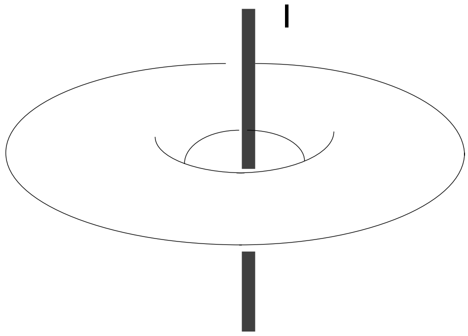

Fig. 2.1: Torus embedded in three space
with a current carrying coil threading the hole

to Aharonov-Bohm gauge fields which are closed 1-form $\boldsymbol{A}$. Since $\mathrm{d}\boldsymbol{A} = 0$ there is no magnetic field associated to $\boldsymbol{A}$ on the surface.

Let $\{c_1, \ldots, c_n\}$, $n = dim\ H_1(M)$ be a basis for the first homology. There is no unique way to choose it (one can always reverse the orientation of cycles, and take linear combinations). So we pick a choice and stick with it, such as e.g. Fig. 2.2. The corresponding periods of a gauge potential $\boldsymbol{A}$ give the flux vector $\phi = (\phi_1, \ldots, \phi_n)$, $\phi_j = \int_{c_j} \boldsymbol{A}$. Since $\boldsymbol{A}$ is closed, deformations of the cycles do not affect ϕ.

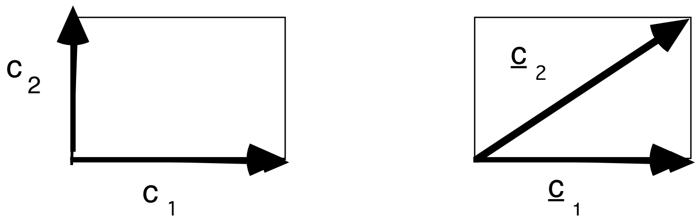

Fig. 2.2: A torus with two choices of bases for the first homology

Examples:

a. The unit circle, traversed counterclockwise generates the first homology of the punctured plane $\boldsymbol{R}^2/\{0\}$. Any closed 1-form can be written as

$$\boldsymbol{A} = \frac{\phi}{2\pi}\, i\, \mathrm{e}^{\mathrm{i}\theta} \left(\mathrm{d}\mathrm{e}^{-\mathrm{i}\theta}\right) + \mathrm{d}\chi, \quad \phi \in \boldsymbol{R}.$$

θ the polar angle. Clearly $\mathrm{d}\boldsymbol{A} = 0$ on $\boldsymbol{R}^2/\{0\}$, and

$$\int_c \boldsymbol{A} = \begin{cases} \phi, & c \text{ encircles origin;} \\ 0, & \text{otherwise.} \end{cases}$$

2.3. Gauge Fixing

Consider a fixed set of n coils. The n-vector of currents through the coils, I, is associated with a (linear) map to the space of Aharonov-Bohm gauge fields:

$$I \longrightarrow \sum \phi^j \ a_j, \quad \phi = (\phi^1, \ldots, \phi^n) \in \boldsymbol{R}^n.$$

ϕ is linear in I and the a_j are I independent. We may think of the a_j as being associated with the given set of coils. Fig. 2.3 shows schematically what is meant by identical systems with different coils.

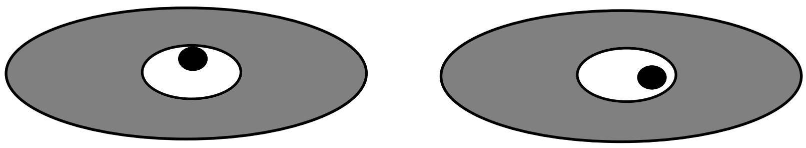

Fig. 2.3: Identical coils, which carry time independent currents, placed in different positions inside the holes of identical systems.

Choosing a basis for the first cohomology means that any closed 1-form has a unique decomposition

$$\sum \phi^j \ a_j + \mathrm{d}\chi, \quad \phi \in \boldsymbol{R}^n, \quad \int_{c_j} a_k = \delta_{jk}.$$

c_j and a_j are the basis elements for the first homology and cohomology respectively. (In this decomposition χ may be ϕ dependent.)

The periods (fluxes) of a 1-form (Aharonov-Bohm gauge field) determines $\boldsymbol{A}$ up to an exact 1-form. Fixing a gauge, e.g. by setting $\chi = 0$ above, removes this freedom. Choosing a fixed set of coils, does the same thing. Hence, choosing a basis for the cohomology and fixing a gauge are all related notions.

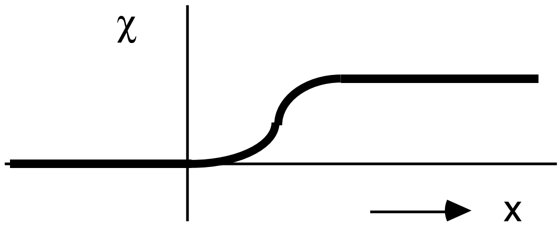

Fig. 2.4: A switching function.

2.4. Harmonic gauge

The Harmonic forms provide a basis of the first cohomology of a closed Riemann surface. We normalize the basis so that the harmonic forms have periods which are either 0 or 1.

Examples:

Let T^2 be the square $[0, 2\pi] \times [0, 2\pi]$ with constant metric. Let (c_1, c_2) and $(\underline{c}_1, \underline{c}_2)$ be two choices of a bases for the first homology groups, shown in Fig. 2.2. The corresponding normalized, harmonic gauge fields are

$$a_1 = \mathrm{d}x,\ a_2 = \mathrm{d}y, \quad \underline{a}_1 = \mathrm{d}x - \mathrm{d}y,\ \underline{a}_2 = \mathrm{d}y.$$

On R^2, harmonicity and normalization are in conflict. $a_1 = \mathrm{d}x, \quad a_2 = \mathrm{d}y$ are harmonic but not normalizable. A convenient choice of Aharonov-Bohm gauge field that we shall use is as follows: take any monotonic function of one coordinate χ so that $\chi(\infty) - \chi(-\infty) = 1$, as in Fig. 2.4. We shall call this a switching function.

$$a_1(x, y) = \chi'(x)\,\mathrm{d}x, \quad a_2(x, y) = \chi'(y)\,\mathrm{d}y,$$

are closed and normalized. If ϕ it time dependent then $\dot\phi^1\, a_1 + \dot\phi^2\, a_2$ describes a gauge field that corresponds to a voltage drop $\dot\phi^1$ between $-\infty$ and $+\infty$ along the x axis.

2.5. Singular Gauge

Let c_j be a base for $H_{n-1}(M)$. Set

$$a_j = \delta_{c_j}(x)\ \mathrm{d}x_t,$$

with $\mathrm{d}x_t$ "transversal" to c_j and $\delta_c(x)$ a surface delta function. For example, in the Euclidean plane:

$$a_1(x) = \delta(x)\ \mathrm{d}x, \quad a_2(x) = \delta(y)\ \mathrm{d}y.$$

2.6. Pure Gauge Fields

A *pure gauge field* associated with *time independent* (smooth) gauge transformations U is given by $\boldsymbol{A} = \mathrm{i}\,(\mathrm{d}U)\,U^\dagger$. Pure gauge fields are closed:

$$\mathrm{d}\boldsymbol{A} = -\mathrm{i}(\mathrm{d}U)\,(\mathrm{d}U^\dagger) = \mathrm{i}\ (\mathrm{d}U)\ U^\dagger\,(\mathrm{d}U)\,U^\dagger = \mathrm{i}\,\boldsymbol{A}\wedge\boldsymbol{A} = 0.$$

For example, on $\boldsymbol{R}^2/\{0\}$, $U(z) = \exp\ \mathrm{i}\ \phi\ \theta$, with $\phi \in \boldsymbol{Z}$ give $\boldsymbol{A} = -\phi\ \mathrm{d}\theta$. We see from this example that the set of *pure gauge fields make a lattice.*

2.7. The Flux Torus

The flux torus, Φ, is the space of Aharonov-Bohm gauge fields with an equivalence relations:

$$\Phi \simeq \{closed\ 1 - forms\}/\{pure\ gauge\ fields\}.$$

For example, a Riemann surface with h holes has $\Phi \simeq T^{2h}$.

Once a gauge and normalization have been fixed any Aharonov-Bohm gauge field (a point in Φ) can be uniquely represented as

$$\boldsymbol{A} = \sum \phi^j\, a_j + pure, \quad 0 \leqslant \phi^j < 1.$$

2.8. A Riemann Metric

With time dependent fluxes, $\dot\phi$ represents a vector of emf's. The question we want to address here is the notion of length for such vectors. One such choice is to make the length square of vectors $\dot\phi$ proportional to the electromagnetic energy which comes from electric field density, $\int \mathrm{d}\,Vol\ E\cdot E = \int \mathrm{d}\,Vol\ \dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{A}}$. (There is no magnetic field energy density for Aharonov-Bohm fields on M.) This motivates choosing for the Riemann metric, g_Φ, on flux space

$$(g_\Phi)_{jk} = \int_M a_j \wedge (*a_k) = \int_M (a_j\cdot a_k)\ \mathrm{d}\ Vol.$$

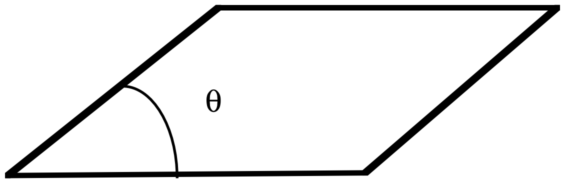

Fig. 2.5: A torus with a skew angle

g_Φ depends on the metric of M, on the basis for the homology groups, and on the choice of gauge for the Aharonov-Bohm fluxes.

For example, consider the unit torus, $\boldsymbol{R}^2/\boldsymbol{Z}^2$, with metric

$$g_M = \begin{pmatrix} a^2 & ab\,\cos\theta \\ ab\,\cos\theta & b^2 \end{pmatrix}, \quad a, b, \theta \in \boldsymbol{R}.$$

This torus is equivalent to the skewed torus with lengths a and b with Euclidean metric shown in the Fig. 2.5.

Choosing the obvious cycles, we have $a_1 = \mathrm{d}x, \quad a_2 = \mathrm{d}y$, harmonic and normalized.

$$g_\Phi = \frac{1}{\sin\theta}\begin{pmatrix} b/a & -\cos\theta \\ -\cos\theta & a/b \end{pmatrix}, \quad \det\ g_\Phi = 1.$$

The tori in coordinate and flux space are related as the unit cell of a crystal is related to its Brillouin zone. The duality between the Bloch momenta, and the fluxes, reappears in various other places in this theory.

3. Magnetic Fields

Magnetic fields, $\boldsymbol{B}$, are closed 2-forms. If M is two dimensional, then any 2-form is closed. Constant magnetic fields in two dimensions are proportional to the area form. For example constant magnetic fields on the Euclidean and Poincaré plane are:

$$\boldsymbol{B} = \begin{cases} B\,\mathrm{d}x \wedge \mathrm{d}y & \text{Euclidean;} \\ (B/y^2)\,\mathrm{d}x \wedge \mathrm{d}y & \text{Poincaré.} \end{cases}$$

On a curved surface there is, in general, no constant vector field. You can't comb the airs of a peach, see Fig. 3.1. It follows that, unlike the situation

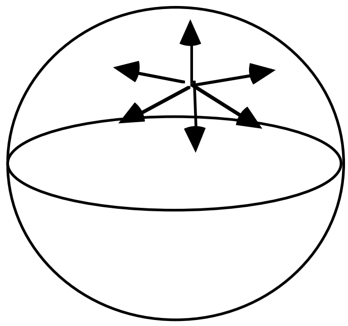

Fig. 3.1: You can't comb the hairs of a peach.

for magnetic fields, there is no natural notion of constant electric field on a curved surface.

3.1. Dirac Quantization

On a closed two dimensional manifold

$$\int_M \boldsymbol{B} = 2\pi \text{ Integer.}$$

If M is embedded in E^3 this says that only magnetic monopoles with quantized magnetic charges may be placed inside M. On punctured surfaces, such as the one in Fig. 1.4, the magnetic flux can be varied continuously at the expense of the fluxes in the punctures.

In Gaussian units the magnetic monopole g and electric monopoles e have the same dimension: $g = e/\alpha$, where $\alpha = e^2/\hbar c$. Dirac says that $2g$ is integral multiple of the quantum flux unit.

The area of a smooth compact h-handled surface with Gaussian curvature -1, is $4\pi\,(h-1), \quad h \geqslant 2$. Dirac quantization conditions says that *constant* magnetic fields are certain fractions: $2B(h-1) \in \boldsymbol{Z}$.

4. Schrödinger Operators on Graphs

4.1. Summary

Here we describe the quantum mechanical kinetic energy operator on spaces that looks like graphs. Graphs represent an idealized version of two dimen-

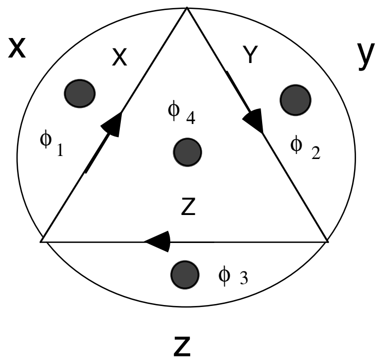

Fig. 4.1: A graph with three vertices and six edges

sional systems of small area whose linear dimensions are large. Since magnetic fields set a length scale the notion of small area corresponds to the weak magnetic field limit for system of finite area.

Schrödinger operators on graphs provide a class of models which can be explicitly and completely analyzed and for which one can get detailed and explicit information about spectral and transport properties [7, 8]. One role they play in the in the logical structure of the theory of adiabatic transport (described in section 11) is to provide examples for which the general theory can be applied, and for which one can verify that the theory leads to non-trivial transport properties.

4.2. The Kinetic Energy

On the edges (links) the kinetic energy is the differential operator:

$$\frac{1}{2}\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} - A(x)\right)^2 .$$

At each vertex, v, we impose boundary conditions that guarantee Kirchoff law. Let ℓ label the edges that are incident on v. (We denote this by $\partial\ell \ni v$.) Then the boundary conditions are:

$$\psi_\ell(v) = \psi(v), \quad \forall\, \ell$$

$$\sum_{\partial\ell\ni v}\Big(\partial - \mathrm{i}A(x)\Big)\psi_\ell(x)\Big|_v = \lambda\psi(v), \quad \lambda \in \boldsymbol{R}.$$

Kirchoff follows from

$$\sum_{\partial\ell\ni v} j_\ell(v) = \sum_{\partial\ell\ni v} \mathrm{Im}\ \bar{\psi}(\partial - \mathrm{i}A)\psi = \sum_{\partial\ell\ni v} \mathrm{Im}\ \lambda|\psi(v)|^2 = 0.$$

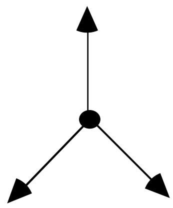

Fig. 4.2: Vertex with incident oriented edges

$\lambda = 0$ corresponds to Neumann, $\lambda = \infty$ to Dirichlet. Finite λ introduces a length scale. Dirichlet disconnects the graph. Neumann is the boundary condition we shall choose in the illustrations below.

4.3. Spectral Analysis

Spectral question on graphs reduce to questions about finite matrices. We focus on compact graphs. Non-compact graphs, like the one in Fig. 1.5 lead to a scattering problem that we shall not study here see e.g. [8, 26, 31].

Fix an energy $E > 0$, $k = \sqrt{2E} > 0$. For a given edge ℓ, let

$$\phi(\ell) = \int_\ell A.$$

The vector $\Psi = (\psi_1, \cdots, \psi_n)$ gives the values of the eigenfunction at the vertices of the graph. Define a matrix $h(\phi, k)$ whose entries are

$$h_{uv}(\phi, k) = \begin{cases} \displaystyle\sum_{\partial\ell=(u,v)} \frac{\exp -\mathrm{i}\phi(\ell)}{\sin\ k\ell}, & u \neq v; \\ \displaystyle -\sum_{\partial\ell \ni u} \cot(k\ell), & u = v. \end{cases}$$

Here u, v denote vertices in the graph.

For example, suppose all edges for Fig. 4.1 have length $\ell = 1$. Let the six complex numbers of modulus one, x, X, y, Y, z and Z correspond to $\exp i\phi(\ell)$ for the six edges. The matrix h is given by

$$h = \frac{1}{\sin\ k} \begin{pmatrix} \alpha & Z + z & \bar{Y} + \bar{y} \\ \bar{Z} + \bar{z} & \alpha & \bar{X} + \bar{x} \\ Y + y & X + x & \alpha \end{pmatrix}, \quad \alpha = -4 \cos\ k.$$

In this example the information on the Aharonov-Bohm gauge fields is in the six complex numbers of modulus 1, x, X, y, Y, z and Z. The fluxes carried by the four flux tubes threading the holes are determined by

$$x\bar{X} = \exp \mathrm{i}\,\phi_1, \quad y\bar{Y} = \exp \mathrm{i}\,\phi_2, \quad z\bar{Z} = \exp \mathrm{i}\,\phi_3, \quad XYZ = \exp \mathrm{i}\,\phi_4.$$

The four fluxes do not determinex, X, y, Y, z, Z uniquely: There is a two dimensional space of gauge transformations given by diagonal unitaries with determinant one.

The spectral properties follow from:

1. The vector of amplitudes at the vertices, Ψ, and k (provided $\sin k\ell \neq 0$), determines the eigenfunction at every point of the graph. For $x \in \ell$ the wave function is determined in terms of its values at the vertices $\{u, v\} = \partial\ell$
$$\psi(x) = \frac{\exp \mathrm{i} \int^x \boldsymbol{A}}{\sin k\ell} \Big(\psi(v)\, \exp\big(-\mathrm{i}\phi(\ell)\big)\, \sin kx + \psi(u)\, \sin k(\ell - x)\Big).$$
2. The eigenvalue equation is
$$\det\ h(k, \phi) = 0.$$
It gives energy bands on the flux torus. The corresponding eigenvectors Ψ lie in the kernel of $h(k, \phi)$.
3. The Hilbert space metric induces a (non-trivial) metric i.e.
$$||\Psi||^2 = \sum \bar{\psi}_u\, g_{uv}(k, \phi)\psi_v.$$

5. Schrödinger Operators on Manifolds

5.1. Summary

This chapter describes the kinetic energy operator for *non relativistic, spinless* quantum particles on (curved) manifolds in the presence of magnetic and Aharonov-Bohm gauge fields. The notion of *periodic boundary conditions* turns out to be subtle when gauge fields are present.

We describe the quantum kinetic energy operator of a single charged particle. The generalization that gives the Hamiltonian describing many interacting particles on curved manifolds is soft and easy (using the additivity of the kinetic energy). The spectral analysis of the two kinds of operators is a different ball game, of course. For the kinetic energy of a single particle (with homogeneous magnetic fields) one has, as we shall see in section 6, explicit spectral information on the low lying eigenvalues. There is no comparable information for interacting electrons.

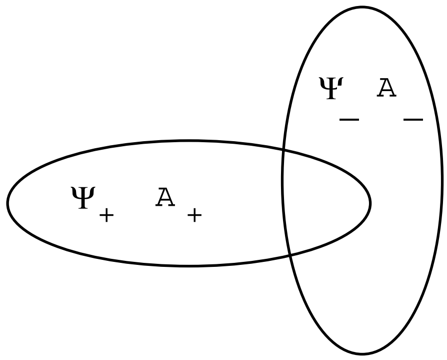

Fig. 5.1: Overlapping patches

5.2. Landau Hamiltonians

On a manifold with gauge fields *define* the velocity operator by the 1-form

$$\boldsymbol{v} = -\mathrm{i}\mathrm{d} - \boldsymbol{A}. \tag{5.1}$$

If we think of the wave function $|\psi\rangle$ as a function (0-form), then $|\boldsymbol{v}\psi\rangle$ is a 1-form. The expectation value of (twice) the kinetic energy associated to the state is:

$$\langle \boldsymbol{v}\psi|\boldsymbol{v}\phi\rangle = \int \mathrm{d}\,Vol\; \overline{\boldsymbol{v}\psi}(x)\cdot\boldsymbol{v}\phi(x) = \int \overline{*\boldsymbol{v}\psi}(x)\wedge\boldsymbol{v}\phi(x).$$

($\bar{\cdot}$ denotes complex conjugation.) The kinetic energy operator now follows by integrating by parts, which we write formally as:

$$H = \frac{1}{2}\,(*\boldsymbol{v}*)\boldsymbol{v}. \tag{5.2}$$

This is the generalization of (− half) the Laplacian $-\Delta = (*\mathrm{d}*)\mathrm{d}$. We call the kinetic energy operator for constant B *Landau Hamiltonian.*

For example, on a two dimensional manifold with conformal metric the Landau Hamiltonian is

$$H = \frac{1}{2\sqrt{g}}\Big((-\mathrm{i}\partial_x - A_x)^2 + (-\mathrm{i}\partial_y - A_y)^2\Big), \quad \boldsymbol{A} = A_x\mathrm{d}x + A_y\mathrm{d}y.$$

5.3. Periodic Boundary Conditions

Wave functions and gauge fields on overlapping patches are related by gauge transformations:

$$A_+ - A_- = \mathrm{d}\chi, \quad \psi_+ = \Big(\exp\ \mathrm{i}\chi\Big)\ \psi_-.$$

Example: Consider the torus in configuration space shown in Fig. 2.5. In the symmetric gauge $\boldsymbol{A} = (B/2)\,(x\mathrm{d}y - y\mathrm{d}x)$, comparing the gauge potentials on the left/right and up/down boundaries of the polygon gives:

$$\begin{aligned} A_L - A_R &= (Ba/2)\ \mathrm{d}y \Longrightarrow \chi_{LR} = (Ba/2)\ y + \phi_1, \\ A_U - A_D &= -(Bb/2)\ \mathrm{d}x \Longrightarrow \chi_{UD} = -(Bb/2)\ x + \phi_2. \end{aligned}$$

It follows that the periodic boundary conditions are:

$$\begin{aligned} \psi(a,y) &= \Big(\exp\ \mathrm{i}\phi_1 \exp\ (\mathrm{i}Bay/2)\Big)\ \psi(0,y); \\ \psi(x,b) &= \Big(\exp\ \mathrm{i}\phi_2 \exp\ (-\mathrm{i}Bbx/2)\Big)\ \psi(x,0). \end{aligned}$$

5.4. Magnetic Translation

The magnetic translations on $\boldsymbol{R}^2$, [???], in the symmetric gauge, are defined by:

$$(\boldsymbol{U}_a\psi)\,(x) = \Big(\exp \mathrm{i}(B\times a\cdot x/2)\Big)\ \psi(x-a), \quad x, a \in \boldsymbol{R}^2.$$

The operators satisfy a Weyl-Heisenberg algebra $\boldsymbol{U}_a\ \boldsymbol{U}_b = \mathrm{e}^{\mathrm{i}\ B\cdot a\times b}\ \boldsymbol{U}_b\ \boldsymbol{U}_a$, and are generated by:

$$\mathrm{i}\ \Big(\boldsymbol{d}_a \boldsymbol{U}_a\Big)\ \boldsymbol{U}_a^\dagger = -\mathrm{i}\mathrm{d} + \boldsymbol{A}.$$

5.5. Automorphic Factors

In the Poincaré upper half plane fix the gauge

$$\boldsymbol{A} = B\frac{\mathrm{d}x}{y}.$$

Let γ be the Möbius transformation that relates the two edges of the polygon such as the octagon in Fig. 1.2. The boundary conditions are determined using Eqs. (1.1)

$$\frac{\mathrm{d}x'}{y'} - \frac{\mathrm{d}x}{y} = \mathrm{Re}\ \left(\frac{\mathrm{d}z'}{y'} - \frac{\mathrm{d}z}{y}\right) = \mathrm{Re}\left(\frac{-2\mathrm{i}c\ dz}{cz+d}\right) = 2\ \mathrm{Im}\ \mathrm{d}\ln\ (cz+d).$$

The gauge transformation that matches wave functions is

$$\chi(\gamma) = -\mathrm{i}B \ \ln \frac{cz+d}{c\bar{z}+d} + \phi(\gamma).$$

Hence, the periodic boundary conditions are [35]

$$\psi(z') = \mathrm{e}^{\mathrm{i}\ \chi(\gamma)}\psi(z) = \mathrm{e}^{\mathrm{i}\ \phi(\gamma)} \left(\frac{cz+d}{c\bar{z}+d}\right)^{B} \ \psi(z).$$

6. Spectral Properties of Landau Hamiltonians

6.1. Summary

There is detailed spectral information about the ground state of Landau Hamiltonians, for a large class of surfaces. This is a consequence of certain index theorems. This chapter is an introduction to this. For more on this see e.g. [19, 35]. Quantum scattering on non-compact Riemann surfaces is also a highly developed subject, but we shall not discuss it at all see e.g. [2, 21, 35, 46].

The theory of adiabatic transport that we aim at is general and is certainly independent of whether the models are explicitly soluble or not. The role played by soluble models, is twofold: First, the theory makes certain assumptions, and it is useful to have a class of models where these assumptions can be verified. This makes the theory non-empty. Second, the models serve to guarantee that the theory is non-trivial: Soluble models show that non-zero non-dissipative transport coefficients arise.

6.2. The Flat Torus

For the flat torus:

$$\begin{aligned} 0 \leqslant 2H = (-\mathrm{i}\partial_x + By/2)^2 + (-\mathrm{i}\partial_y - Bx/2)^2 = D^\dagger D - B; \\ D = 2\partial - B\bar{z}/2,\ D^\dagger = -2\bar{\partial} - Bz/2. \end{aligned}$$

Let $Spec(H)$ denote the spectrum of H. Since B is constant

$$Spec(2H) = Spec(D^\dagger D) - B = Spec(DD^\dagger) + B.$$

Positivity of the kinetic energy gives $Ker\, D = 0,\ if\ B > 0$. ($Ker\, D$ is the vector space which is annihilated by D.) The Index theorem says:

$$dim\, Ker\, D^\dagger - dim\, Ker\, D = \frac{B\, Area}{2\pi}. \tag{6.1}$$

(In the section 6.3 we shall describe where this comes from.) It follows that the ground state energy is $B/2$ and its degeneracy is

$$dim\, Ker\, D^\dagger = \frac{B\, Area}{2\pi}. \tag{6.2}$$

The method gives no information for zero magnetic fields.

6.3. Riemann surfaces

Let $D(B) = 2\partial - \mathrm{i}B/y, \quad D^\dagger(B) = -2\bar{\partial} - \mathrm{i}(B+1)/y$. The Landau Hamiltonian can be written as:

$$2\, H(B) = y^2\ D(B)\ D^\dagger(B-1) + B = y^2\ \ D^\dagger(B-1) D(B) - B.$$

Positivity of the kinetic energy says that $Ker\ D(B) = 0$ for $B > 0$. The index theorem says that (see the next section):

$$\begin{aligned} &dim\, Ker\, D^\dagger(B-1) - dim\, Ker\, D(B-1) = \\ &\quad (2B-1)\ \frac{Area}{4\pi}. \end{aligned} \tag{6.3}$$

Combining the two gives, for $B > 1$, the ground state energy $B/2$ and degeneracy, [5],

$$dim\, Ker\, D^\dagger(B-1) = (2B-1)\ \frac{Area}{4\pi}.$$

It is instructive to restate the result in terms of the number of flux quanta. $B > 1$, which is where the method applies, correspond to total flux that is larger than $2(h-1)$. The ground state degeneracy is $\big(\#\ flux\ quanta\big) - (h-1)$. *The method* distinguishes a region of weak magnetic fields, where the total magnetic flux is less than $2(h-1)$, and where the method does not apply, and a region of strong magnetic fields where the total magnetic flux is larger than $2(h-1)$ and the method fixes the ground state energy and it degeneracy.

The dynamics and spectral properties in weak magnetic fields is complicated, because,the negative curvature dominates the Lorentz force.

6.4. Index and Heat Kernel

Given operators A and B, the spectra of AB and BA almost coincide, that is:

$$Spec(AB)/\{0\} = Spec(BA)/\{0\}.$$

Hence, if $DD^\dagger$ is an operator with discrete spectrum, all its non-zero eigenvalues coincide with those of $D^\dagger D$. It follows that for any $\beta > 0$,

$$dim\, Ker D^\dagger - dim\, Ker D = Tr\left(\mathrm{e}^{-\beta DD^\dagger} - \mathrm{e}^{-\beta D^\dagger D}\right).$$

Since the right hand side is independent of β we compute it in the limit $\beta \to 0$. To leading order in β:

$$Tr\left(\mathrm{e}^{-\beta DD^\dagger} - \mathrm{e}^{-\beta D^\dagger D}\right) \to \beta\, Tr\left([D^\dagger, D]\, \mathrm{e}^{-\beta D^\dagger D}\right).$$

When $\beta \to 0$ the "heat" propagates only little around a small neighborhood where the behavior of space is approximately Euclidean [15, 32]. This suggests

$$\langle x|\mathrm{e}^{-\beta D^\dagger D}|y\rangle \to \frac{1}{4\pi\beta}\mathrm{e}^{-(x-y)^2/2\beta}.$$

Suppose now that $[D^\dagger, D]$ is a multiplication operator (it is in the applications above), then

$$dim\, Ker D^\dagger - dim\, Ker D = \frac{1}{4\pi}\int_{polygon} \mathrm{d}^2x\left([D^\dagger, D]\right)(x).$$

The commutator

$$[D^\dagger(B-1), D(B-1)] = \begin{cases} 2(B-1) & \text{flat torus;} \\ (2B-1)/y^2 & \text{Poincaré,} \end{cases}$$

gives the index formulas Eqs. (6.1-3).

7. Currents

7.1. Summary

This section discusses two notions of current operator operator in quantum mechanics. Every textbook in quantum mechanics tells us what the *current*

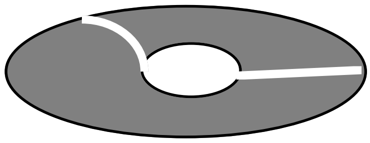

Fig. 7.1: The currents through homologous cuts on a disc

density operator is and the innocent reader may wonder if he does not already know all there is to know about the current operator. A point we stress here is that there is no one current operator; the current is an (operator valued) function on cross sections, and more precisely on the closed $(n-1)$ chains, (n is the dimension). See Fig. 7.1 where two distinct cross sections for the current are shown. The duality between homology and cohomology leads to a more general notion of currents as a function on the closed 1-forms. These may be viewed at currents averaged over a distribution of cross sections. We call this generalized notion loop current. On a surface with h holes, it is convenient to consider the loop current as a 1-form whose h components correspond to the h loop-currents. The loop current operator is the operator valued 1-form $\boldsymbol{d}H(\boldsymbol{A})$, where $\boldsymbol{d}$ is the exterior differential with respect to fluxes, H is the Hamiltonian and $\boldsymbol{A}$ is the 1-form of the Aharonov-Bohm gauge fields. This definition is of course independent of whether H is solvable or not and in particular holds for H that describe interacting, multiparticle, Schrödinger operator.

7.2. Classical

Current densities $j(x)$ are 1-forms. Incompressible current densities satisfy $\mathrm{d} * j = 0$. The currents are (co) periods of the current densities:

$$I(c) = \int_c *j, \quad c \in H_{m-1}(M).$$

Different choices of c correspond to different currents. However, in the case of incompressible flow $I(c) = I(c')$ if c is homologous to c'. Indeed, if $c - c' = \partial C$, then

$$I(c) - I(c') = \int_{c-c'} *j = \int_C \mathrm{d} * j = 0.$$

7.3. Loop Currents

The notion of current can be generalized by considering it as a function on closed (and normalized) 1-forms. Let $a \in H^1(M)$ and define the loop current by:

$$I(a) = \int_M a \cdot j \, \mathrm{d}Vol = \int_M a \wedge *j.$$

We may think of the 1-form a as associated with an Aharonov-Bohm gauge field (with normalized periods). In general, different choices of flux tubes such as in Fig. 2.3, lead to different currents. If a is in a singular δ like 1-form concentrated on the closed chain c, (with normalized period), $I(a) = I(c)$. In the cases that a is a smooth 1-form, this is a more general notion of a current (which may be thought of as averaged over cross sections). The current depends on a. However, for incompressible flow it is a function on the cohomology, i.e. $I(a) = I(a')$ if $a - a' = \mathrm{d}\chi$, the current is independent of the choice of flux tubes. .

7.4. The Current Operator

The velocity operator in quantum mechanics was defined as the 1-form, Eq. (5.1). In a state $|\psi\rangle$ the expectation of the current density is

$$\langle\psi|j(x)|\psi\rangle = -\ \mathrm{Re}\ \bar{\psi}(x)(\boldsymbol{v}\psi)(x).$$

The loop current involves a choice of weight a and is defined by

$$I(a,\psi) = -\frac{1}{2}\,\langle\psi|a\cdot v + v\cdot a|\psi\rangle.$$

Let

$$\boldsymbol{A} = \sum \phi^j\ a_j + \boldsymbol{A_o}, \quad \boldsymbol{d} = \sum \mathrm{d}\phi_j \partial_j.$$

with a_j and $\boldsymbol{A_o}$ independent of ϕ. Then $\boldsymbol{dv} = -\sum a_j\ \mathrm{d}\phi^j$, and

$$I(a,\psi) = \sum I(a_j,\psi)\ \mathrm{d}\phi^j = \\ -\frac{1}{2}\,\langle\psi|\ \boldsymbol{dv}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{dv}|\psi\rangle = \langle\psi|\boldsymbol{d}H|\psi\rangle\,, \tag{7.1}$$

with H the non relativistic, spinless, Schrödinger operator.

If H is time independent and $|\psi\rangle$ a (normalized, nondegenerate) eigenstate of H with energy E the conventional expression for the persistent currents [18]

$$\sum (\partial_j E)\ \mathrm{d}\phi^j, \tag{7.2}$$

follows from Eq. (7.1) and Feynman-Hellman. This is a case where the current is (guaranteed to be) incompressible, and independent of the flux tube.

8. Adiabatic Theorems

8.1. Summary

It is a basic fact that the evolution generated by a time independent Schrödinger operator reduces to spectral analysis. Adiabatic theorems say that the evolution generated by slowly varying time-dependent Schrödinger operators, also reduce to spectral analysis, but of *a family* of Schrödinger operators. A strategy we shall describe here, which goes back to Kato, [36] is to compare the true evolution with a fictitious evolution, with geometric significance. Adiabatic theorems compare the two evolutions.

8.2. Hilbert Space Projections

Let P be a projection operator: $P^2 = P$, so $Spec(P) = \{0, 1\}$. P is orthogonal if $P^\dagger = P$.

$$dim\ P \equiv dim\, RangeP = Tr\ P = dim\ Ker\ (P - 1).$$

Differentiating $P^2 = P$ gives

$$P\ (\mathrm{d}P)\ P = 0. \tag{8.1}$$

For example, a one dimensional projection (non-orthogonal) is: $P = |\psi\rangle\langle\phi|$, with $\langle\phi|\psi\rangle = 1$. Eq. (8.1) reads: $\langle \mathrm{d}\phi|\psi\rangle + \langle\phi|\mathrm{d}\psi\rangle = 0$. Spectral projections commute with H, $PH = HP$ and have an integral representation:

$$P = -\frac{1}{2\pi \mathrm{i}} \oint R(z)\mathrm{d}z.$$

It follows that P is orthogonal whenever H is self-adjoint. Unless otherwise stated we shall assume that P is associated with an energy interval bordered by gaps, as shown in Fig. 8.1.

8.3. The Adiabatic Setting

Consider a family of Schrödinger operators, $H(s)$, with parameter s which is time independent in the past and future. Let τ be the time scale, and

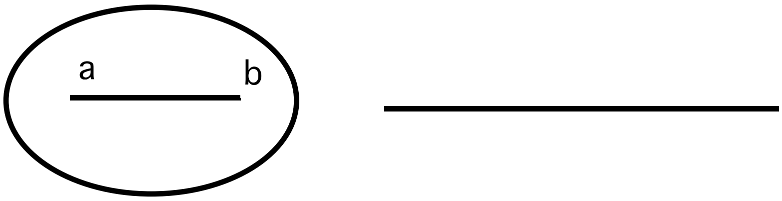

Fig. 8.1: The spectrum and the contour associated to a spectral projection

$s = t/\tau$ the scaled time. Consider the evolution equations $\mathrm{i}\,\partial_t\; U(t) = H(t/\tau)\; U(t)$, or equivalently,

$$\frac{\mathrm{i}}{\tau}\,\partial_s\; U(s) = H(s)\; U(s), \tag{8.2}$$

with initial condition $U(0) = 1$ for the evolution operator in the limit $\tau \to \infty$.

8.4. Adiabatic Evolution

Kato [36] introduced fictitious evolutions, U_A, which are defined so that they respect the spectral splitting of the Hamiltonian, that is, with strictly zero tunneling expressed by $U_A(s)\; P(0) = P(s)\; U_A(s)$. The corresponding generator, $H_A = (\mathrm{i}/\tau)(\mathrm{d}U_A)\; U_A^\dagger$, satisfies $[H_A, P] = (\mathrm{i}/\tau)\,\dot{P}$.

The most general solution of this commutator equation is given by

$$H_A(s) = f(H) + \frac{i}{\tau}[\dot{P}, P],$$

with f an arbitrary function. Kato chose $f = 0$.

$$H_K(s) = \frac{\mathrm{i}}{\tau}[\dot{P}, P], \tag{8.3}$$

turns out to be purely geometric in character (see 9.3). Another natural choice [10] is to take $f(x) = x$, which gives

$$H_{ASY}(s) = H(s) + \frac{\mathrm{i}}{\tau}[\dot{P}, P], \tag{8.4}$$

a generator which is formally close to the physical generator.

8.5. Comparison of Dynamics

The essence of adiabatic theorems is the comparison of the physical and fictitious dynamics for times order τ. One way to do that is to consider how close is

$$S(s) \equiv U_A^\dagger(s)\, U(s)$$

to the identity. Kato showed that

$$S(s) - 1 = O(1/\tau). \tag{8.5}$$

Stronger results are described in [17, 37, 39, 41].

9. The Adiabatic Curvature

This chapter describes the geometry of families of Hilbert space projections. The adiabatic curvature, for a one dimensional projection, is a close relative of Berry's phase [16, 50]. It is a measure of the sensitivity of the *quantum states* to parametric changes in the Hamiltonian. Of course, a quantum system can display sensitivity in more than one way, such as sensitivity of the spectrum (as a set) and the eigen-energies. In the applications to charge transport the parameters are Aharonov Bohm fluxes and the adiabatic curvature is related to non-dissipative transport.

9.1. Parallel Transport and Riemanian Geometry

Let P denote the operator associated to family of finite dimensional and smooth projections. When *Range* P is one (complex) dimensional,

$$\textit{Range}\ P \simeq \boldsymbol{C} \simeq \boldsymbol{R}^2.$$

Parallel transport (or connection) is concerned with the question of orienting planes in the family.

Since Riemanian geometry provides a basic paradigm, and the historical motivation, for the more abstract setting, it is useful to recall how these issues get sorted out in Riemanian geometry. Consider, for example Fig. 9.1, which shown a bundle of tangent planes.

The Levi-Civita connection stipulates that tangent vectors is parallel transported along a geodesic if it keeps its length and angle with respect to the geodesic. This rule can be used to transport a frame from one plane to another along a geodesic. The rule can be expressed in terms of a first order

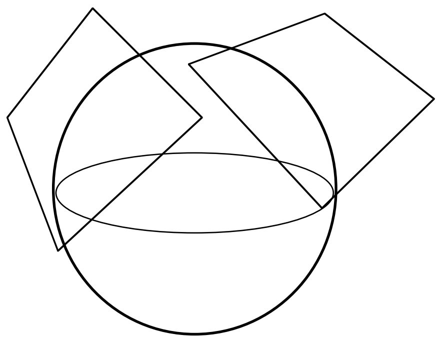

Fig. 9.1: A bundle of tangent planes to the sphere

differential equation involving the *covariant derivative* D, as the following example shows: Consider the Poincaré plane. With $v \in \boldsymbol{C}$ we associate the tangent vector $(v_x, v_y) = (Re\, v, Im\, v)$. Since vertical lines are geodesics, the Levi-Civita connection says that a constant vector field on vertical lines v has parallel vectors, so the equation of parallel transport is $\partial_y v = 0$. Similarly, semi-circles with $r = const$ are geodesics, and Levi-Civita says that the vector field $\mathrm{e}^{\mathrm{i}\theta}$ has parallel vectors. The corresponding differential equation is $\partial_\theta v = \mathrm{i}\, v$. In Cartesian coordinates these two equations read

$$\partial_y v = 0, \quad \left(\partial_x + \frac{\mathrm{i}}{y}\right) v = 0.$$

This fixes the covariant derivative to be

$$D = \left(\partial_x + \frac{\mathrm{i}}{y}, \partial_y\right).$$

The Levi-Civita connection coincides (up to a factor i) with the quantum mechanical velocity operator in the gauge field $A = B/y$ for $B = 1$.

9.2. The Canonical Connection

We shall denote by $\boldsymbol{d}$ the exterior derivative with respect to the parameters and think of these as fluxes.

When $P = 1$ the constant vector field has all its vectors parallel to each other and the equation of parallel transport is $\boldsymbol{d}|\psi\rangle = 0$. The natural geometric generalization of this to non-trivial projections is to require that

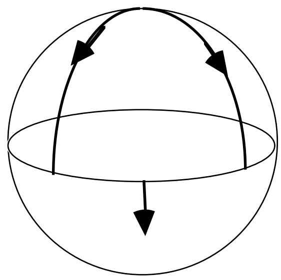

Fig. 9.2: Holonomy of tangent vectors on the sphere

there is no variation of the vector in *Range* P, i.e. $P\,\mathrm{d}\,|\psi\rangle = 0$ along a path γ. This sets the covariant derivative to be $D = P\,\boldsymbol{d}$.
For $|\psi\rangle \in$ *Range* P, $P_\perp \boldsymbol{d}$ is "tensorial", $(P_\perp \equiv 1 - P)$, i.e. for a (scalar valued) function g

$$(P_\perp \boldsymbol{d})\; g\,|\psi\rangle = g\,P_\perp \boldsymbol{d}\;|\psi\rangle = g\,(\mathrm{d}P)\;|\psi\rangle.$$

One has the operator identities on *Range* P

$$D = P\boldsymbol{d} = \boldsymbol{d} - (\boldsymbol{d}P)\;P = \boldsymbol{d} - [\boldsymbol{d}P,\;P].$$

From this sequence of identities one sees that if the parameter is (scaled) time, the equation of parallel transport

$$\mathrm{i}\,D_s\,|\psi\rangle = \Big(\mathrm{i}\,\partial_s - \mathrm{i}[\dot{P}, P]\Big)\;|\psi\rangle = \left(\mathrm{i}\,\partial_t - \frac{\mathrm{i}}{\tau}[\dot{P}, P]\right)\;|\psi\rangle = 0,$$

is Schrödinger equation with the adiabatic generator of Kato Eq. (8.3).

9.3. Curvature

Curvature measures the extent that parallel transport along a closed path fails to bring a vector back to itself. A standard example from Riemanian geometry is associated with the parallel transport of tangent vectors along a spherical triangle, as shown in Fig. 9.2

For example, suppose *RangeP* is one dimensional, and pick an appropriately normalized, vector $|\psi\rangle$ in *Range* P. That is $P = |\psi\rangle\langle\varphi|, \quad \langle\varphi|\psi\rangle = 1.$

Any vector in *Range* P can be written as $|f\rangle = g|\psi\rangle$, g complex valued. The equation of parallel transport is

$$\langle\varphi|D|f\rangle = \boldsymbol{d}g + g\,\langle\varphi|\boldsymbol{d}\psi\rangle = g\big(\boldsymbol{d}\log g + \langle\varphi|\boldsymbol{d}\psi\rangle\big) = 0.$$

For a circle of projections, i.e. for a closed path ℓ in parameter space,

$$\int_\ell \boldsymbol{d}\log\ g = -\int_\ell \langle\varphi|\boldsymbol{d}\psi\rangle = -\int_{\partial^{-1}\ell} \langle \boldsymbol{d}\varphi|\boldsymbol{d}\psi\rangle.$$

$\langle \boldsymbol{d}\varphi|\boldsymbol{d}\psi\rangle$ is the adiabatic curvature and $\partial^{-1}\ell$ is a surface with boundary ℓ.

9.4. Projections of Arbitrary Rank

The example in the previous subsection introduces the adiabatic curvature for projections which are rank one. Here we generalize this to projections of higher rank. This generalization is both natural, and something we shall need for the applications. Some applications concern infinite dimensional projections.

Let ϕ denote the coordinates in parameter space, i.e. the family of projection is represented by a function $P(\phi)$. The curvature, Ω, is a 2-form with components Ω_{jk}:

$$\Omega = \sum \Omega_{jk}\,\mathrm{d}\phi^j \wedge \mathrm{d}\phi^k, \quad \Omega_{jk} = [D_j, D_k].$$

It is remarkable that even though Ω is made from differential operators, it is not a differential operator. In fact (on *Range* P)

$$\begin{aligned}
[D_j, D_k] &= \big(P\partial_j\big)\big(P\,\partial_k\big) - \big(P\partial_k\big)\big(P\,\partial_j\big)\\
&= P\big(\partial_j P\big)\,\partial_k - P\big(\partial_k P\big)\,\partial_j\\
&= P\big(\partial_j P\big)\,P_\perp\partial_k - P\big(\partial_k P\big)\,P_\perp\partial_j\\
&= P\big(\partial_j P\big)\,\big(\partial_k P\big) - P\big(\partial_k P\big)\,\big(\partial_j P\big)\\
&= P\big[(\partial_j P)\,,\,(\partial_k P)\big] = P\big[(\partial_j P)\,,\,(\partial_k P)\big]\,P.
\end{aligned}$$

In form notation we have

$$\Omega = P\ \boldsymbol{d}P \wedge \boldsymbol{d}P\ P.$$

9.5. Miscellaneous Formulas for the Curvature

Suppose P is a spectral projection associated with an eigenvalue e of the Schrödinger operator H, i.e. $PH = eP$. Then

$$\Omega(P) = P\ \boldsymbol{d}H\ \hat{R}^2(e)\,\boldsymbol{d}H\ P, \quad \hat{R}(z) = P_\perp R(z)\,P_\perp. \tag{9.1}$$

This is a relative of formulas of perturbation theory and of Kubo type formulas. The rank one case is a special case of this since by first order perturbation theory

$$\boldsymbol{d}|\psi\rangle = -\hat{R}(E)\ (\boldsymbol{d}H)\ |\psi\rangle.$$

Suppose that the family of projections is represented by the action of a family of unitaries on a fixed projection. That is

$$P = U\ Q\ U^\dagger, \quad \boldsymbol{d}Q = 0.$$

Let $\boldsymbol{A} = (\boldsymbol{d}U)\,U^\dagger$, denote the generator of the family. Using $\boldsymbol{d}P = [\boldsymbol{A}, P]$ and $\boldsymbol{d}P\,P = P_\perp\,\boldsymbol{A}\,P$, we find

$$\Omega(P) = -P\,\boldsymbol{A}P_\perp\,\boldsymbol{A}\,P = -P\,\boldsymbol{A}\wedge\boldsymbol{A}\,P + (P\,\boldsymbol{A}P)\wedge(P\,\boldsymbol{A}\,P)\,.$$

Two special cases that deserve spelling out are:

Finite Rank: If P is finite dimensional and $\boldsymbol{A}$ is bounded, cyclicity of the trace and anticommutativity of 1-forms gives

$$Tr\,\Omega(P) = -Tr\left(P\,\boldsymbol{A}\wedge\boldsymbol{A}\,P\right), \quad dim\,P < \infty,\ \boldsymbol{A}\ bounded. \tag{9.2}$$

If U comes from a Lie group, $\boldsymbol{A}$ is a Lie algebra valued 1-form, and $\boldsymbol{A}\wedge\boldsymbol{A}$ a Lie algebra valued 2-form [48].

Gauge transformation: When U comes from a (smooth) family of (abelian) gauge transformation $\boldsymbol{A}\wedge\boldsymbol{A} = 0$ and:

$$Tr\,\Omega(P) = Tr\left((P\,\boldsymbol{A}P)\wedge(P\,\boldsymbol{A}\,P)\right). \tag{9.3}$$

The curvature vanishes if P is finite dimensional.

9.6. General Properties of the Curvature

The flatness of the full Hilbert space is expressed by:

$$\Omega(P_\perp) = -Tr\,\Omega(P). \tag{9.4}$$

Spectral projections that come from self-adjoint Hamiltonians are automatically orthogonal. For orthogonal projections, $P = P^\dagger$, the adiabatic curvature is pure imaginary:

$$\Omega(P)^\dagger = -\Omega(P); \quad Re\,Tr\,\Omega(P) = 0. \tag{9.5}$$

For this reason one finds that the adiabatic curvature is often defined to be a multiple of $\sqrt{-1}$ of the adiabatic curvature as we have defined it here.

If P is orthogonal and time reversal invariant, i.e. P commutes with a (fixed) anti-unitary, then

$$Tr\,\Omega^{2n+1}(P) = 0, \tag{9.6}$$

for all integral n. If P, Q are commuting orthogonal projections, i.e. $PQ = QP = 0$, then $P + Q$ is an orthogonal projection and

$$Tr\,\Omega(P+Q) = Tr\,\Omega(P) + Tr\,\Omega(Q). \tag{9.7}$$

For $P = U\ Q\ U^\dagger$

$$\mathrm{Tr}\,\Omega(P) = \mathrm{Tr}\,\Omega(Q) + \boldsymbol{d}\ \mathrm{Tr}\,(Q\ U^\dagger \mathrm{d}U). \tag{9.8}$$

10. Constant Curvature

10.1. Summary

Here we give a collection of examples where the adiabatic curvature is a multiple of a natural area form in parameter space. Constant curvature are related to quantized transport.

10.2. Curvature and Quantum Numbers

Consider spin 1/2 in a magnetic field [16]

$$H = B \cdot \sigma, \quad B \in R^3/\{0\}.$$

σ is the triplet of Pauli matrices. The family of eigenvalues, $E_\pm = \pm|B|$, has a conic singularity at the origin, as shown in Fig. 10.1. The family of projections is smooth away from the origin and is given by

$$P_\pm = \frac{1 \pm \hat{B} \cdot \sigma}{2}, \quad \hat{B} = B/|B|.$$

The associated curvature is proportional to the area form of the unit sphere:

$$Tr\,\Omega(P_\pm) = \pm \frac{\mathrm{i}}{2|B|^3} \left(B_1 \mathrm{d}B^2 \wedge \mathrm{d}B^3 + B_2 \mathrm{d}B^3 \wedge \mathrm{d}B^1 + B_3 \mathrm{d}B^1 \wedge \mathrm{d}B^2 \right).$$

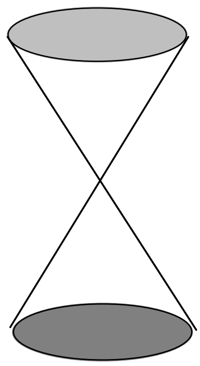

Fig. 10.1: A Conic family of eigenvalues

It is 1/2 the Riemanian (Levi-Civita) curvature of the unit sphere reflecting the fact that 2π rotation of spin 1/2 flips the sign.

10.3. Phase Space Translations

The family of unitaries associated with translations and boosts:

$$U_a = \exp\big(\mathrm{i}pa/\hbar\big), \quad V_b = \exp\big(-\mathrm{i}xb/\hbar\big), \quad [p,x] = -i\hbar, \quad a,b \in \boldsymbol{R}.$$

satisfy the Weyl-Heisenberg algebra $U_a\, V_b = \mathrm{e}^{\mathrm{i}ab/\hbar}\, V_b\, U_a$. $W(a,b) = U_a\, V_b$, the unitary family of phase space translations, is generated by:

$$\mathrm{i}\ (\boldsymbol{d}W)\ W^\dagger = \frac{1}{\hbar}\big(-p\boldsymbol{d}a + (a+x)\boldsymbol{d}b\big).$$

For Q fixed the family of projections $P = WQW^\dagger$ has constant curvature on phase space:

$$\mathrm{Tr}\,\Omega(P) = -\frac{\mathrm{i}}{\hbar}(\mathrm{Tr}\,Q)\ \boldsymbol{d}a \wedge \boldsymbol{d}b.$$

10.4. Landau Hamiltonians on Tori

The previous example can be used to compute the adiabatic curvature associated with Landau levels on tori. Take the torus $[0,2\pi]\times[0,2\pi]$. The magnetic co-translations:

$$U_\phi = \exp\ \mathrm{i}v_1\,\phi/B, \quad V_\phi = \exp\ -\mathrm{i}v_2\,\phi/B,$$

(with v_j the components of the velocity operator Eq. (5.1) in the symmetric gauge), satisfy Weyl-Heisenberg algebra $U_a V_b = \exp(\mathrm{i}ab/B)\, V_b U_a$. A computation like the previous example gives

$$Tr\ \Omega(P) = \frac{\mathrm{i}}{B}\,(Tr\, P)\ \boldsymbol{d}\phi_1 \wedge \boldsymbol{d}\phi_2. \tag{10.1}$$

From Eq. (6.2) we know that the degeneracy of Landau levels for the $[2\pi \times 2\pi]$ torus is: $Tr\ P = B Area/2\pi = 2\pi B$ (which is integral by Dirac) and thus

$$Tr\ \Omega(P) = 2\pi\mathrm{i}\,\boldsymbol{d}\phi_1 \wedge \boldsymbol{d}\phi_2.$$

It follows that *adiabatic curvature is (universal) constant on flux space.* A result of the next chapter is that the Hall conductance is the $\sqrt{-1}$ multiple of the adiabatic curvature. This makes the Hall conductance of a Landau level equal $1/2\pi$. To convert this to the usual units, recall that the unit of conductance is $2\pi\big(e^2/h\big)$ with e the electron charge and h Planck constant.

10.5. Curvature and Gauge Transformations

Consider the family of Schrödinger Hamiltonians on the infinite plane: $H(\boldsymbol{v}, x)$ with

$$\boldsymbol{v}(\phi) = \boldsymbol{v} - \phi_1\, \mathrm{d}\chi(x)\ - \phi_2\, \mathrm{d}\chi(y),$$

and where χ is a switching function. The flux dependence of the family comes from gauge transformations. As a consequence, if we let P project on all states below the Fermi energy E_{F}, then

$$P(\phi) = W(\phi)\, P(0)\, W^\dagger(\phi), \quad W(\phi) = \exp -\mathrm{i}\,\big(\phi_1\, \chi(x) + \phi_2\, \chi(y)\big),$$

with an (abelian) generator $\boldsymbol{A} = \chi(x)\boldsymbol{d}\phi_1\ + \chi(y)\boldsymbol{d}\phi_2$. From Eq. (9.3)

$$Tr\, \Omega = Tr[P(0)\chi(x)P(0),\ P(0)\chi(y)P(0)]\, \boldsymbol{d}\phi_1 \wedge \boldsymbol{d}\phi_2. \tag{10.2}$$

The adiabatic curvature is a constant, independent of the flux. In order to examine it in more detail we shall need a formula for certain integrals of switching functions.

10.6. Area of Triangles

A remarkable integral of switching functions is:

$$\begin{aligned}&\int_{\boldsymbol{R^2}} \mathrm{d}x_1 \wedge\ \mathrm{d}x_2 \Big(\chi(y_1 + x_1)\, \chi(z_2 + x_2) - \chi(y_2 + x_2)\, \chi(z_1 + x_1)\Big)\\ &= y \wedge z\,. \end{aligned}\tag{10.3}$$

Where $y \wedge z = y_1\, z_2 - y_2\, z_1$ is (twice) the areas of triangle with vertices at $(0, y, z)$. The (singular) integral is independent of details of the switching function, and gives a multilinear function of the coordinates on the right hand side whereas the dependence on the coordinates of the integrand is non-linear. We shall meet areas of triangles again in the last section when we discuss conductance as an Index.

10.7. Curvature for Landau Levels

We can say more about the curvature of Landau levels in Eq. (10.2). We note that since the magnetic field is constant, the Hamiltonian, and therefore also the spectral projection on a Landau level, is translation invariant up to gauge transformation. That is, with $a, x, y \in \boldsymbol{R}^2$:

$$\langle x + a|P|y + a\rangle = \mathrm{e}^{-\mathrm{i}\Lambda(x;a)}\,\langle x|P|y\rangle\,\mathrm{e}^{\mathrm{i}\Lambda(y;a)},. \tag{10.4}$$

Eqs. (10.2-3) give:

$$\begin{aligned} &Tr\,\Omega_{12}(P) \\ &= \int_{R^6} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\langle x|P|y\rangle\,\langle y|P|z\rangle\,\langle z|P|x\rangle \times \\ &\qquad \Big(\chi(y_1)\,\chi(z_2) - \chi(y_2)\,\chi(z_1)\Big) \\ &= \int_{R^4} \mathrm{d}y\,\mathrm{d}z\,\langle 0|P|y\rangle\,\langle y|P|z\rangle\,\langle z|P|0\rangle\; y\,\wedge\,z. \end{aligned} \tag{10.5}$$

For the lowest Landau level in the plane one has

$$\langle x|P|y\rangle = (B/2\pi)\;\mathrm{e}^{-B(x-y)^2/4}\,\mathrm{e}^{\mathrm{i}\,B\cdot x\wedge y/2}.$$

The four dimensional integral is essentially Gaussian. It turns out to integrate to $\sqrt{-1}$

11. Conductances

11.1. Summary

The basic formula which relates transport coefficients with the adiabatic curvature is, formally:

$$I(a,\psi) = \boldsymbol{d}E - \,\dot{\phi}\cdot\Big(\sqrt{-1}\,\Omega(P)\Big). \tag{11.1}$$

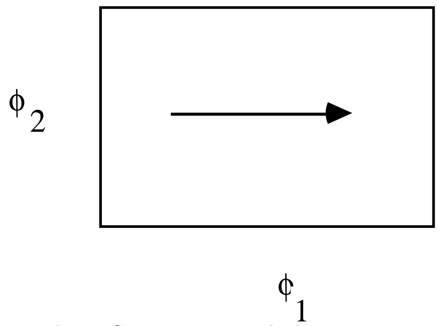

Fig.11.1: A path in flux space and the associated vector $\dot{\phi}$

$I(a,\psi)$ is the expectation value of the (adiabatically evolving), loop current 1-form discussed in chapter 7. a is the Aharonov-Bohm gauge fields which specify a particular loop current operator (see 7.3). ψ is the (instantaneous) eigenstate. $\boldsymbol{d}E = \sum \left(\frac{\partial E}{\partial \phi^j}\right) \mathrm{d}\phi^j$ with E the eigenenergy function gives the contribution of the persistent currents to the loop current, Eq. (7.2). $\dot{\phi}$ is the vector of emf due to adiabatically varying fluxes; $\Omega(P)$ is the adiabatic curvature in the ground state, and $P = |\psi\rangle\langle\psi|$. $\sqrt{-1}\,\Omega$ is a (real) conductance 2-form in flux space. It is proportional to the area form in flux space and *the constant of proportionality is the common conductance.* (The conductance is normally thought of as a function and not as a 2-form.) The notion of conductance as a function clearly depends on the existence of a natural area form in flux space something we introduced in section 2.8.

11.2. Setting

Consider a finite quantum system, with one or many interacting electrons, in the presence of external Aharonov-Bohm gauge fields. We assume that the adiabatic theorem holds (see section 8) and the system is initially in the ground state, which is non-degenerate and associated with a finite vector ψ. When all the gauge fields are held fixed, there are, in general, persistent currents in the ground state. Consider a situation where one, or some, of the fluxes are varied adiabatically in time, e.g. as represented in Fig. 11.1. This generates a small emf around some of the loops in the system and will lead, in general, to additional charge transport, beyond that carried by the persistent current. This is the object we focus on.

11.3. Charge Transport and Adiabatic Curvature

Here is a pedantic formulation of Eq. (11.1).

Theorem: Suppose $\phi_1(s)$ changes monotonically and adiabatically, ϕ_2 fixed, and that at $t = 0$ the system is in a (non degenerate) ground state ψ. The total charge transport around ϕ_2, from time 0 to time $t = s\tau > 0$ with s fixed, is

$$Q_2(s\tau, \phi_2, \psi) = \tau \int_0^s \frac{\partial E}{\partial \phi_2}\, \mathrm{d}s - \mathrm{i} \int_{\phi(0)}^{\phi(s)} \mathrm{d}\phi_1\, \Omega_{12}\big(P(\phi_1, \phi_2)\big) + O(1/\tau).$$

The projection is $P = |\psi\rangle\langle\psi|$.

Remarks

1. The first term, can be interpreted as the contribution of persistent currents. It is the leading term being $O(\tau)$. The contribution of the adiabatic curvature to transport is subleading and $O(1)$. It is geometric in the sense that it depends on the orbit traversed in flux space but not on its history in time.

2. There are situations where charge transport is dominated by the adiabatic curvature. First, in certain classes of models the contribution from the persistent currents vanishes identically. This is the case for the models discussed in chapters 6,10 and 13. More generally, for any model, consider a path of (almost) *fixed emf* in flux space, which winds over many Aharonov-Bohm flux periods $O(\tau^{1/2+\epsilon})$. Since the contribution from the persistent currents have a vanishing average over an Aharonov-Bohm period, the contribution of the persistent currents is $O(\tau^{1/2-\epsilon})$ and the contribution of the adiabatic curvature, being $O(\tau^{1/2+\epsilon})$, dominates. Finally, for any model, and any path in flux space the ϕ_2 *averaged charge* is determined by adiabatic curvature alone.

Proof

Charge transport can always be computed as a boundary value:

$$Q_2 \equiv \int_0^t \mathrm{d}t\ \langle\psi|\, \frac{\partial H}{\partial \phi_2}\, |\psi\rangle = \tau \int_0^s \mathrm{d}s\, \langle\psi|\, \frac{\partial H}{\partial \phi_2}\, |\psi\rangle = \mathrm{i}\, \langle\psi|\frac{\partial \psi}{\partial \phi_2}\rangle \Bigg|_0^s ,$$

where all that was used in the definition of current, the Schrödinger equation $\frac{\mathrm{i}}{\tau}\, \partial_s \psi = H(s, \phi_2)\, \psi$, and integration by parts. We now introduce the evolution operator, $U(s, \phi_2)$, the adiabatic evolution $U_A(s, \phi_2)$ and their comparison $S(s, \phi_2)$ (see section 8):

$$|\psi(s, \phi_2)\rangle = U(s, \phi_2)\, |\psi(0, \phi_2)\rangle = U_A(s, \phi_2)\, S(s, \phi_2)\, |\psi(0, \phi_2)\rangle.$$

In this notation

$$\langle\psi|\frac{\partial\psi}{\partial\phi_2}\rangle\Big|_0^s = Tr\Big\{S(s,\phi_2)\,P(0,\phi_2)\,S^\dagger(s,\phi_2)\times$$
$$\Big(U_A^\dagger(s,\phi_2)\left(\frac{\partial U_A}{\partial\phi_2}\right)(s,\phi_2)+\left(\frac{\partial S}{\partial\phi_2}\right)(s,\phi_2)\,S^\dagger(s,\phi_2)\Big)\Big\}\,.$$

By the adiabatic theorem $S(s,\phi_2)=1+O(1/\tau)$, so

$$\begin{aligned}Q_2 &= \mathrm{i}\,Tr\,\left\{P(0,\phi_2)\,U_A^\dagger(s,\phi_2)\,\frac{\partial U_A}{\partial\phi_2}(s,\phi_2)\right\}+O(1/\tau)\\ &= \mathrm{i}\,\langle\psi_A|\frac{\partial\psi_A}{\partial\phi_2}\rangle\,\Big|_0^s+O(1/\tau).\end{aligned}$$

$|\psi_A\rangle$ evolves according to the adiabatic evolution $\frac{\mathrm{i}}{\tau}\,\partial_s\psi_A = H_{ASY}(s,\psi_2)\,\psi_A$ and

$$|\frac{\partial\psi_A}{\partial s}\rangle = \dot\phi_1\;|\frac{\partial\psi_A}{\partial\phi_1}\rangle.$$

We now unwind the computation above using:

$$\partial_s\Big(\langle\psi_A|\frac{\partial\psi_A}{\partial\phi_2}\rangle\Big)=\frac{\partial}{\partial\phi_2}\langle\psi_A|\frac{\partial\psi_A}{\partial s}\rangle+\Big(\langle\frac{\partial\psi_A}{\partial s}|\frac{\partial\psi_A}{\partial\phi_2}\rangle-\langle\frac{\partial\psi_A}{\partial\phi_2}|\frac{\partial\psi_A}{\partial s}\rangle\Big).$$

The first bracket gives the adiabatic curvature term. The second brackets give the persistent currents since

$$\begin{aligned}\partial_{\phi_2}\langle\psi_A|\frac{\partial\psi_A}{\partial s}\rangle\;&= -\mathrm{i}\tau\,\partial_{\phi_2}\langle\psi_A|H_{ASY}\,\psi_A\rangle =\\ &-\mathrm{i}\tau\,\partial_{\phi_2}\langle\psi_A|H\,\psi_A\rangle=-\mathrm{i}\,\tau\,\partial_{\phi_2}\,E\,.\end{aligned}$$

Remarks

1. van-Kampen criticized the interchange of thermodynamic and weak field limit that goes into the standard linear response theory, such as in Kubo's formulas. The framework here involve no interchange of limits.

2. As we shall see in the next section, the adiabatic curvature has quantized periods. Consequently, constant curvature implies quantized curvature. In particular, a necessary and sufficient conditions for linear response to hold with quantized conductances is

No persistent currents $\wedge$ *Constant curvature* $\Longleftrightarrow$ *Quantized conductances*

Exercise: Show that for the Aharonov-Bohm gauge field $\boldsymbol{A}=\sum\phi^j\,a_j(x)$, the conductance is a periodic function of the fluxes with the Aharonov-Bohm period.

11.4. Covariance

Two identical systems with Aharonov-Bohm gauge fields $\phi^j\, a_j$ in one and $\phi^j \tilde{a}_j$ in the other, have unitarily equivalent Schrödinger operators. Nevertheless, the two systems *do not*, in general have the same conductances. Let

$$\mathrm{d}\chi_j = a_j - \tilde{a}_j, \quad U = \prod \mathrm{e}^{\mathrm{i}\phi^j \chi_j}, \quad Q = U\,P\,U^\dagger.$$

Then,

$$Tr\,\big(\Omega(Q) - \Omega(P)\big) = \boldsymbol{d}\,Tr\,(PU^\dagger \boldsymbol{d}U) = \mathrm{i}\boldsymbol{d}\,Tr\,(P\chi_j)\,\boldsymbol{d}\phi^j.$$

There is no reason for the right hand side to vanish in general. The conductances of the two systems would then be different. In particular, the conductances are not a function on the flux torus as defined in 2.7.

This result is disturbing at first, because it appears to conflict with gauge invariance. It does not. The two systems are gauge equivalent when the fluxes are viewed as parameters. However, when the fluxes actually change in time, even adiabatically, the two systems have different electric fields and are not gauge equivalent.

12. Chern Numbers

12.1. Summary

In this section we show that the periods of the adiabatic curvature are integrals multiples of $2\pi\mathrm{i}$. First we consider the classical situation where the projection lives on a closed manifold (specifically, a sphere). Then we consider a slight generalization, where the projection is periodic *up to gauge transformations.* We then discuss few applications, including a Diophantine equation for the periods. Seminal papers on Chern numbers in condensed matter theory are [24, 45, ???, ???]

12.2. Chern Number of a Complex Line Bundle on S^2:

Suppose P is one dimensional so *Range* $P \simeq \boldsymbol{C}$. On the upper hemisphere, (see Fig. 1.6), we can choose a smooth family of normalized vectors, $|\psi_+\rangle \in$ *Range* P. Similarly, in the lower hemisphere we choose $|\psi_-\rangle \in$ *Range* P.

The period of the adiabatic curvature associated to P is

$$\int_{S^2} \mathrm{Tr}\,\Omega = \int_{S_+} \mathrm{d}\langle\psi_+|\mathrm{d}\psi_+\rangle + \int_{S_-} \mathrm{d}\langle\psi_-|\mathrm{d}\psi_-\rangle$$
$$= \int_{\partial S_+} \Big(\langle\psi_+|\mathrm{d}\psi_+\rangle - \langle\psi_-|\mathrm{d}\psi_-\rangle\Big).$$

But, since $Rank\, P$ is one dimensional

$$\psi_+ = \left(\mathrm{e}^{\mathrm{i}\chi}\right)\ \psi_-,$$

with χ a real valued, smooth function, on the equator. Direct computation shows

$$\langle\psi_+|\mathrm{d}\psi_+\rangle - \langle\psi_-|\mathrm{d}\psi_-\rangle = \mathrm{i}\,\mathrm{d}\chi.$$

So finally,

$$\int_{S^2} \mathrm{Tr}\,\Omega = \mathrm{i}\int_{\partial S_+} \mathrm{d}\chi = 2\pi\mathrm{i}\ \text{Integer}.$$

A similar demonstration works when P is a function on the torus, and, for that matter, any closed surface. In the case $Range\, P \simeq \boldsymbol{C}^n$ similar ideas apply.

We shall denote

$$c(P,S) \equiv \frac{1}{2\pi\mathrm{i}} \int_S Tr\,\Omega(P)$$

the Chern number, (an integer), associated to the projection P and the closed surface S (a sphere in the example). The surface S is in parameter space, (and is distinct from the surface in configuration space). The geometry of the quantum system in coordinate space does not enter explicitly. (It affects P, of course.)

For example, consider the 2×2 hermitian matrix valued function on the 2-sphere [50]:

$$H(\phi) = \sum\ g_{jk}\,\phi^j\,\sigma^k, \quad \det g \neq 0, \quad |\vec{\phi}| = 1.$$

The Chern numbers associated with the projections of the families of lower/upper eigenvalues are

$$c(P_\pm, S^2) = \begin{cases} \pm\, sgn\ \det\ g & S^2\ \text{surrounds origin;} \\ 0 & \text{otherwise.}\end{cases}$$

The important thing to note from this example is that non-zero Chern numbers are associated with surfaces in parameter space that encloses the conic singularity, Fig. 10.1, where eigenvalues cross. For spheres that do not enclose the point of level crossing the Chern number vanishes: *Chern numbers are born where the adiabatic limit dies.*

12.3. Chern numbers for the Flux Torus

In applications to charge transport, P is a spectral projection on flux space. It is not, in general, a periodic function of the fluxes but rather it is periodic up to pure gauge fields (section 2.6). The integrality of the Chern numbers requires a slight generalization of the previous demonstration which works under the following assumptions, [???]:

Suppose

$$P(\phi_1 + 2\pi, \phi_2) = U\, P(\phi_1, \phi_2)\, U^\dagger, \quad P(\phi_1, \phi_2 + 2\pi) = V\, P(\phi_1, \phi_2)\, V^\dagger,$$

with U, V *independent* of ϕ and commuting $U\,V = V\,U$. Then

$$\int_T \mathrm{Tr}\,\Omega = 2\pi\,\mathrm{i}\ \mathrm{Integer}, \quad T = [0, 2\pi] \times [0, 2\pi].$$

In the context of conductance, U, V are gauge transformations, associated with pure gauge fields.

12.4. Diophantine Equations

Symmetry arguments can give information on the Chern numbers of charge transport, through Diophantine equation[13, 23, 38, ???] .

On the torus in coordinate space $T_x = [0, 2\pi] \times [0, 2\pi]$, consider the family of Hamiltonian $H(v - \phi, x)$, where $\boldsymbol{v}$ is the velocity (1-form) operator (see 10.3), and $\phi = \phi^1\,\mathrm{d}x + \phi^2\,\mathrm{d}y$ 1-form of fluxes. Suppose $H(\boldsymbol{v} - \phi, x)$ is a periodic function of x with $m \times n$ unit cells in T_x which may originate from, say, a periodic potential. The Chern numbers for any smooth family of projection, $c(P, T_\phi)$ satisfies

$$(2\pi B)\ c(P, T_\phi) - Tr\, P = 0\ mod\ (mn), \tag{12.1}$$

provided mn and B are relatively prime. T_ϕ is the unit torus in flux space. (By Dirac $2\pi B$ is integral.)

The magnetic co-translations, W, can be used to shift the flux dependence from the kinetic energy term to the potential energy term:

$$H(v - \phi, \{x, y\}) = W(\phi)\, H\big(v, \{x - \phi_1/b, y + \phi_2/b\}\big)\, W^\dagger(\phi).$$

Let $P(\phi)$ be a spectral projection for H and let $Q(\phi) = W^\dagger(\phi)\, P(\phi)\, W(\phi)$. The corresponding curvature are related by Eq. (9.8):

$$Tr\big(\Omega(P)\big) = Tr\big(\Omega(Q)\big) + \frac{\mathrm{i}}{B} Tr(P)\ \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2.$$

Integrating this identity over the square in flux space $[0, 2\pi B] \times [0, 2\pi B]$ gives

$$(2\pi B)^2\, c(P, T_\phi) - (2\pi B)\, Tr\, P = mn\, Integer.$$

If mn and $2\pi B$ are relatively prime, Eq. (12.1) follows.

13. Counting electrons

13.1. Summary

In this chapter we describe a method for comparing the (infinitely many) number of electrons that lie below the Fermi energy of two extended system. When the method applies, this difference is a finite integer (and an Index of a certain operator). For homogeneous systems this integer coincides with the adiabatic curvature. The recognition that conductance is an Index is due to Bellissard [12].

13.2. Landau Levels

The number of electrons that occupy a full Landau level in the plane is infinite, of course. We want to compare the number of electrons in full Landau levels in situations that differ by one extra flux tube carrying one unit of quantum flux. This is described by the family of Hamiltonians parameterized by the flux in the tube:

$$H(\phi) = -\Delta_{rr} + \left(-\frac{\mathrm{i}}{r}\,\partial_\theta - \frac{B\, r^2}{2} + \phi\right)^2.$$

The simplifying feature of this example is the existence of a constant of motion, which makes the problem explicitly soluble, for all values of ϕ, [6]. Increasing ϕ by one unit, induces a flow that takes the spectrum back to itself. But it does so in such a way that if the Fermi energy is just above the lowest Landau level, then an extra electron is added to the system in the process, see Fig. 13.1.

In this example the difference is one, an integer. This need not always be. The corresponding problem for the Laplacian (without magnetic fields) has

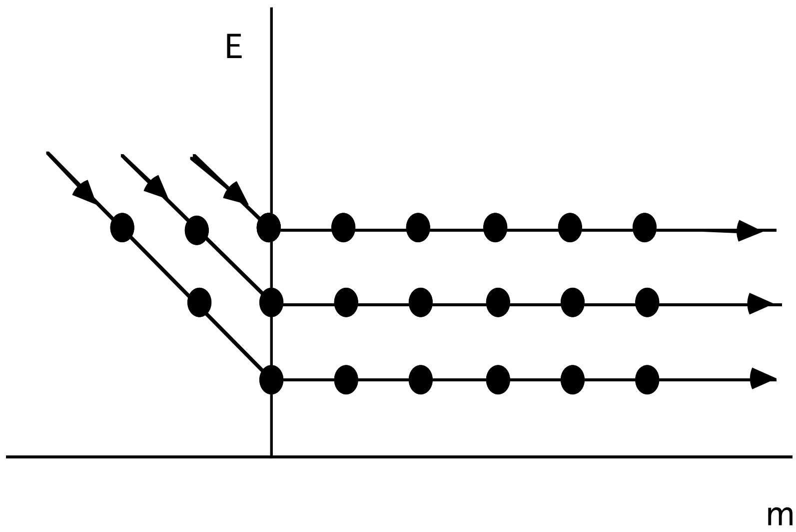

Fig. 13.1: The energy spectrum as function of flux

been analyzed by A. Moroz [40]. (There are related results of Akkermans.) It is harder (because the Fermi energy lies in the spectrum) and requires a clever regularization. He finds for the spectral projection below any fixed positive Fermi energy:

$$dim\, P_E(\phi) - dim\, P_E(0) = -\frac{1}{2}\phi(1-\phi), \quad 0 \leqslant \phi \leqslant 1.$$

The right hand side is never a (non-zero) integer.

13.3. Comparing dimensions

Let $Index(P,Q)$ compare the dimensions of P and Q. In the finite dimensional case

$$\mathrm{Index}\,(P,Q) = \dim P - \dim Q = \mathrm{Tr}\,(P-Q) \in \boldsymbol{Z}.$$

A comparison of dimensions should satisfy:

$$\begin{aligned} Index\,(P,Q) &= -Index\,(Q,P) = -Index\,(P_\perp, Q_\perp) \\ &= Index\,(UPU^\dagger, UQU^\dagger) \\ &= Index\,(P,R) + Index\,(R,Q). \end{aligned}$$

U unitary and $P_\perp = 1 - P$ and R a projection.

The comparison we introduce satisfies these conditions, [9]. However, for the method to apply P and Q must be close in an appropriate sense, more precisely, $P - Q$ needs to be compact.

13.4. Compact operators

Definition: C, a self adjoint, bounded, operator, is compact if it has discrete spectrum and 0 is the only point of accumulation of its eigenvalues.
Examples:

1. The resolvent of the Harmonic oscillator $(-\Delta + x^2)^{-1}$ is a compact operator.
2. If P and Q are spectral projections below a gap in the spectrum of non-relativistic Schrödinger operators H and H' in two dimensions, which differ by one Aharonov-Bohm flux tube, then $P - Q$ is compact.

If C is compact, with eigenvalues λ_n then $\sum |\lambda_n|$ may or may not converge. We say that C^N is *trace class* if for N (possibly large)

$$\sum |\lambda_n|^N < \infty.$$

13.5. Index

Suppose $P - Q$ compact. Define

$$Index\,(P, Q) = dim\,Ker\,(P - Q - 1) - dim\,Ker\,(P - Q + 1).$$

Since $P - Q$ is compact, ± 1 have at most finite multiplicity. Each of the terms on the right hand side is a finite integer and the index is a finite integer too.

It is not obvious from the definition that the Index is a good comparison of dimensions or even that it reduces to what it should be in the finite dimensional case. To see that it does, let us introduce

$$S = P - Q_\perp, \quad C = P - Q$$

Then

$$S^2 + C^2 = 1, \quad SC + CS = 0.$$

Theorem: $Spec\,C/\{\pm 1\}$ is invariant under reflection.

Proof:

$$C\,e_n = \lambda_n\,e_n$$

$$C\,(S\,e_n) = -S\,(C\,e_n) = -\lambda_n(S\,e_n).$$

It remains to check $S\,e_n \neq 0$.

$$S\,e_n = 0, \Longrightarrow C^2\,e_n = e_n, \Longrightarrow C\,e_n = \pm e_n.$$

It follows:

$$\text{Index}\,(P,Q) = \lim_{\epsilon\to 0} \sum_{|\lambda_m|>\epsilon} \lambda_m = \mathit{Trace}(P-Q) = \text{Tr}\,(P-Q)^{2N+1}.$$

13.6. Projection Related by Unitaries

In finite dimensions, if two projections are related by invertible transformations, they have the same dimensions. This is, fortunately, not the case for Hilbert space projections.

Theorem: Suppose P is a spectral projection for a Schrödinger operator, associated with a gap, $Q = UPU^\dagger$, U a unit flux tube. Then $Index(P,Q)$ is an integer given by

$$\begin{aligned} &Index\,(P,Q) = Tr\,(P-Q)^3 = \\ &\int_{R^6} \mathrm{d}x\mathrm{d}y\,\mathrm{d}z\,\langle x|\,P\,|y\rangle\,\langle y|\,P\,|z\rangle\,\langle z|\,P\,|x\rangle \times \\ &\qquad \left(1-\frac{u(x)}{u(y)}\right)\left(1-\frac{u(y)}{u(z)}\right)\left(1-\frac{u(z)}{u(x)}\right). \end{aligned} \tag{13.1}$$

Comments:

1. The gap condition guarantees, for a large class of Schrödinger operator, exponential decay:

$$|\langle z|\,P\,|x\rangle| < C\exp -c|z-x|, \quad c>0.$$

2. The convergence near the diagonal comes from:

$$|u(z)-u(x+z)| = |\exp\,\mathrm{i}\theta_1 - \exp\mathrm{i}\theta_2| = O(\Big|\frac{x}{z}\Big|), \quad |z|\to\infty.$$

Fig. 13.2 gives the geometric content of the estimate. The three factors give $O(\big|\frac{1}{z}\big|^3)$ which is enough for two dimensions.

13.7. Homogeneous Systems

For Schrödinger operator invariant under translation the Index formula Eq. (13.1) coincides with the adiabatic curvature Eq. (10.2). Landau Hamiltonians with constant magnetic fields are examples, but a version of what

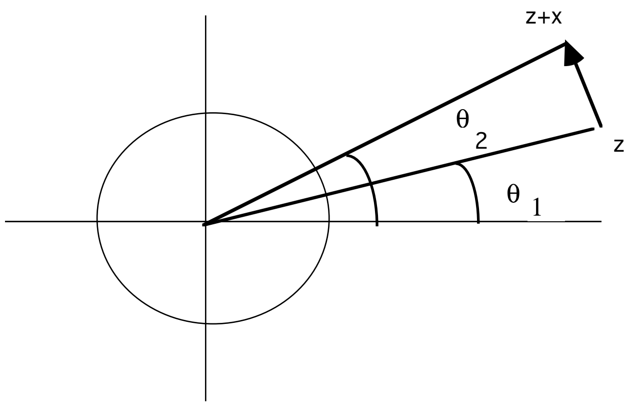

Fig. 13.2: Thin triangles

I shall describe below works for random potentials as well. Translation invariance says that Eq. (10.4) holds. We use this to derive a formula of Connes [22]:

$$Index\,(P,Q) = 2\pi \mathrm{i} \int \mathrm{d}y\, \mathrm{d}z\, \langle 0|P|y\rangle\langle y|P|z\rangle\, \langle z|P|0\rangle\, y \wedge z.$$

This coincides with the expression for the adiabatic curvature Eq. (10.5)
Proof:

$$Index = \int_{R^4} \mathrm{d}y\, \mathrm{d}z\, \langle 0|\, P\, |y\rangle\, \langle y|\, P\, |z\rangle\, \langle z|\, P\, |0\rangle\, f(y,z),$$

where

$$f(y,z) = \int_{R^2} \mathrm{d}x \left(1 - \frac{u(x)}{u(y-x)}\right)\left(1 - \frac{u(y-x)}{u(z-x)}\right)\left(1 - \frac{u(z-x)}{u(x)}\right).$$

It is remarkable that this integral can be explicitly computed to give $f(x,y) = \pi \mathrm{i} y \wedge z$. To see that note first that the integrand is

$$\begin{aligned}&\left(\frac{u(z-x)}{u(y-x)} - \frac{u(y-z)}{u(z-x)}\right) + \left(\frac{u(x)}{u(z-x)} - \frac{u(z-x)}{u(x)}\right) + \\ &\left(\frac{u(y-x)}{u(x)} - \frac{u(x)}{u(y-x)}\right) = 2\mathrm{i}\, \left(\sin\alpha + \sin\beta + \sin\gamma\right)\,.\end{aligned}$$

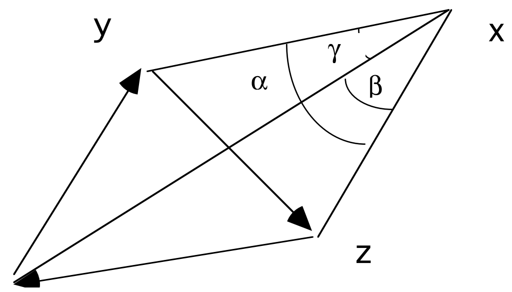

Fig. 13.3: The angles related to the oriented $\{0, y, z\}$, triangle

The angles are chosen so that $-\pi \leqslant \alpha, \beta, \gamma \leqslant \pi$ and inherit sign from the orientation of the line segments and the orientation of the plane. The rest of Connes's computation has been remarkably simplified by Colin loop Verdiére [???]. Since

$$\alpha + \beta + \gamma = \begin{cases} 2\pi, & x \in \text{ Triangle} \\ 0, & \text{outside,} \end{cases}$$

one immediately has

$$\int \mathrm{d}x(\alpha + \beta + \gamma) = 2\pi Area.$$

To complete the computation note that $\sin\alpha - a = O(|x|^{-3})$ and therefore the integral of $\sin\alpha - a$ is absolutely convergent. But since the integrand is antisymmetric, it vanishes

$$\int \mathrm{d}^2x\,(\sin\alpha - a) = 0.$$

This shows that the Index coincides with the adiabatic curvature.

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