5d theories on curved backgrounds, Yang-Mills deformations and instantons

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- 5d gauge theories are naively non-renormalizable
- For SUSY th. one can compute the exact effective action on the Coulomb branch
- In view of it, surprisingly sometimes one can send the bare coupling to infinity and have a sensible theory with no scale (other than the Coulomb branch modulus, of course)

5d fixed point theories do exist

Seiberg

Warning: in some cases 5d gauge theories emanate from 6d fixed points. Instead we will concentrate on purely 5d theories.

- Therefore it is natural to think of 5d theories starting from a fixed point and considering the flows emanating from its deformations

*** what are their properties? where can we define them?**

- In general such fixed point theory can be something very complicated (non-lagrangian)

- Since $1/g_{YM}^2 \sim \text{mass}$ one such mass deformation is in particular turning on a Maxwell kinetic term (leading to a gauge theory)

    - For instance: take the rank 1 fixed point theory with global SU(2) symmetry known as the E1 theory

    - Upon (positive) mass deformation it flows to a conventional SU(2) gauge theory

- Upon deformation to a gauge theory we have a lagrangian description (easy). However not always possible!

*** when can we do it?**

- In the gauge th. there is a topological $j \sim \star F \wedge F$ current whose electrically charged objects are instantons (particles!)

- This current is many times enhanced in the fixed point theory

Turning it the other way around: the fixed point theory has a certain symmetry broken by the mass deformation

*** can one understand properties of these “exotic symmetries” from the gauge theory deformation?**

Contents

- Motivation
- Putting fixed point theories in arbitrary backgrounds
- Mass deformations to gauge theory
- Instanton operators and gauge theories
- Conclusions

Putting fixed point theories on arbitrary backgrounds

Spoiler alert: see Johannes' talk!

- It is interesting to consider fixed point theories on arbitrary backgrounds
  - $\mathbb{R} \times S^4$ and the index
  - S^5 and the partition function
  - relations to other dimensions, AGT-like…
- A natural way to do this is to couple the theory to (off-shell) SUGRA, find the SUSY backgrounds and then freeze out gravity dynamics

Festuccia & Seiberg

- Since we are after 5d SCFT’s it is natural to use 5d superconformal gravity

Fujita & Ohashi
Bergshoeff, Cucu, Derix, de Wit, Halbersma & Van Proeyen
Bergshoeff, Cucu, de Wit, Gheerardyn, Vandoren & Van Proeyen

- The system to consider is then SUGRA+SCFT off-shell. Then the SUGRA sector can be considered alone providing, after freezing, SUSY backgrounds for the SCFT sector

  - this is really independent on the SCFT, lagrangian or not

- The 5d Weyl multiplet contains

e_μ^a	$V_\mu^{(ij)}$	$T_{\mu\nu}$	D	$\psi^i_{\mu\alpha}$	χ^i_α
vielbein	$SU(2)$ R-symmetry	antisymm.		gravitino	dilatino

- The vanishing of the relevant SUSY variations leads to (we already substituded the superconformal spinor)

$$
\begin{aligned}
0 &= \mathcal{D}_\mu \epsilon^i - \frac{1}{4}\gamma_{\mu\nu}\mathcal{D}^\nu \epsilon^i + \imath \gamma_{\mu\kappa\lambda} T^{\kappa\lambda}\epsilon^i - 3\imath T_{\mu\nu}\gamma^\nu \epsilon^i, \\
0 &= \frac{1}{128}\epsilon^i(32D + R) + \frac{1}{15}T_{\mu\nu}T^{\mu\nu}\epsilon^i + \frac{1}{8}\mathcal{D}^\mu \mathcal{D}_\mu \epsilon^i + \frac{3\imath}{40}\gamma_{\kappa\lambda\mu}T^{\kappa\lambda}\mathcal{D}^\mu \epsilon^i + \frac{11\imath}{40}\gamma^\mu T_{\mu\nu}\mathcal{D}^\nu \epsilon^i \\
&\quad + \frac{\imath}{4}\gamma_{\mu\kappa\lambda}\nabla^\mu T^{\kappa\lambda}\epsilon^i + \frac{\imath}{2}\gamma^\mu \nabla^\nu T_{\mu\nu}\epsilon^i - \frac{1}{5}\gamma^{\kappa\lambda\mu\nu}T_{\kappa\lambda}T_{\mu\nu}\epsilon^i.
\end{aligned}
$$

[1] We will be interested on euclidean solutions
[2] We impose a Majorana reality condition on the spinor parameter

- In order to find all SUSY backgrounds we need to find all solutions to these equations

Kuzenko, Novak & Tartaglino-Mazzucchelli
Alday, Genolini, FLuder, Richmond & Sparks
Pini, D.R-G. & Schmude

- Note that with the spinor we can form

$s = \epsilon^i C \epsilon_i$	$v_\mu = \epsilon^i C \gamma_\mu \epsilon_i$	$\Theta^{ij}_{\mu\nu} = \epsilon^i C \gamma_{\mu\nu}\epsilon^j$
scalar	1-form	2-form

- It will be useful to parametrize the covariant derivatives with intrinsic torsions

$$\nabla_\mu \epsilon^i \equiv P_{\mu\nu}\gamma^\nu \epsilon^i + Q^{ij}_\mu \epsilon_j \qquad sP_{\mu\nu} = \epsilon^i \gamma_\nu \nabla_\mu \epsilon_i = \frac{1}{2}\nabla_\mu v_\nu , \qquad sQ^{ij}_\mu = 2\epsilon^{(i}\nabla_\mu \epsilon^{j)}.$$

- One can see that the gravitino eq. is solved by (cf. Johannes' talk!)

$$s^2(P-4\imath T)_{[\mu\nu]} = \frac{1}{3}\left[(v\wedge\Theta^{ij})_{\mu\nu\rho} + 2\Theta^{ij}_{\mu\nu} v_\rho\right](Q-V)^\rho_{ij}. \quad s\Pi_\mu{}^\nu (Q-V)_\nu{}^i{}_j = -\frac{1}{2}[(Q-V)^\nu, \Theta_{\mu\nu}]^i{}_j.$$

- In addition

$$P_{(\mu\nu)} = \frac{1}{5} g_{\mu\nu} P^\lambda{}_\lambda. \qquad 0 = (P-4\imath T)^+.$$

It follows that v must in general be a **conformal Killing vector**. Only if the Tr$P = 0$ v becomes actual Killing.

- This implies the parametrization

$$(Q-V)^{ij}_\mu = s^{-1}\left(v_\mu \Delta^{ij} + W^\lambda \Theta^{ij}_{\lambda\mu}\right) \qquad \text{s.t.} \qquad v(W) = 0, \Delta^{ij} = \Delta^{ji}.$$

- In turn, the dilatino eq. (much harder) is, at the end of the day, solved by (cf. Johannes' talk!)

$$\pounds_v \Delta^i{}_j = -\frac{2}{5} s P^\mu{}_\mu \Delta^i{}_j - [\iota_v Q + P^{[\mu\nu]}\Theta_{\mu\nu}, \Delta]^i{}_j. \quad \pounds_v W_\kappa = \frac{1}{50}\Pi_\kappa{}^\lambda (3s^2 P^\mu{}_\mu W_\lambda - 34 P^\rho{}_\rho P_{[\lambda\mu]} v^\mu - 20 s \nabla_\lambda P^\rho{}_\rho).$$

- At least locally this can be solved by going to a frame where the vector is Killing (and not only conformal Killing) where

$$0 = [\iota_v Q + P^{[\mu\nu]}\Theta_{\mu\nu}, \Delta]^i{}_j,$$

- These equations completely characterize a generic solution to 5d conformal SUGRA: characterize a generic background where to put a 5d SCFT

Example: $\mathbb{R}^5$

- With zero background fields we can have 2 types of spinors

$$\epsilon^i = \epsilon^i_0 \qquad \epsilon^i = x_\mu \gamma^\mu \epsilon^i_0$$

- For the Poincare SUSY the intrinsic torsions are zero, while for the superconformal ones

$$Q^{ij}_\mu = -\frac{2}{sx^2} x_\kappa \Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = \frac{1}{sx^2}(x \wedge v)_{\mu\nu}, \qquad P_{(\mu\nu)} = s^{-1} x_\kappa v^\kappa \delta_{\mu\nu}.$$

The trace of P is not zero and hence v is only conformal Killing!

Example: $\mathbb{R} \times S^4$

- Relevant for index computations

Kim, Kim & Lee

- We can again find 2 sets of spinors satisfying

$$\nabla_\mu \epsilon^q = -\frac{1}{2}\gamma_\mu\, \gamma_5 \epsilon^q\,, \qquad \nabla_\mu \epsilon^s = \frac{1}{2}\gamma_\mu\, \gamma_5 \epsilon^s.$$

- This corresponds to a solution with

$$Q_\mu^{ij} = \pm\frac{1}{2s} w_\kappa \Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = \mp\frac{1}{2s}\,(w \wedge v)_{\mu\nu}, \qquad P_{(\mu\nu)} = \mp\frac{1}{2s}\, w_\kappa v^\kappa\, g_{\mu\nu},$$

Upper (lower) sign to ϵ^q (ϵ^s)

- v is conformal Killing (not along $\mathbb{R}$)
- $w = d\tau$ –τ is the $\mathbb{R}$ direction–

Example: S^5

- Relevant for partition function computations

Hosomichi, Seong & Terashima
Kallen, Qiu & Zabzine

- We can again find 2 sets of spinors

$$\epsilon^i_q = \frac{1}{\sqrt{1+\vec{x}^2}}\epsilon^i_0, \qquad \epsilon^i_s = \frac{1}{\sqrt{1+\vec{x}^2}}\not{x}\eta^i_0,$$

- They fit into our discussion as

$$Q^{ij}_\mu = \frac{1}{2s}x_\kappa \Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = -\frac{1}{2s}\left(x\wedge v\right)_{\mu\nu}, \qquad P_{(\mu\nu)} = -\frac{1}{2s}\,x_\kappa v^\kappa\, g_{\mu\nu}.$$

$$Q^{ij}_\mu = -\frac{1}{2sx^2}x_\kappa \Theta^{ij\kappa}{}_\mu, \qquad P_{[\mu\nu]} = \frac{1}{2sx^2}\left(x\wedge v\right)_{\mu\nu}, \qquad P_{(\mu\nu)} = \frac{1}{2sx^2}\,x_\kappa v^\kappa\, g_{\mu\nu}.$$

Example: topological twist on $\mathbb{R} \times \mathcal{M}_4$

- Relevant for partition function computations on arbitrary manifolds. Perhaps more applications? *cf.* Qiu & Zabzine
- In general we need background SUGRA fields. Note that being the space a direct product, we can have a notion of chirality inherited from 4d
- The "minimal" set-up is to turn on the SU(2) R-symmetry gauge field cancelling the spin connection acting on left (right) spinors
- The spinors are constant and covariantly constant and trivially fit our discussion

Mass deformations to gauge theory

- Sometimes the fixed point theory can be deformed into a gauge theory…it is natural to wonder when can that happen

- To gain some insight, consider e.g. an SU(2) theory on the Coulomb branch. The effective action looks

$$S \sim \int \mathrm{Tr}\Big[-\frac{1}{4}\sigma F \wedge \star F + \cdots \Big] \sim \int \mathrm{Tr}\Big[\sigma \mathcal{L}_{YM}\Big]$$

- We have a cubic theory with the effective YM coupling given by the inverse of the Coulomb branch scalar

- However one can imagine adding a background vector multiplet with the coupling

$$S \sim \int \mathrm{Tr}\Big[-\frac{1}{4}\sigma_B F_D \wedge \star F_D + \cdots \Big]$$

- Thus the VEV of the scalar for this background multiplet turns on a YM coupling for the fixed point theory, mass-deforming it into a gauge theory

$$\langle \sigma_B \rangle \sim \frac{1}{g_{YM}^2}$$

- Of course, this must be done in a way compatible with supersymmetry!

- The relevant (background) vector multiplet SUSY variation is

$$\delta\Omega^i_B = -\frac{\imath}{2}\,\nabla\!\!\!/\,\sigma_B\,\epsilon^i + Y^i_{B\,j}\,\epsilon^j + \sigma_B\,\gamma\cdot T\epsilon^i + \sigma_B\,\eta^i\,,$$

We set to zero all fields in the background vector multiplet other than the scalar

- Substituting the data of the background one finds the condition

$$\pounds_v\sigma_B + \frac{2\,s}{5}\,P^{\mu}{}_{\mu}\sigma_B = 0\,,$$

(constant) YM coupling can be turned on iff the vector is Killing and not only conformal Killing

- Going back to the examples

  - $\mathbb{R}^5$case

    - The superconformal spinors involve a conformal Killing vector: the YM coupling breaks those SUSY's

      → 5d gauge theories are nor conformal because the coupling is dimensionful

  - Topological twist

    - The spinors are constant, the intrinsic torsions are zero and hence the vector is Killing

      → One can turn on (for free) the YM coupling

- S^5 case
  - Naively no way to have a YM coupling (both Poincare and superconformal involve non-traceless P).

However one can consider a combination such that the effective P trace vanishes

$$\xi^1 = \epsilon_q^1 + \epsilon_s^2, \qquad \xi^2 = \epsilon_q^2 - \epsilon_s^1, \quad \nabla_\mu \xi^i = -\frac{\imath}{2}\gamma_\mu (\sigma^2)_i^j \xi_j$$

Hosomichi, Seong & Terashima

- $\mathbb{R} \times S^4$ case
  - No way to combine spinors such that the effective P trace is zero

No YM coupling can be turned on!!!!

c.f. Kim, Kim, Lee & Park

However one can turn on a **position-dependent** coupling

$$\sigma_B = g_{YM}^{-2}\, e^{\tau}$$

The localization etc. would be pretty much the same, so naively the localization computation would go pretty much unchanged.

Instanton operators and gauge theories

- 5d gauge theories admit an automatically conserved topological current

$$j \sim \mathrm{Tr} \star F \wedge F$$

- The electrically charged particles are instanton particles (the wv. is a line)
- The mass of those instanton particles is $m \sim g_{YM}^2$

Become massless at infinite coupling: signals enhanced symmetries in the fixed point theory

- Indeed, instantonic “objects” contribute crucially to the index
  - Localization admits locii where instantons sit at N/S poles of the sphere
  - Each contributes a copy of the (K-th) Nekrasov instanton partition function
  - Crucial to show symmetry enhancement (e.g. E-series)

Kim, Kim & Lee
Bergman, D.R-G. & Zafrir
Bergman & Zafrir
Zafrir

- It is natural to introduce “instanton operators”: an operator inserting one unit of instanton flux on a sphere surrounding it

Lambert, Papageorgakis & Schmidt-Sommerfeld

$$ds^2 = dr^2 + r^2 d\Omega_4^2 \qquad I = \frac{1}{8\pi^2}\int_{S^4} F \wedge F$$

Basically the so-called Yang monopole

- Order zero question: is this SUSY? To that matter, let's consider the vector multiplet SUSY variation

$$\delta\Omega^i = -\frac{1}{4}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^i, \qquad F_{r\mu} = 0, \qquad F = \pm \star_4 F \quad \rightsquigarrow \quad \Gamma_5\epsilon^i = \pm\epsilon^i$$

Asume no background T (actually it wouldn't help…)

- These ops. will be SUSY only if the theory is on a background admitting 4d chiral spinors

- Unfortunately, this is not the case on $\mathbb{R}^5$

Instanton operators are not SUSY

Lambert, Papageorgakis & Schmidt-Sommerfeld
Schmude & D.R-G

- They can be nevertheless supersymmetrized

Consider the topologically twisted theory

Schmude & D.R-G

- Then the background Killing spinors are constant, covariantly constant and “chiral”!

- Now there is a background SU(2) R-symmetry gauge field. For its field strength

$$\int_{S^4} R_i^{\ j} \wedge R_j^{\ i} = 8\pi^2.$$

It is itself a Yang monopole! (for R-symm.)

- The Yang monopole configuration satisfies

$$\mathrm{Tr}(F\wedge F)=96\,\frac{\rho^4}{\Big((1+\rho^2)+(1-\rho^2)\,\cos\alpha_1\Big)^4}\,dr\wedge\omega_4\,;$$

α_1 is the polar angle of the sphere

ρ controls the isotropy of the configuration

- For any non-zero ρ this is a regular configuration and the previous discussion applies. But for zero ρ this becomes a delta function supported at N/S. There we only need to solve the SUSY condition at N/S

- One can see that the spinors are chiral at N/S!

"Collimated" instanton operators **are** SUSY

Bergman & D.R-G.

Moreover they live at N/S, just as expected from the index

- Instanton operators insert flux at a point, but in order to study time-evolution we would like to change to cartesian coordinates

$$ds^2 = dr^2 + d\Omega_4^2 \quad \rightarrow \quad ds^2 = dt^2 + d\vec{x}^2$$

$$r^2 = t^2 + \vec{x}^2 \qquad t = -r\cos\alpha_1$$

- Then

$$\mathrm{Tr}(F \wedge F) = -96\,\frac{\mu_{eff}^4}{(\vec{x}^2 + \mu_{eff}^2)^4}\,dt \wedge d^4\vec{x}, \qquad \mu_{eff} = \rho\,(\sqrt{t^2 + \vec{x}^2} + t)$$

- This is just like a standard BPST instanton only that with a time-dependent size

- “Collimated” instantons are not time-dependent (and SUSY)
- One can now study and quantize their zero modes

Tachikawa
Zafrir
Yonekura

- Let’s concentrate on the SU(2) example: one can see that there are 8 zero modes

How come 8 zero modes?? This naively seems to suggest that all SUSY’s are broken (8 broken SUSY’s+ Goldstone th.= 8 zm.)

- Suppose we were in the fixed point theory: then 16/2=8 as expected. Quantization of these zero modes gives a current multiplet
- The theory remembers the fixed point: mapping into the sphere leads to a position-dependent coupling, which assymptotically leads to the fixed point theory

Conclusions

- 5d gauge theories are to be understood as deformations of a fixed point theory
- By coupling to 5d conformal gravity we can study where such fixed point theories can be supersymmetrically construced
- We constructed the generic backgrounds (see Johannes' talk for a more complete account) . They require that the Killing spinors define a conformal Killing vector.
- We studied when the mass-deformation to a gauge theory is allowed: geometries whose spinors define a Killing (and not only conformally Killing) vector.

- On conformally Killing vectors we can still have a gauge theory, yet with a position-dependent coupling: interesting implications for e.g. the index
- We studied instanton operators: generically non-SUSY
- There are two exceptions
  - Turn on the topological twist

    → Applications/implications yet to be understood!

  - Consider collimated instantons

    → Relevant for zero-mode construction

Many thanks!!!