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Fluctuations in the Diamagnetic Response of Disordered Metals.

E. AKKERMANS (*) and B. SHAPIRO (**)

(*) *Centre National de la Recherche Scientifique, Centre de Recherches sur les Très Basses Températures - B.P. 166X, 38042 Grenoble Cedex, France*

(**) *Department of Physics, Technion Israël Institute of Technology 32000 Haïfa, Israel*

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Abstract. – We study fluctuations in the diamagnetic response of disordered mesoscopic samples. For weak magnetic fields, we derive a convenient expression for the susceptibility χ of any specific sample and show the existence of large anomalous fluctuations in χ, in agreement with other recent calculations. For strong magnetic fields, we show that the fluctuations are considerably suppressed.

Recently there has been much interest in the physics of mesoscopic systems [1]. A mesoscopic sample can be viewed as one huge «molecule» and the electronic wave functions are sensitive to the specific configuration of impurities in the sample. Therefore mesoscopic systems can exhibit anomalous fluctuations from sample to sample. For instance, the relative fluctuation in the electrical conductance is anomalously large and decreases with the sample size L only as L^{-1}, instead of the standard $L^{-3/2}$ decrease, in three dimensions [2].

Similar anomalous fluctuations are expected to exist also in the static magnetic susceptibility of disordered mesoscopic samples. It is the purpose of the present letter to study such fluctuations, which are due to the orbital motion of electrons in the presence of the disordered potential of impurities (we do not consider magnetic effects due to the spin). We consider both weak and strong magnetic fields. The case of weak fields (Landau diamagnetism) has been discussed in recent works of Cheishvili [3], Fukuyama [4] and Serota [5] which had appeared while our work was in progress. For this case, we derive results similar to those of ref. [4, 5] (our calculation is based on a convenient expression for the susceptibility which, to our knowledge, has not been given in the literature). For strong magnetic fields, we demonstrate that fluctuations are strongly suppressed.

We, thus, consider an isolated metallic sample, with the Hamiltonian (in obvious notations)

$$\hat{H} = \sum_i \left\{ \frac{1}{2m} \left[\boldsymbol{p}_i - \frac{e}{c} \boldsymbol{A}(\boldsymbol{r}_i) \right]^2 + V(\boldsymbol{r}_i) \right\}, \tag{1}$$

where $V(\boldsymbol{r})$ is the impurity potential, the sum is over all electrons in the sample, and the electron-electron interactions are neglected. For weak magnetic field standard perturbation theory, to first order in $\boldsymbol{A}$, leads to the following expression for the α-th component ($\alpha = x, y, z$) of the current density

$$j_\alpha(\boldsymbol{r}) = \int \mathrm{d}^3 r' \sum_\beta Q_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}') A_\beta(\boldsymbol{r}') - \frac{e^2}{mc} \rho(\boldsymbol{r}) A_\alpha(\boldsymbol{r}), \tag{2}$$

where $\rho(\boldsymbol{r})$ is the carrier density and the kernel $Q_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}')$ is defined as

$$Q_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') = 2(1/c)\left(\frac{e\hbar}{m}\right)^2 \frac{1}{2\pi i} \int \mathrm{d}\varepsilon f(\varepsilon)\left[\frac{\partial G_\varepsilon^{\mathrm{A}}(\boldsymbol{r}, \boldsymbol{r}')}{\partial r_\alpha} \frac{\partial G_\varepsilon^{\mathrm{A}}(\boldsymbol{r}', \boldsymbol{r})}{\partial r'_\beta} - \text{c.c.}\right]. \tag{3}$$

Here $G_\varepsilon^{\mathrm{A}}$ denotes the advanced Green's function of the unperturbed problem (*i.e.* in the absence of the magnetic field), $f(\varepsilon)$ is unity for occupied states (and zero otherwise), and the (c.c.)-term denotes complex conjugation of the first term (it can be written exactly as the first term, but with retarded Green's functions instead of the advanced ones). The same expression for $j_\alpha(\boldsymbol{r})$ can be derived using, from the start, Green's functions with magnetic field and then expanding in $\boldsymbol{A}$ (see *e.g.* ref. [6]).

One could now easily write down an expression for the susceptibility χ using the relation $\boldsymbol{M} = (1/2c)\int \mathrm{d}^3 r\, \boldsymbol{r} \times \boldsymbol{j}(\boldsymbol{r})$ between the currents and the magnetic moment $\boldsymbol{M}$ of the sample. Such an expression, however, would not be of particular use for our purpose. For instance, in the absence of disorder (and in a homogeneous magnetic field) current flows only at the boundaries of the sample so that calculating χ from currents would require a careful study of boundary effects (see *e.g.* ref. [7] for an illuminating discussion of this point). Below we derive a more convenient expression for χ. Let us first write eq. (2) as

$$j_\alpha(\boldsymbol{r}) = \sum_\beta \int \mathrm{d}^3 r'\, Q_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}')[A_\beta(\boldsymbol{r}') - A_\beta(\boldsymbol{r})]. \tag{4}$$

This result is a consequence of a sum rule:

$$\int \mathrm{d}^3 r'\, Q_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') = (e^2/mc)\rho(\boldsymbol{r})\delta_{\alpha\beta},$$

which follows from gauge invariance and can be also verified directly. The change in energy, due to the magnetic currents, is given by

$$E = -\frac{1}{2c}\int \mathrm{d}^3 r\, \boldsymbol{j}\cdot\boldsymbol{A}(\boldsymbol{r}) = -\frac{1}{2c}\int \mathrm{d}^3 r\, \mathrm{d}^3 r' \sum_{\alpha,\beta} Q_{\alpha,\beta}(\boldsymbol{r}, \boldsymbol{r}') A_\alpha(\boldsymbol{r})[A_\beta(\boldsymbol{r}') - A_\beta(\boldsymbol{r})]. \tag{5}$$

Next, we symmetrize this expression by splitting it into two equal parts, changing in the second part $r \rightleftarrows r'$, $\alpha \rightleftarrows \beta$ and using the symmetry relation $Q_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') = Q_{\beta\alpha}(\boldsymbol{r}', \boldsymbol{r})$. Finally, choosing the Landau gauge $A_y = Bx$ (the magnetic field B is in the z-direction), we obtain

$$\chi \equiv -\frac{1}{\Omega}\frac{\partial^2 E}{\partial B^2} = -\frac{1}{2c}\frac{1}{\Omega}\int \mathrm{d}^3 r\, \mathrm{d}^3 r' (x - x')^2 Q_{yy}(\boldsymbol{r}, \boldsymbol{r}'), \tag{6}$$

where the integration is over the sample volume Ω. Let us emphasize that no averaging over disorder was made so far: eq. (6) refers to any specific sample.

For free electrons (*i.e.* no disorder) a straightforward calculation leads to the Landau diamagnetic susceptibility, $\chi_0 = -\mu_B^2 g_0/3$, where μ_B is the Bohr magneton and g_0 is the density of states at the Fermi level. In the presence of disorder the average Green's function $\langle G_E^A(\boldsymbol{r}, \boldsymbol{r}')\rangle$ differs from the free-electron Green's function by a factor $\exp[-|\boldsymbol{r}-\boldsymbol{r}'|/2l]$, where l is the elastic mean free path. This leads only to small, of order $(k_F l)^{-2}$, corrections to the average susceptibility $\langle\chi\rangle$, consistent with the old result of Dingle [8]. (k_F is the Fermi wave vector, and $k_F l \gg 1$ is assumed.)

It is important to realize that the susceptibility remains essentially unaffected by disorder only *on the average*. In a given sample disorder can have a large effect on susceptibility. The point is that the magnetic currents $\boldsymbol{j}(\boldsymbol{r})$ in the bulk vanish only on the average. For any specific sample $\boldsymbol{j}(\boldsymbol{r})$ in the bulk does not vanish. Moreover, due to diffusion, long-range correlations in the spatial current distribution exist, which leads to large fluctuations in susceptibility.

Calculation of $\langle\delta\chi^2\rangle$ from eq. (6), with $Q_{yy}(\boldsymbol{r}, \boldsymbol{r}')$ given by in eq. (3), requires averaging of a product of four derivatives of Green's functions. This averaging is done by the diagram technique, and the diagrams are similar to those describing the conductance fluctuations [2]. An example of a diagram with two ladders is shown in fig. 1, where solid lines denote

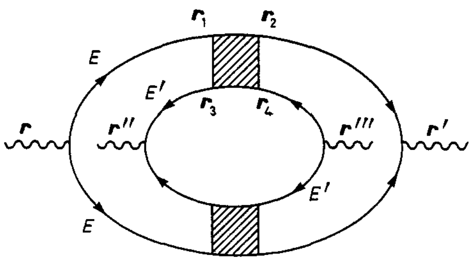

Fig. 1. – A diagram with two ladders. Arrows to the right (left) denote an advanced (retarded) Green's function.

averaged Green's functions and shaded boxes represent diffusion ladders. For distances larger than l, a diffusion ladder $\mathscr{L}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4) = \delta(\boldsymbol{r}_1 - \boldsymbol{r}_3)\,\delta(\boldsymbol{r}_2 - \boldsymbol{r}_4)\,T(\boldsymbol{r}_1, \boldsymbol{r}_2; \Delta E)$, where $\Delta E \equiv E - E'$ and $T(\boldsymbol{r}_1, \boldsymbol{r}_2; \Delta E)$ satisfies the diffusion equation

$$\left(-\nabla^2 - i\frac{\Delta E}{\hbar D}\right) T(\boldsymbol{r}_1, \boldsymbol{r}_2; \Delta E) = \left(\frac{\hbar^2}{2m}\right)^2 \frac{12\pi}{l^3}\,\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,, \tag{7}$$

with $D = (1/3)\, v_F\, l$ being the diffusion coefficient ($v_F = \hbar k_F/m$).

For an infinite medium the solution of (7) is

$$T(\boldsymbol{r}_1; \boldsymbol{r}_2; \Delta E) = \left(\frac{\hbar^2}{2m}\right)^2 \frac{3}{l^3}\,\frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \exp\left[(-1 + i)\sqrt{\frac{3|\Delta E|}{2\hbar v_F\, l}}\,|\boldsymbol{r}_1 - \boldsymbol{r}_2|\right]. \tag{8}$$

The other parts of the diagram (*i.e.* other than the ladders) are short range and decay at a distance of order l. They can be replaced by δ-functions with appropriate weights. As a result, we obtain the following contribution from the diagram in fig. 1:

$$\langle\delta\chi^2\rangle = -2\chi_0^2\left(\frac{8\,k_F^2\, l^3}{\hbar^3\, v_F^3\,\Omega}\right)^2 \operatorname{Re}\int_{-\infty}^{0} \mathrm{d}E \int_{0}^{\infty} \mathrm{d}E' \int \mathrm{d}^3 r\, \mathrm{d}^3 r'\,(x - x')^4\, T^2(\boldsymbol{r}, \boldsymbol{r}'; \Delta E)\,. \tag{9}$$

If one uses the expression (8) for the ladder in eq. (9), one obtains $\langle \delta\chi^2 \rangle \simeq \chi_0^2 (l/L)^2$, which is Fukuyama's result [4] (there are similar contributions from other diagrams). Similar calculations in two dimensions lead to [4] $\langle \delta\chi^2 \rangle \simeq \tilde{\chi}_0^2 (k_F l)^2$, where $\tilde{\chi}_0$ is the Landau susceptibility in two dimensions.

Serota [5] has pointed out that the above estimate for $\langle \delta\chi^2 \rangle$ would be correct if the sample were coupled to the outside world by leads (1). For a truly isolated sample there exists an eigenfunction of the diffusion equation (7) which is constant throughout the sample. This eigenfunction contributes to the ladder a constant $T_0(\Delta E) = \Omega^{-1}(\hbar^2/2m)^2(12\pi/l^3) \cdot (\hbar D/i\,\Delta E)$ which is not included in the expression (8) for the ladder. Taking into account this zero mode, together with the appropriate energy cut-off $\Delta \simeq L^{-d} g_0^{-1}$ in addition to T given by eq. (8), leads to the Serota's [5] result $\langle \delta\chi^2 \rangle \simeq (k_F l)^4 \chi_0^2$, in any dimension d.

Let us now consider the case of a very strong magnetic field (in this part we restrict ourselves to the two-dimensional case only). Let us assume that all electrons reside in the lowest Landau band (the lowest Landau level broadened by the disordered potential $V(\boldsymbol{r})$) and that the filling factor is close to one. Thus, the electron concentration is $n \approx 1/2\pi\lambda^2$, where $\lambda \equiv (\hbar c/eB)^{1/2}$ is the magnetic length. For a sufficiently smooth potential (the typical variation length L_v of the potential is much larger than λ) one can use the quasi-classical approximation for the guiding centre motion. (This approximation has been used, by a number of authors, in the theory of the quantum Hall effect [9].) The energy correction, ΔE, due to disorder, is then given by

$$\Delta E = \frac{eB}{2\pi\hbar c} \int \mathrm{d}^2 r\, V(\boldsymbol{r}) . \tag{10}$$

Equation (10) simply means that each of the (approximately) $\mathrm{d}^2 r/2\pi\lambda^2$ electrons within the area $\mathrm{d}^2 r$ contributes an energy $V(\boldsymbol{r})$. The fluctuating part of the magnetic moment ΔM is therefore

$$\Delta M = -\left(\frac{e}{2\pi\hbar c}\right) \int \mathrm{d}^2 r\, V(\boldsymbol{r}) . \tag{11}$$

The same result can be derived with the help of the local current density $\boldsymbol{j}(\boldsymbol{r})$. Indeed, within the guiding centre approximation, the particle drifts along an equipotential line with a velocity $\boldsymbol{v}(\boldsymbol{r}) \approx (c/e)(1/B^2)(\boldsymbol{B} \wedge \boldsymbol{\nabla} V)$. The current density is thus: $j(\boldsymbol{r}) \approx e(2\pi\lambda^2)^{-1} v(\boldsymbol{r}) = (e/2\pi\hbar)\hat{Z} \wedge \boldsymbol{\nabla} V$, where $\hat{Z}$ is a unit vector in the Z direction (the direction of $\boldsymbol{B}$). The magnetic moment related to these bulk currents is

$$\Delta M = \frac{1}{2c} \int \mathrm{d}^2 r\, \boldsymbol{r} \wedge \boldsymbol{j}(\boldsymbol{r}) = \frac{e}{4\pi\hbar c} \hat{Z} \int \mathrm{d}^2 r\, (\boldsymbol{r} \cdot \boldsymbol{\nabla} V) , \tag{12}$$

since, for the two-dimensional geometry considered here, $\boldsymbol{r}$ is perpendicular to $\hat{Z}$. Let us recall that $V(\boldsymbol{r})$ describes the fluctuating potential in the bulk and does not include the confining potential at the boundaries of the sample (the latter gives rise to the surface currents which produce the main, average, part of the magnetic moment). Taking $V(\boldsymbol{r}) = 0$ at

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(1) There is no need for actual leads. If, for instance, the electrons scatter elastically within the sample but undergo an inelastic collision when they reach the boundary, then Fukuyama's estimate for $\langle \delta\chi^2 \rangle$ should be valid.

the sample boundary, we can rewrite eq. (12) as

$$\Delta M = -\frac{e}{4\pi\hbar c}\int \mathrm{d}^2 r\, V(\boldsymbol{r})\,\mathrm{div}\,\boldsymbol{r} = -\frac{e}{2\pi\hbar c}\int \mathrm{d}^2 r\, V(\boldsymbol{r})\,, \tag{13}$$

which coincides with eq. (11). Writing now the potential correlation function $\langle V(\boldsymbol{r})\cdot V(\boldsymbol{r}')\rangle = V_0^2 f(\boldsymbol{r}-\boldsymbol{r}')$, where V_0 is a characteristic magnitude of $V(\boldsymbol{r})$ and $f(\boldsymbol{r})$ is a function of order 1 decaying on a length L_v, we find

$$\langle \Delta M^2\rangle = \left(\frac{eV_0 L L_v}{2\pi\hbar c}\right)^2 . \tag{14}$$

Let us recall now that, in this regime of extremely strong magnetic fields, the average magnetization corresponds to one Bohr magneton per electron (indeed, the energy per electron is just $\hbar\omega_c/2 = \hbar eB/2mc$). Therefore the average magnetization is

$$\langle M\rangle = \frac{L^2}{2\pi\lambda^2}\,\frac{e\hbar}{2mc} \tag{15}$$

and the relative fluctuation is

$$\delta \equiv \left[\frac{\langle \Delta M^2\rangle}{\langle M\rangle^2}\right]^{1/2} \simeq \frac{L_v}{L}\,\frac{V_0}{\hbar\,\omega_c}. \tag{16}$$

Thus, in contrast to the weak-magnetic-field case (where δ was size independent and large) here the relative fluctuation is small and decreases with the sample size.

In conclusion, we have studied fluctuations due to disorder, in the magnetic response of mesoscopic metallic samples. For weak magnetic fields, we derived a convenient expression for the susceptibility χ and recovered the results of Fukuyama [4] and Serota [5] for $\langle\delta\chi^2\rangle$. For strong fields, we show that fluctuations in the magnetic response are considerably suppressed. It would be interesting to consider the closely related problem of persistent currents in disordered rings [10], when the magnetic field acts not only in the annulus but also on the ring. It would also be of interest to discuss the disorder-induced fluctuations in the de Haas-van Alphen effect and their interplay with the oscillations due to the edge states [11].

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