Macroscopic Approach to Correlations in the Electronic Transmission and Reflection from Disordered Conductors

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Recently, a macroscopic theory of N-channel disordered conductors treated the evolution (with the length L) of the probability distribution of the transfer matrix for the full conductor and allowed a theoretical description of the universal conductance fluctuations. Those results are used here to calculate the correlation function between transmission as well as reflection coefficients: In the case $L \gg W$ (width of the sample), the former essentially coincides with the one obtained from microscopic perturbative calculations. The latter, on the other hand, is a prediction of the present model.

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Much effort has recently been devoted to the understanding of the intensity fluctuations of waves multiply scattered from disordered media. In quantum electronic transport, multiple scattering leads to anomalously large (the so-called "universal") conductance fluctuations[1-9] (UCF): The variance of the dimensionless conductance g (in units of e^2/h) is of order unity and, to a large extent, does not depend on the size of the sample nor on the degree of disorder. Multiple scattering is equally important in light propagation through a random medium, where it is responsible for various effects in intensity statistics (the speckle pattern).[6,10-19] It turns out,[6,10] for instance, that the variance of the transmission coefficient $T_b \equiv \sum_a T_{ab}$ (towards the right, say, when a single mode b is excited on the left) is of the order l/NL, which is again much larger than what a naive statistical consideration would suggest. Here l is the (elastic) mean free path, L the length of the system, and N the total number of channels.

Another interesting fact was pointed out is Refs. 3 and 5: The naive assumption that (in the $N \gg 1$ limit) the various transmission factors T_{ab} are statistically independent violates the notion of UCF. It was then suggested in Ref. 5 that lack of correlation between reflection factors R_{ab} might be consistent with UCF. The correlation coefficient $C^T_{ab,a'b'}$ between T_{ab} and $T_{a'b'}$ was explicitly evaluated later,[6] by use of diagrammatic perturbation theory. As far as we know, there is no explicit evaluation of the correlation coefficient $C^R_{ab,a'b'}$ between reflection factors.

The standard theoretical description of the abovementioned problems, usually based on a perturbative treatment or on numerical simulations, is of a *microscopic* nature. In Refs. 3 and 7–9, on the other hand, the input to the analysis is the transfer matrix for the *full* conductor: The approach was thereby named *macroscopic* in Refs. 8 and 9. The theory of Ref. 8 is based on the properties of flux conservation, time-reversal invariance, and the appropriate combination law when two wires are put together. The distribution associated with systems of very small length is selected on the basis of a *maximum-entropy* criterion; the combination law then shows that the "evolution" of the distribution with the length L is governed by a Fokker-Planck or diffusion equation in N dimensions. In Ref. 9 it was shown that for quasi-1D systems [$L \gg W$ (width of the sample)], UCF are a consequence of the theory of Ref. 8. It thus appears that, at least in this context, the macroscopic approach contains the same physical information as the detailed microscopic calculations.

In the present Letter we shall first be concerned with the calculation of the correlation coefficient $C^T_{ab,a'b'}$ between transmission factors, within the *macroscopic* approach of Refs. 8 and 9. We show that for quasi-1D systems the result essentially coincides with the one obtained from the *microscopic* calculation of Ref. 6. We then extend the analysis to the study of the reflection coefficients. We show that $\langle R_{ab}\rangle$ is enhanced by a factor of 2 in the backward direction, a result interpreted as a weak localization effect. We finally calculate the correlation coefficient $C^R_{ab,a'b'}$ and contrast these new results with the behavior suggested in Ref. 5.

Let the disordered system be placed between two perfect leads: There, the quantized transverse states define N channels for propagating modes, so that the wave function is specified by a $2N$-component vector. The transfer matrix $\mathcal{R}$ relates the vector on the right with that on the left. Under the restrictions of flux conservation and time-reversal invariance, an $\mathcal{R}$ matrix can be represented in the form[8]

$$\mathcal{R} = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} (1+\lambda)^{1/2} & \lambda^{1/2} \\ \lambda^{1/2} & (1+\lambda)^{1/2} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}, \qquad (1)$$

where u,v are arbitrary $N \times N$ unitary matrices and λ is a real, diagonal matrix with N positive elements

$\lambda_1, \ldots, \lambda_N$. One can write the various quantities of interest in terms of the parameters of Eq. (1). For instance, the $N \times N$ transmission and reflection matrices (when incidence is from the left) are given by

$$t = u(1+\lambda)^{-1/2} v, \tag{2}$$

$$\gamma = -v^T[\lambda/(1+\lambda)]^{1/2} v. \tag{3}$$

An ensemble of $\mathcal{R}$ matrices is described by the differential probability[8] $dP_L(\mathcal{R}) = p_L(\mathcal{R}) d\mu(\mathcal{R})$, where L is the length of the system and $d\mu(\mathcal{R})$, the invariant measure associated with the group of $\mathcal{R}$'s, is given by $J(\boldsymbol{\lambda})\prod_a d\lambda_a \, d\mu(u)\, d\mu(v)$, where $J(\boldsymbol{\lambda}) = \prod_{a<b}|\lambda_a - \lambda_b|$, and $d\mu(u)$ [and $d\mu(v)$] is the invariant measure of the unitary group U(N). In Ref. 8 the probability density $p_L(\mathcal{R})$ was considered to be *isotropic*, i.e., independent of the unitary matrices u,v of Eq. (1) and dependent only upon $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$: $p_L(\mathcal{R}) = p_L(\boldsymbol{\lambda})$. Without making use of any specific statistical distribution $p_L(\boldsymbol{\lambda})$, we shall see that *just from the isotropic assumption* we can obtain the structure of the averages and correlations of transmission and reflection factors, as a function of channel indices. Specific values for the various coefficients will then be taken from the results of Ref. 9, based on the diffusion equation that $p_L(\boldsymbol{\lambda})$ must obey.

The transmission factor T_{ab} is defined as $|t_{ab}|^2$, t_{ab} being the ab matrix element of Eq. (2). Its average is given by

$$\langle T_{ab}\rangle = \sum_{\alpha\alpha'} M^{a\alpha}_{a\alpha'} M^{\alpha b}_{\alpha' b} \langle (\tau_\alpha \tau_\beta)^{1/2}\rangle_L, \tag{4}$$

where $\tau_a = (1+\lambda_a)^{-1}$. The last factor in Eq. (4) is an average performed with $p_L(\boldsymbol{\lambda})$, for which we need not be more specific for the time being. The factors M occurring in Eq. (4) are a particular case of the general average[20,21]

$$M^{a_1\alpha_1, \ldots, a_m\alpha_m}_{a_1'\alpha_1', \ldots, a_m'\alpha_m'} = \langle (u_{a_1'\alpha_1'} \cdots u_{a_m'\alpha_m'})(u_{a_1\alpha_1} \cdots u_{a_m\alpha_m})^* \rangle_0, \tag{5}$$

performed with the invariant measure of the unitary group (indicated by the index 0). In Ref. 20 it is shown that

$$M^{a\alpha}_{a'\alpha'} = N^{-1}\delta_{a'a}\delta_{\alpha'\alpha}, \tag{6}$$

so that Eq. (4) becomes

$$\langle T_{ab}\rangle = N^{-2}\langle T\rangle, \tag{7}$$

where $T = \sum_{ab} T_{ab}$ is the total transmission factor into all channels, when the incident channels are fed with N incoherent unit fluxes.

Next we calculate, from Eq. (2), the crossed second moment

$$\langle T_{ab} T_{a'b'}\rangle = \sum_{\alpha\beta\alpha'\beta'} M^{a\alpha, a'\beta}_{a\alpha', a'\beta'} M^{\alpha b, \beta b'}_{\alpha' b, \beta' b'} \langle (\tau_\alpha \tau_\beta \tau_{\alpha'} \tau_{\beta'})^{1/2}\rangle_L. \tag{8}$$

In Refs. 20 and 21 the M coefficients of Eq. (8) are shown to be

$$M^{a\alpha, b\beta}_{a'\alpha', b'\beta'} = (N^2-1)^{-1}(\delta^a_{a'}\delta^b_{b'}\delta^\alpha_{\alpha'}\delta^\beta_{\beta'} + \delta^b_{a'}\delta^a_{b'}\delta^\beta_{\alpha'}\delta^\alpha_{\beta'}) - N^{-1}(N^2-1)^{-1}(\delta^a_{a'}\delta^b_{b'}\delta^\beta_{\alpha'}\delta^\alpha_{\beta'} + \delta^b_{a'}\delta^a_{b'}\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}). \tag{9}$$

The correlation coefficient $C^T_{ab,a'b'} = \langle T_{ab} T_{a'b'}\rangle - \langle T_{ab}\rangle\langle T_{a'b'}\rangle$ can now be calculated, with the result

$$C^T_{ab,a'b'} = (N^2-1)^{-2}\{[(1+N^{-2})\langle T^2\rangle_L - 2N^{-1}\langle T_2\rangle_L]\delta_{aa'}\delta_{bb'} + [(1+N^{-2})\langle T_2\rangle_L - 2N^{-1}\langle T^2\rangle_L](\delta_{aa'}+\delta_{bb'}) + [\mathrm{var}T + N^{-2}\langle T^2\rangle_L + 2N^{-2}\langle T\rangle^2_L - N^{-4}\langle T\rangle^2_L - 2N^{-1}\langle T_2\rangle_L]\}. \tag{10}$$

Here we have defined $T_2 = \sum_a (1+\lambda_a)^{-2}$. Equation (10) is exact. As a check, we can easily verify that the sum of (10) over a,b,a',b' gives precisely var$T$.

In Ref. 6, Eqs. (3), three types of terms are also obtained: If we set $W \ll L$ (quasi-1D systems), they are seen to have essentially the structure provided by the δ functions of our Eq. (10). The difference is that our Kronecker δ's (that we can write as $\delta_{aa'} = \delta_{\Delta q_a = 0}$, with $\Delta q_a = |q_a - q_{a'}|$ [q_a being the transverse wave vector labeling the channel (the eigenmode) a] are replaced by some "smeared" (on a distance $\Delta q \sim 1/L$) δ functions in Ref. 6.

The above conclusions made use of the isotropy assumption only. To be more specific about the coefficients of the δ functions in Eq. (10), we now use the results of Ref. 9, where the diffusion equation satisfied by $p_L(\boldsymbol{\lambda})$ was employed. To leading order in $N \gg 1$, and for $s \equiv L/l \gg 1$, it was shown in Ref. 9 that $\langle T\rangle_L \approx N/s$, $\langle T^2\rangle_L \approx (N/s)^2$, var$T \approx \frac{2}{15}$, and $\langle T_2\rangle_L \approx \frac{2}{3}N/s$. Within these same approximations, Eq. (10) becomes

$$C^T_{ab,a'b'} = \langle T_{ab}\rangle_L \langle T_{a'b'}\rangle_L [\delta_{aa'}\delta_{bb'} + \tfrac{2}{3}\langle T\rangle_L^{-1}(\delta_{aa'}+\delta_{bb'}) + \tfrac{2}{15}\langle T\rangle_L^{-2}], \tag{11}$$

in agreement with Eqs. (3) of Ref. 6 when $W \ll L$.

Thus, while our approach may miss some subtle correlations between nearby the same channels, it does correctly describe the "global" fluctuations in transmission and conductance. Assume, for instance, that a single mode, b, is excited on the left and we are interested in fluctuations of T_{ab} and $T_b = \sum_a T_{ab}$, which represent, respectively, the transmission coefficient to a single channel a and to all channels on the right. From (11) we find var$T_{ab} = \langle T_{ab}\rangle^2$, just as in Eq. (5)

of Ref. 5, and $\mathrm{var}T_b = \frac{2}{3} N^{-1} s^{-1}$, in agreement with Refs. 6 and 17.

We now turn to the study of the reflection coefficient $R_{ab} = |r_{ab}|^2$, r_{ab} being the ab matrix element of Eq. (3). The average of R_{ab} is given by

$$\langle R_{ab} \rangle = \sum_{\alpha\beta} M^{\beta a, \beta b}_{\alpha a, \alpha b} \langle (\rho_\alpha \rho_\beta)^{1/2} \rangle_L, \tag{12}$$

where $\rho_\alpha = \lambda_\alpha (1+\lambda_\alpha)^{-1}$. Making use of Eq. (9) we have

$$\langle R_{ab} \rangle = (1+\delta_{ab}) N^{-1} (N+1)^{-1} \langle R \rangle_L, \tag{13}$$

R being the total reflection coefficient $\sum_{ab} R_{ab}$. Result (13) means that backward scattering to the same channel is enhanced by a factor of 2 as compared with the scattering to any other channel. Except for a smeared-out cone, just as we mentioned after Eq. (10), this is precisely the *enhanced backscattering* predicted by weak-localization theory.[22] From $\langle R \rangle = N - \langle T \rangle$ and[9] $\langle T \rangle \simeq N s^{-1}$, Eq. (13) becomes, for $N \gg 1$, $(1+\delta_{ab}) N^{-1} (1 - s^{-1})$, just as in Eq. (12) of Ref. 5, with the backward enhancement included.

We can similarly calculate the crossed second moment

$$\langle R_{ab} R_{a'b'} \rangle = \sum_{\alpha\beta\alpha'\beta'} M^{\beta a, \beta b, \beta' a', \beta' b'}_{\alpha a, \alpha b, \alpha' a', \alpha' b'} \langle (\rho_\alpha \rho_\beta \rho_{\alpha'} \rho_{\beta'})^{1/2} \rangle_L. \tag{14}$$

The M factor needed in Eq. (14) was calculated in Ref. 21. We merely present here the result for the correlation coefficient $C^R_{ab,a'b'} = \langle R_{ab} R_{a'b'} \rangle - \langle R_{ab} \rangle \langle R_{a'b'} \rangle$, giving, for the coefficient of each δ function, the term to leading order in N and in the limit $s \gg 1$:

$$C^R_{ab,a'b'} = \langle R_{ab} \rangle \langle R_{a'b'} \rangle [(1+\delta_{ab})^{-1} (\delta_{aa'} \delta_{bb'} + \delta_{ab'} \delta_{a'b}) + \langle R \rangle_L^{-1} (\delta_{ab} \delta_{a'b'} \delta_{aa'} - \delta_{aa'} - \delta_{bb'} - \delta_{ab'} - \delta_{ba'}) + \tfrac{32}{15} \langle R \rangle_L^{-2}]. \tag{15}$$

We can easily verify that the sum of (15) over a,b,a',b' gives $\frac{2}{15} = \mathrm{var}R = \mathrm{var}T$. From Eq. (15) we also find the fluctuation of R_{ab} as $\mathrm{var}R_{ab} = \langle R_{ab} \rangle^2$, just as in Eq. (10) of Ref. 5.

We see from Eq. (15) that we are not entitled to neglect correlations between reflection coefficients (as was suggested in Ref. 5), any more than we are in the case of transmission coefficients. In particular, both have a "long-range" term (the one with no Kronecker δ's) of the same order of magnitude: $\frac{32}{15} N^{-4}$ and $\frac{2}{15} N^{-4}$, respectively. We emphasize the consistency between the results of Eqs. (11) and (15): They are a consequence of the same model and, indeed, give rise to the same value for the UCF.

It would be very nice if one could measure, probably in optical experiments, the correlations studied above, which, as we saw, have expressions that are far from obvious.

Finally, we make a few comments on the validity of the model that we are employing.

The isotropy assumption seems reasonable and is mathematically very convenient, but it cannot hold generally. Indeed, it is not surprising that the model gives excellent results for the quasi-1D case ($L \gg W$). If we think of the channels as localized in real space, it is clear that the condition $L \gg W$ is necessary for perfect mixing of channels: Indeed, when $L \ll W$, there can be no coupling between a channel close to the upper corner on the left side and a channel close to the lower corner on the opposite side, so that the isotropy assumption fails. We also remark that in obtaining moments of T in Ref. 9, one has to assume $N/s = Nl/L \gg 1$ (otherwise the expansion made there is not valid), so that L must be much smaller than the 1D localization length $\xi_1 \sim Nl$ in the wire (this is a necessary condition for the metallic regime). We thus have a lower and an upper bound for the validity of the model: $W \ll L \ll (k_F W)^{d-1} l$.

In 2D there is the additional requirement that W should be smaller than the 2D localization length $\xi_2 \sim l \exp(k_F l)$. This is again natural, since pieces of size larger than ξ_2 are essentially decoupled from each other, so that in 2D the isotropy assumption fails at a scale larger than ξ_2.

How to relax the condition of isotropy in order to extend the range of validity of our approach is still not clear at this moment.

Another question, which was already mentioned in the text, is how to improve on our Kronecker δ's in the channel indices and get a "smeared-out cone," as in microscopic calculations. This question is perhaps related once again to the isotropy assumption: Indeed, we saw that this assumption alone gives rise to the δ functions.

The study of these questions will have to be left for the future.

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