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Vortices in small superconducting disks

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Abstract

We study the Ginzburg–Landau equations in order to describe a two-dimensional superconductor in a bounded domain. Using the properties of a particular integrability point ($\kappa = 1/\sqrt{2}$) of these nonlinear equations which allows vortex solutions, we obtain a closed expression for the energy of the superconductor. The presence of the boundary provides a selection mechanism for the number of vortices. A perturbation analysis around $\kappa = 1/\sqrt{2}$ enables us to include the effects of the vortex interactions and to describe quantitatively the magnetization curves recently measured on small superconducting disks [A.K. Geim, I.V. Grigorieva, S.V. Dubonos, J.G.S. Lok, J.C. Maan, A.E. Filippov, F.M. Peeters, Nature (London) 390 (1997) 259]. We also calculate the optimal vortex configuration and obtain an expression for the confining potential away from the London limit. © 2000 Elsevier Science B.V. All rights reserved.

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The dimensionless Ginzburg–Landau energy functional $\mathscr{F}$ of a superconductor depends on only one parameter [2], the ratio κ between the London penetration depth λ and the coherence length ξ:

$$\mathscr{F} = \int_{\Omega} \frac{1}{2}|B|^2 + \kappa^2|1-|\psi|^2|^2 + |(\vec{\nabla} - i\vec{A})\psi|^2. \quad (1)$$

The order parameter ψ is dimensionless as well as the magnetic field B measured in units of $\phi_0/4\pi\lambda^2$, with $\phi_0 = hc/2e$. Lengths are measured in units of $\lambda\sqrt{2}$. The expression (1) assumes that both the order parameter and the vector potential have a slow spatial variation. The integral is over the volume $\Omega = \pi R^2 d$ of a thin disk of radius R and thickness d.

Outside the superconducting sample, the order parameter vanishes and the magnetic field is the solution of the Maxwell equation. The boundary condition between a superconductor and an insulator is $(\vec{\nabla} - i\vec{A})\psi|_{\vec{n}} = 0$ where $\vec{n}$ is the unit vector normal to the surface of the disk. The presence of a boundary precludes a complete analytical solution of the 3D Ginzburg–Landau equations for a thin disk. We are then led to make some simplifying assumptions, based upon numerical results [4].

The thickness d of the sample considered in the experiments [1] fulfills $d \ll \xi$ and $d \leq \lambda$. If the curvature of the magnetic flux lines, given by R/λ_e^2 (where $\lambda_e(d,R,\lambda)$ stands for the effective screening length), is smaller than $1/\lambda_e$, i.e., if $R \ll \lambda_e$, then both ψ and the vector potential $\vec{A}$ can be considered to be constant across the thickness and the disk is effectively two-dimensional. The expression of the effective screening length $\lambda_e(d,R,\lambda)$ is not known,

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except for the case $R \to \infty$ where [5,6] $\lambda_e \simeq \lambda^2/d$. In the London limit (i.e., $\kappa \to \infty$), such a system has been described using Pearl's solution [5]. Finally, since ψ and $\vec{A}$ are constant over the thickness, the covariant Neumann boundary condition, stated above, is automatically satisfied on the upper and lower surface of the disk.

The Ginzburg–Landau equations are nonlinear, second-order differential equations whose solutions are usually unknown. However, for the special value $\kappa = 1/\sqrt{2}$ known as the dual point [7,8], the equations for ψ and $\vec{A}$ reduce to first order differential equations and the minimal free energy can be calculated exactly for an infinite plane. This relies on the identity true for two dimensional systems $|(\vec{\nabla} - i\vec{A})\psi|^2 = |\mathscr{D}\psi|^2 + \vec{\nabla} \times \vec{j} + B|\psi|^2$ where $\vec{j}$ is the current density and the operator $\mathscr{D}$ is defined as $\mathscr{D} = \partial_x + i\partial_y - i(A_x + iA_y)$. At the dual point, the expression (1) for $\mathscr{F}$ is rewritten using this identity as follows:

$$\mathscr{F} = \int_{\Omega} \frac{1}{2} |B - 1 + |\psi|^2|^2 + |\mathscr{D}\psi|^2 + \oint_{\partial\Omega} \left(\vec{j} + \vec{A}\right) \cdot \vec{dl} \quad (2)$$

where the last integral over the boundary $\partial\Omega$ of the system results from Stokes theorem.

For an infinite plane, we impose that the system is superconducting at large distance, i.e., $|\psi| \to 1$ and $\vec{j} \to 0$ at infinity so that the boundary term in Eq. (2) coincides with the London fluxoid. It is quantized and equal to $\oint_{\partial\Omega} \vec{\nabla}\chi \cdot \vec{dl} = 2\pi n$, where χ is the phase of the order parameter. The integer n is the winding number of the order parameter ψ and as such is a topological characteristic of the system (for a discussion of topological aspects see Ref. [9]). The extremal values of $\mathscr{F}$ are $\mathscr{F} = 2\pi n$, and are obtained when the bulk integral in Eq. (2) vanishes identically giving rise to two first-order differential equations. These two equations can be decoupled to give for $|\psi|$ a second-order nonlinear equation which admits families of vortex solutions [10]. However, for the infinite plane, there is no mechanism to select the value of n, which only plays the role of a classifying parameter.

The extension of these results to finite size systems, namely the existence and stability of vortex solutions and their behaviour as a function of the applied field received a partial numerical answer. Numerical simulations of the Ginzburg–Landau equations [11] show the existence of stationary vortex solutions whose number depends on the applied magnetic field. Moreover, these simulations indicate that the physical picture derived for $\kappa = 1/\sqrt{2}$ remains qualitatively valid for quite a large range of values of κ, with a small corresponding change of free energy [12].

We consider finite size systems at the dual point, i.e., for $\kappa = 1/\sqrt{2}$. There, the edge currents screen the external magnetic field, therefore producing a magnetic moment opposite to the direction of the field, whereas vortices in the bulk of the system produce a magnetic moment along the direction of the applied field. Assuming cylindrical symmetry, the current density $\vec{j}$ has only an azimuthal component, with opposite signs in the bulk and on the edge of the system. Thus, there exists a circle Γ on which $\vec{j}$ vanishes.[2] This allows us to separate the domain Ω into two concentric subdomains Ω_1 and Ω_2 whose boundary is the circle Γ. Therefore, one can extend to the subdomain Ω_1 the results obtained for the infinite case. The existence of vortices in a finite domain such as Ω_1 was checked numerically [14]. It was shown that $|\psi|$ vanishes as a power law at the center of the disk, hence there is a (multi-)vortex in the center whose multiplicity is determined by the exponent of the power law. The magnetic flux $\Phi(\Omega_1) = n$ is quantized and the free energy in Ω_1 is $\mathscr{F}(\Omega_1) = 2\pi n$.

The contribution of $\mathscr{F}(\Omega_2)$ to the free energy can be written, using the phase and the modulus of the order parameter ψ, as:

$$\int_{\Omega_2} (\nabla|\psi|)^2 + |\psi\left(\vec{\nabla}\chi - \vec{A}\right)|^2 \frac{B^2 + \left(1 - (|\psi|)^2\right)^2}{2}. \quad (3)$$

We know, from the London equation, that both the magnetic field and the vector potential decrease rapidly away from the boundary $\partial\Omega$ of the system over a distance of order $\lambda\sqrt{2}$. Over the same distance, at the dual point, $|\psi|$ saturates to unity. One can thus estimate the integral (3) using a saddle-point

[2] This property seems to remain true even in the absence of cylindrical symmetry [17].

method. We assume cylindrical symmetry, and we neglect the term $(\nabla|\psi|)^2$ on the boundary because of the boundary conditions, so that the relation (3) is now given by an integral over the boundary of the system. To go further, we need to implement boundary conditions for the magnetic field $B(R)$ and the vector potential $A(R)$. The choice $B(R)=B_e$, where B_e is the external imposed field, corresponds to the geometry of an infinitely long cylinder, where the flux lines are not distorted outside the system [3]. A more suitable choice for a flat thin disk is provided by demanding $\phi=\phi_e$. This boundary condition implies that the vector potential is identified by continuity to its external applied value $\vec{A}_e$. It should be noticed that the magnetic field $\vec{B}$ has then a non-monotonous variation: it is low in the bulk, larger than B_e near the edge of the system, because of the distortion of flux lines, and eventually equal to its applied value far outside the system [15].

Finally, the minimization of the free energy with respect to $|\psi|$ gives $1-|\psi|^2=|\vec{\nabla}\chi-\vec{A}|^2$, such that, performing the integral over the boundary of the system, we obtain:

$$\frac{1}{2\pi}\mathscr{F}(\Omega_2)=\frac{\lambda\sqrt{2}}{R}(n-\phi_e)^2-\frac{1}{2}\left(\frac{\lambda\sqrt{2}}{R}\right)^3(n-\phi_e)^4. \tag{4}$$

We have neglected the contribution of the B^2 term, which is smaller by a factor of the order $(\lambda/R)^2$.

The thermodynamic Gibbs potential $\mathscr{G}$ of the system is then:

$$\frac{1}{2\pi}\mathscr{G}(n,\phi_e)=n+a(n-\phi_e)^2-\frac{a^3}{2}(n-\phi_e)^4-a^2\phi_e^2 \tag{5}$$

where we have defined $a=\lambda\sqrt{2}/R$. The relation (5) consists in a set of quartic functions indexed by the integer n. The minimum of the Gibbs potential is the envelop curve defined by the equation $(\partial\mathscr{G}/\partial n)|_{\phi_e}=0$, i.e., the system chooses its winding number n in order to minimize $\mathscr{G}$. This provides a relation between the number n of vortices in the system and the applied magnetic field ϕ_e.

We consider the limit of large enough R/λ, such that the quartic term is negligible. The Gibbs potential then reduces to a set of parabolas. The vortex number n is then given by the integer part:

$$n=\left[\phi_e-\frac{R}{2\sqrt{2}\lambda}+\frac{1}{2}\right] \tag{6}$$

while the magnetization $M=-\partial\mathscr{G}/\partial\phi_e$, is given by:

$$-M=2a(\phi_e-n)-2a^2\phi_e. \tag{7}$$

For ϕ_e smaller that $R/(2\sqrt{2}\lambda)$, we have $n=0$ and $(-M)$ increases linearly with the external flux. This corresponds to the London regime. The field H_1 at which the first vortex enters the disk corresponds to $\mathscr{G}(n=0)=\mathscr{G}(n=1)$, i.e., to:

$$H_1=\frac{\phi_0}{2\pi\sqrt{2}R\lambda}+\frac{\phi_0}{2\pi R^2}. \tag{8}$$

The subsequent vortices enter one by one for each crossing $\mathscr{G}(n+1)=\mathscr{G}(n)$; this happens periodically in the applied field, with a period equal to $\Delta H=\phi_0/\pi R^2$ and a discontinuity of the magnetization $\Delta M=(2\sqrt{2}\lambda)/R$.

There is a qualitative similarity between the results we derived using the properties of the dual point and those obtained from a linearized version of the Ginzburg–Landau functional [16]. However, the two approaches differ in their quantitative predictions due to the importance of the nonlinear term.

Within the previous approximations, the expression (5) captures the main features observed experimentally, i.e., the behaviour of the magnetization at low fields (before the first discontinuity), the periodicity and the linear behaviour between the successive jumps. From the experimental parameters [1] namely $R=1.2$ μm and $\lambda(T)=84$ nm at $T=0.4$ K, we compute from our expressions $H_1=25$ G and $\Delta H=4.6$ G. These values agree with the experimental results [4] to within a few percent. We emphasize that H_1 scales like $1/R$, whereas ΔH scales like $1/R^2$ in accordance with the experimental data [1]. We calculate the ratio of the magnetization jumps to the maximum value of M to be 0.20 as compared to a measured value of 0.22. The total number of jumps scales like R^2 and the upper critical field is independent of R in our theory in agreement with the experimental data.

At the duality point $\kappa = 1/\sqrt{2}$, the contribution of the vortices to the free energy is topological and does not depend either on the precise shape of the vortices or on the form of their interaction. This property does not hold for other values of κ. For a 2D film with an infinitesimal current sheet, the vortex configuration has been computed by Pearl [5], using the London equation, and differs qualitatively from the present model. Indeed, away from the dual point, both the shape of the vortices and their interaction modify the free energy and the magnetization. For instance, in the London limit ($\kappa \to \infty$), radially symmetric solutions become unstable and different geometrical configurations of the vortices have different energies. This results from two contributions to the energy, arising from the interaction between the vortices themselves and between the vortices and the edge currents. We have performed a perturbative analysis around the dual point, and we obtained that the bulk free energy $\mathscr{F}(\Omega_1)$ is given by:

$$\frac{1}{2\pi}\mathscr{F}(\Omega_1) = n\left(1 + \frac{1}{2}\left(\kappa\sqrt{2} - 1\right)\right) + \beta\left(\kappa\sqrt{2} - 1\right)\sum_{i<j}\mathscr{U}(r_{ij}). \tag{9}$$

The part that is linear in n is independent of the position of the vortices. It has been evaluated using a variational ansatz [12]. The two-body interaction potential near the dual point is well approximated by a function $\mathscr{U}(r_{ij}) = \mathscr{U}(r)$ where $\mathscr{U}(0) = 1, \mathscr{U}(\infty) = 0$ with $\beta = 1/4$. In particular, for a configuration where all the vortices are close to the center of the disk, the bulk free energy is:

$$\frac{1}{2\pi}\mathscr{F}(\Omega_1) = n\left(\frac{5}{8} + \kappa\frac{3\sqrt{2}}{8}\right) + \frac{1}{8}n^2\left(\kappa\sqrt{2} - 1\right). \tag{10}$$

Thus, away from the dual point, the linear term in the bulk free energy is not topological anymore and is modified by the interaction. The attractive or repulsive character of the interaction between vortices depends on the sign of $(\kappa\sqrt{2} - 1)$.

To obtain the edge contribution to the energy, we consider first the case of a single vortex placed at a distance x (in units of $\lambda\sqrt{2}$) from the center of the disk. Then, the phase of the order parameter is given by [17] $\tan\,\chi = (\sin\,\theta)/(\cos\,\theta - y)$, where $y = ax = x/R$ (whereas one has $\chi = n\theta$ for the symmetrical case). Starting from expression (3), we obtain:

$$\frac{1}{2\pi}\mathscr{F}(\Omega_2, y) = a(n - \phi_e)^2 - \frac{a^3}{4\kappa^2}(n - \phi_e)^4 + f(a, y, \phi_e) \tag{11}$$

where the function $f(a, y, \phi_e)$ can be explicitly calculated and represents a vortex confining potential inside a finite superconductor for $\kappa \simeq 1/\sqrt{2}$. This corresponds to the well-known Bean–Livingston confining energy barrier in a 3D superconductor that has been obtained in the extreme type II limit [18,19] using the London equation. It is important to emphasize that around the dual point, vortices are not point-like and therefore the usual expression of the Bean–Livingston energy barrier does not hold.

The possible equilibrium configurations of vortices result from the competition between the bulk and edge contributions to the free energy derived above. It is either a giant vortex at the center of the disk, a situation that preserves the cylindrical symmetry, or a polygonal pattern of small vortices. In order to evaluate the energy of these configurations, we generalize the relation (11) to the case of a polygonal configuration of vortices placed at a distance x from the center of the disk. The resulting energy [17] is n-times the barrier contribution $f(a, y, \phi_e)$ obtained in Eq. (11) provided the following substitutions are made: $a \to na$, $y \to y^n$ and $\phi_e \to \phi_e/n$.

In conclusion, we have investigated the question of the existence and stability of vortices in small two-dimensional bounded superconducting systems. We have shown [13] that starting from the exact solution of the Ginzburg–Landau equations for an infinite plane and for the special value $\kappa = 1/\sqrt{2}$, it is possible to derive an analytical expression for the free energy in a bounded system. The resulting expression provides a satisfactory quantitative description of the magnetization measured on small superconducting aluminium disks in the low magnetic field regime. For larger fields, we cannot neglect anymore the interaction effects due to the vortices and the edge currents. Perturbation theory [17] around the value $\kappa = 1/\sqrt{2}$ has allowed us to derive an expression for both the confining potential barrier of the vortices and the strength of the interaction between vortices. This provides a more refined description of the measured magnetization at larger fields.

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