

Chapter 13

Interactions and diffusion

ν_0 is the average density of states per spin direction and $\rho_0 = \nu_0/\Omega$ is the density of states per unit volume. The energy $\Delta = 1/(\rho_0\Omega) = 1/\nu_0$ is the average level spacing per spin direction. Most results will be presented in the CGS system, in the form most commonly found in the literature. Unless specified, we take $\hbar = 1$.

13.1 Introduction

Up to this point, electron-electron interaction has been neglected in the description of spectral properties and electronic transport. Although electrons interact through the Coulomb interaction, the free electron model constitutes a very good approximation for the description of many physical properties. This is due to the screening of the Coulomb interaction which occurs on a length of the order of the average distance between electrons. However, the electron-electron interaction has important physical consequences which can be classified in two categories :

- Each electron is sensitive not only to the disorder potential but also to the electronic density fluctuations induced by other electrons. As a result, the energy levels are shifted and the thermodynamic and transport properties are modified, particularly the density of states and the conductivity. The change in density of states is maximum around the Fermi level, thus constituting a direct signature of the interaction. Moreover, this change is important since it affects the orbital magnetism of the electron gas and the persistent current (Chapter 14). The change in conductivity is of the same order of magnitude as the weak localization correction, but its nature is quite different. In particular, it does not depend on the magnetic field, making it more difficult to observe.
- The interaction between electrons is an inelastic process (the total energy is conserved but the energy of each electron is modified). Each electron

stays in an eigenstate of the single particle Schrödinger equation only during a finite time, and thus phase sensitive processes are affected. The electron-electron interaction, like the interaction with other degrees of freedom (*e.g.* phonons), destroys the phase coherence after a characteristic time τ_{ϕ}^{ee} . We shall see that this phase coherence time can be also understood as resulting from a fluctuating electromagnetic field that dephases the trajectories paired in either the Diffuson or the Cooperon, in a way qualitatively similar to the dephasing induced by the motion of scatterers (chapter 6). The equivalence between the effects of the Coulomb interaction and of a fluctuating electromagnetic field is far from being obvious. It relies on the fluctuation-dissipation theorem.

The diffusive nature of the electronic motion plays an essential role because it *strenghtens* the interaction effect. This can be understood in the following qualitative way. As a result of the diffusive motion, the probability that two electrons interact is enhanced since an electron moves less rapidly than if its motion were ballistic. The effective interaction between two electrons is thus enhanced since each of them has an increased probability of staying in the interaction region. The modification of physical quantities due to the interaction must be proportional to the time spent in this region. More precisely, for a physical quantity $X(E)$ that depends on some energy scale E , we expect a modification proportional to the probability of return (5.5) into the interaction region during the time \hbar/E ,

$$\frac{\delta X(E)}{X} \propto \frac{1}{\rho_0 \hbar \Omega} \int_0^{\hbar/E} Z(t) dt \quad (13.1)$$

This chapter is devoted to the study of the interplay between disorder and interaction. The latter is treated as a perturbation to the model of independent electrons in a random potential.

13.2 Screened Coulomb interaction

The Coulomb interaction between two electrons at a distance R is described by the potential ¹ $U_0(\mathbf{R}) = e^2/R$. In $d = 3$, its Fourier transform is $U_0(\mathbf{q}) = 4\pi e^2/q^2$. The case of other space dimensions is discussed in Appendix A13.1. In a metal, the Coulomb interaction is screened due to the presence of the other electrons. In the Thomas-Fermi approximation, it becomes [314]

$$U(\mathbf{q}) = \frac{4\pi e^2}{q^2 + \kappa^2} \quad (13.2)$$

or

$$U(\mathbf{R}) = \frac{e^2}{R} e^{-\kappa R} . \quad (13.3)$$

¹In CGS units.

The Thomas-Fermi wavevector κ , reciprocal of the screening length, is given by [314]

$$\kappa^2 = 8\pi e^2 \rho_0 = 8\pi e^2 \frac{\nu_0}{\Omega} \quad (13.4)$$

where ρ_0 is the density of states per unit volume and per spin direction. The $\mathbf{q} = 0$ value of the screened interaction is simply related to the density of states by

$$U = U(\mathbf{q} = 0) = \int U(\mathbf{R}) d\mathbf{R} = \frac{1}{2\rho_0} \quad (13.5)$$

For a metal, $\kappa \simeq k_F$, the screening is quite efficient and acts over a length of order λ_F (see remark p. 492). In the weak disorder limit, the screening length κ^{-1} is much smaller than the elastic mean free path l_e .

In a diffusive system, the screening is not instantaneous and it is important to describe its dynamics properly. The effective interaction between electrons is a function $U(\mathbf{q}, \omega)$ of wavevector *and* of frequency. The charge reorganization is described by means of the dielectric function $\epsilon(\mathbf{q}, \omega)$ which is related to the density-density response function $\bar{\chi}_0(\mathbf{q}, \omega)$ by the relation $\epsilon(\mathbf{q}, \omega) = 1 + U_0(\mathbf{q})\bar{\chi}_0(\mathbf{q}, \omega)$. We have [314]

$$U(\mathbf{q}, \omega) = \frac{U_0(\mathbf{q})}{\epsilon(\mathbf{q}, \omega)} = \frac{U_0(\mathbf{q})}{1 + U_0(\mathbf{q})\bar{\chi}_0(\mathbf{q}, \omega)} \quad (13.6)$$

In section A7.1.2, we have shown that, in the diffusion approximation,

$$\bar{\chi}_0(\mathbf{q}, \omega) = 2\rho_0 \frac{Dq^2}{-i\omega + Dq^2} \quad (13.7)$$

Therefore the effective interaction depends also on frequency : the interaction is said to be *dynamically screened*, a consequence of the diffusive nature of the electronic motion. Using (13.6), we obtain

$$U(\mathbf{q}, \omega) = \frac{4\pi e^2}{q^2} \frac{-i\omega + Dq^2}{-i\omega + Dq^2 + D\kappa^2} \quad (13.8)$$

The dielectric function $\epsilon(\mathbf{q}, \omega)$ deduced from (13.7) is

$$\epsilon(\mathbf{q}, \omega) = 1 + 2\rho_0 \frac{4\pi e^2}{q^2} \frac{Dq^2}{-i\omega + Dq^2} \quad (13.9)$$

or

$$\epsilon(\mathbf{q}, \omega) = 1 + 4\pi\sigma_0 P_d(\mathbf{q}, \omega) \quad (13.10)$$

where $\sigma_0 = 2e^2 D\rho_0$ is the Drude conductivity given by the Einstein relation (7.14) and P_d is the Diffuson (4.89) written in the diffusion approximation, $ql_e \ll 1$ so that *a fortiori* $q \ll \kappa$. Finally the dielectric function becomes

$$\epsilon(\mathbf{q}, \omega) \simeq \frac{4\pi\sigma_0}{-i\omega + Dq^2} \quad (13.11)$$

and the dynamical effective interaction is simply related to the density-density response function $\bar{\chi}_0(\mathbf{q}, \omega)$ by

$$U(\mathbf{q}, \omega) \simeq \frac{1}{\bar{\chi}_0(\mathbf{q}, \omega)} = \frac{1}{2\rho_0} \frac{-i\omega + Dq^2}{Dq^2} . \quad (13.12)$$

The static limit, $\bar{\chi}_0 = 2\rho_0$, is obtained by taking $\omega = 0$.

Remark : the parameter r_s

A measure of the strength of the electronic correlations is provided by the dimensionless parameter r_s , which is the ratio between Coulomb potential energy and kinetic energy. The potential energy is of order e^2/a where $a = n^{-1/d}$ is the average distance between electrons in a gas of density n in d dimensions, while the kinetic energy is the Fermi energy $p_F^2/2m \propto \hbar^2/(2ma^2)$. The parameter r_s is thus defined as the ratio

$$r_s = \frac{a}{a_0} \propto \frac{\text{potential energy}}{\text{kinetic energy}} , \quad (13.13)$$

where $a_0 = \hbar^2/(m_0e^2)$ is the Bohr radius and m_0 is the free electron mass (see section 2.1.1). The parameter r_s is thus proportional to the average distance between electrons. In three dimensions,

$$\frac{4\pi}{3} r_s^3 a_0^3 = \frac{1}{n} . \quad (13.14)$$

Since $k_F^3 = 3\pi^2 n$,

$$r_s = \left(\frac{9\pi}{4} \right)^{1/3} \frac{1}{k_F a_0} . \quad (13.15)$$

The screening vector κ is given by $\kappa^2 = 8\pi e^2 \rho_0$, where $\rho_0 = mk_F/(2\pi^2 \hbar^2)$ is the density of states at the Fermi level, per unit volume and per spin direction. The ratio κ/k_F is thus given by :

$$\frac{\kappa}{k_F} = \left(\frac{16}{3\pi^2} \right)^{1/3} \left(\frac{m}{m_0} \right)^{1/2} \sqrt{r_s} \simeq 0.81 \left(\frac{m}{m_0} \right)^{1/2} \sqrt{r_s} . \quad (13.16)$$

For a metal such as copper, we have $r_s = 2.67$ and $m/m_0 = 1.3$, so that $\kappa/k_F \simeq 1.51$. The screening is thus very efficient in this metal. For other metals, see Table 1.1 in reference [314], taking note of the different definitions.

13.3 Hartree-Fock approximation

In order to describe the effects of the Coulomb interaction, we now use the Hartree-Fock approximation whose main lines are recalled here. For more details, see [314]. First, the Hartree approximation consists in finding the solutions (ϵ_i, ϕ_i) of the non-linear equation

$$\epsilon_i \phi_i(\mathbf{r}) = -\frac{1}{2m} \Delta \phi_i(\mathbf{r}) + V_{ion}(\mathbf{r}) \phi_i(\mathbf{r}) + \int U(\mathbf{r} - \mathbf{r}') n(\mathbf{r}') \phi_i(\mathbf{r}) d\mathbf{r}' , \quad (13.17)$$

where $V_{ion}(\mathbf{r})$ is the one-body potential describing the interaction of the electrons with the lattice and the impurities, and $U(\mathbf{r} - \mathbf{r}')$ is the two-body screened Coulomb interaction between electrons. The electronic density is

$$n(\mathbf{r}) = 2 \sum_j f(\epsilon_j) |\phi_j(\mathbf{r})|^2 \quad (13.18)$$

and $f(\epsilon)$ is the Fermi factor. Electrical neutrality implies

$$\bar{V}_{ion}(\mathbf{r}) + \int U(\mathbf{r} - \mathbf{r}') \overline{n(\mathbf{r}')} d\mathbf{r}' = 0 \quad , \quad (13.19)$$

where we have replaced the spatial average by the disorder average. The Hartree equation is then rewritten as

$$\epsilon_i \phi_i(\mathbf{r}) = -\frac{1}{2m} \Delta \phi_i(\mathbf{r}) + V(\mathbf{r}) \phi_i(\mathbf{r}) + \int U(\mathbf{r} - \mathbf{r}') (n(\mathbf{r}') - \bar{n}) \phi_i(\mathbf{r}) d\mathbf{r}' \quad . \quad (13.20)$$

in which $V(\mathbf{r}) = V_{ion}(\mathbf{r}) - \bar{V}_{ion}$ is the disorder potential defined in Chapter 2, and $\bar{n} = \overline{n(\mathbf{r})}$. In the Hartree approximation, the non-linear equation (13.20) is an effective Schrödinger equation in which the potential seen by one electron depends on the electronic density, that is on the wave functions of the other electrons. In this approximation, the total wave function is the product of single particle wave functions and does not satisfy the Pauli principle.

In order to take into account the antisymmetry of the total wave function, we have to add the Fock term, which describes *exchange* between particles of same spin. Equation (13.20) then becomes [314, 315] :

$$\begin{aligned} \epsilon_i \phi_i(\mathbf{r}) = & -\frac{1}{2m} \Delta \phi_i(\mathbf{r}) + V(\mathbf{r}) \phi_i(\mathbf{r}) + \int U(\mathbf{r} - \mathbf{r}') (n(\mathbf{r}') - \bar{n}) \phi_i(\mathbf{r}) d\mathbf{r}' \\ & - \sum_j f(\epsilon_j) \int U(\mathbf{r} - \mathbf{r}') \phi_j^*(\mathbf{r}') \phi_j(\mathbf{r}) \phi_i(\mathbf{r}') d\mathbf{r}' \quad . \end{aligned} \quad (13.21)$$

In principle, the non-linear equation (13.21) must be solved self-consistently, which is a difficult problem having no analytical solution in the presence of disorder. Here we shall consider the interaction U as a perturbation and we shall limit ourselves to lowest order. The unperturbed states are the eigenstates $\{\epsilon_i, \phi_i(\mathbf{r})\}$ of the Hamiltonian (2.1) in the presence of disorder². We calculate the effect of the Coulomb interaction in the framework of the Hartree-Fock approximation, following the lines of the method developed in chapter 3. In principle, we have to evaluate first the diagonal Green's function in the state ϕ_i by writing a Dyson equation analogous to (3.67). A self-energy $\Sigma_i = \delta\epsilon_i + i\Gamma_i$ is then obtained, whose real part $\delta\epsilon_i$ measures the displacement of the energy level ϵ_i and whose imaginary part Γ_i gives the width of this level, that is its inverse lifetime³. These two components of the self-energy give the correction to the one particle density of states and the electronic lifetime. Here we present a simplified version of this formalism.

²We could have followed the same line as in Chapter 3 and treated on the same footing the disorder potential and the interaction perturbatively on the basis of plane waves. This choice would not be convenient since a Fermi golden rule argument such as in (3.1) would introduce a width \hbar/τ_e resulting from the disorder effect. As we shall see, this width is much larger than the width due to interactions. It is thus more convenient to treat the disorder in a non perturbative way.

³For more details on the N -body problem, see [316] and more specifically [317] for the Hartree-Fock approximation in the presence of disorder.

13.4 Density of states anomaly

13.4.1 Static interaction

We first evaluate the one-particle density of states in the presence of electron-electron interaction by calculating the shift $\delta\epsilon_i$ of the energy levels for a static screened interaction $U(\mathbf{r}-\mathbf{r}')$ given by (13.3) in $d=3$. The shift $\delta\epsilon_i$ is obtained from (13.21) and takes the form $\delta\epsilon_i = \delta\epsilon_i^H + \delta\epsilon_i^F$ where the Hartree contribution $\delta\epsilon_i^H$ is given, in the lowest order, by

$$\delta\epsilon_i^H = \int U(\mathbf{r}-\mathbf{r}') |\phi_i(\mathbf{r})|^2 (n(\mathbf{r}') - \bar{n}) d\mathbf{r} d\mathbf{r}' . \quad (13.22)$$

Similarly, the exchange, or Fock, term $\delta\epsilon_i^F$ writes

$$\delta\epsilon_i^F = - \sum_{j,\sigma} f(\epsilon_j) \int U(\mathbf{r}-\mathbf{r}') \phi_j^*(\mathbf{r}') \phi_j(\mathbf{r}) \phi_i^*(\mathbf{r}) \phi_i(\mathbf{r}') d\mathbf{r} d\mathbf{r}' . \quad (13.23)$$

The total energy E_T is

$$\begin{aligned} E_T &= E_T^0 + \frac{1}{2} \int U(\mathbf{r}-\mathbf{r}') n(\mathbf{r}) \delta n(\mathbf{r}') d\mathbf{r} d\mathbf{r}' \\ &\quad - \sum_{i,j} f(\epsilon_i) f(\epsilon_j) \int U(\mathbf{r}-\mathbf{r}') \phi_j^*(\mathbf{r}') \phi_j(\mathbf{r}) \phi_i^*(\mathbf{r}) \phi_i(\mathbf{r}') d\mathbf{r} d\mathbf{r}' \end{aligned} \quad (13.24)$$

where E_T^0 is the total energy in the absence of interaction and $\delta n(\mathbf{r}') = n(\mathbf{r}') - \bar{n}$. The factor 1/2 avoids double counting of the interaction in the total energy. The mean shift of an energy level ϵ is defined by

$$\Delta_\epsilon = \frac{1}{\nu_0} \overline{\delta(\epsilon - \epsilon_i) \delta\epsilon_i} . \quad (13.25)$$

Since, on average, each energy level ϵ is changed into $\epsilon + \Delta_\epsilon$, the distance between two levels ϵ_1 and ϵ_2 becomes $(\epsilon_2 - \epsilon_1) [1 + \partial\Delta_\epsilon/\partial\epsilon]$. This shift leads to a relative change in the density of states

$$\frac{\delta\nu}{\nu_0} = - \frac{\partial\Delta_\epsilon}{\partial\epsilon} . \quad (13.26)$$

Consider first the *exchange term* (13.23). By inserting the relation $f(\epsilon_j) = \int d\epsilon' \delta(\epsilon' - \epsilon_j) f(\epsilon')$ and by using (3.26) to transform the product of four wave functions into a product of two non-local density of states, this term can be rewritten as ⁴:

$$\Delta_\epsilon^F = - \frac{1}{\nu_0} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int U(\mathbf{r}-\mathbf{r}') \overline{\rho_\epsilon(\mathbf{r}, \mathbf{r}') \rho_{\epsilon-\omega}(\mathbf{r}', \mathbf{r})} d\mathbf{r} d\mathbf{r}' . \quad (13.27)$$

⁴The zero of the energies is taken at the Fermi level.

Expressing the correlation function of the non-local density of states $\rho_\epsilon(\mathbf{r}, \mathbf{r}')$ with the help of (4.207) and using (13.26), we obtain ⁵

$$\delta\nu^F(\epsilon) = \frac{\rho_0}{\pi} \int_{-\infty}^{\infty} f'(\epsilon - \omega) d\omega \int U(\mathbf{r} - \mathbf{r}') \operatorname{Re} P_d(\mathbf{r}, \mathbf{r}', \omega) d\mathbf{r} d\mathbf{r}' \quad (13.28)$$

The interaction (13.3) being short-range (much shorter than l_e), the integral factorizes

$$\delta\nu^F(\epsilon) \simeq \frac{U\rho_0}{\pi} \int_{-\infty}^{\infty} f'(\epsilon - \omega) d\omega \int \operatorname{Re} P_d(\mathbf{r}, \mathbf{r}, \omega) d\mathbf{r} \quad (13.29)$$

where the parameter U is defined by (13.5). Introducing the temporal Fourier transform of relation (5.5), $\int \operatorname{Re} P_d(\mathbf{r}, \mathbf{r}, \omega) d\mathbf{r} = \int_0^\infty Z(t) \cos \omega t dt$, the density of states correction due to the interaction can be written as a function of the return probability $Z(t)$. Using (15.109) and $U\rho_0 = 1/2$, we obtain ⁶

$$\delta\nu^F(\epsilon) = -\frac{1}{2\pi} \int_0^\infty \frac{\pi T t}{\sinh \pi T t} Z(t) \cos \epsilon t dt \quad (13.30)$$

The contribution of the *Hartree term* to the density of states correction is obtained from the relations (13.22) and (13.25)

$$\Delta_\epsilon^H = \frac{2}{\nu_0} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int U(\mathbf{r} - \mathbf{r}') \overline{\rho_\epsilon(\mathbf{r}, \mathbf{r}) \rho_{\epsilon - \omega}(\mathbf{r}', \mathbf{r}')^c} d\mathbf{r} d\mathbf{r}' \quad (13.31)$$

The product of the local densities that appears in this expression is given by (4.210). Due to the short range potential, the main contribution has only one Diffuson. The relation (13.26) for the density of states gives

$$\delta\nu^H(\epsilon) = -2 \frac{\rho_0}{\pi} \int_{-\infty}^{\infty} f'(\epsilon - \omega) d\omega \int g^2(\mathbf{R}) U(\mathbf{R}) \operatorname{Re} P_d(\mathbf{r}, \mathbf{r}, \omega) d\mathbf{r} d\mathbf{r}' \quad (13.32)$$

where the function $g(\mathbf{R})$ is defined by (3.98). The short range term $g^2(\mathbf{R})U(\mathbf{R})$ can be integrated separately, and the Hartree contribution writes

$$\delta\nu^H(\epsilon) = -\frac{F}{\pi} \int_{-\infty}^{\infty} f'(\epsilon - \omega) d\omega \int \operatorname{Re} P_d(\mathbf{r}, \mathbf{r}, \omega) d\mathbf{r} \quad (13.33)$$

where we have introduced the parameter (Figure 13.5)

$$F = \frac{\int g^2(\mathbf{R}) U(\mathbf{R}) d\mathbf{R}}{\int U(\mathbf{R}) d\mathbf{R}} = \frac{1}{U} \int g^2(\mathbf{R}) U(\mathbf{R}) d\mathbf{R} \quad (13.34)$$

⁵We keep only the contribution of the Diffuson, and we also check that the product of average values gives a negligible contribution.

⁶Notice that the result does not depend on the coupling constant e^2 . This is because the screening length κ^{-1} is much smaller than l_e , and thus the relation (13.29) contains the coefficient $U = U(q=0) = 4\pi e^2 / \kappa^2$. This ratio no longer depends on e^2 .

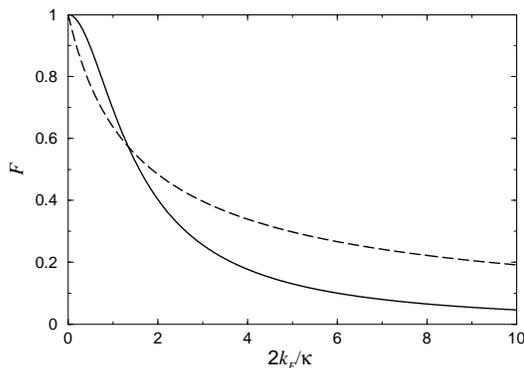


Figure 13.1: Variation of the parameter F as a function of the ratio $2k_F/\kappa$, in dimensions $d = 2$ (dashed line) and $d = 3$ (full line).

with $U = 1/2\rho_0$ (relation 13.5). The parameter F varies between 0 for strong screening ($\kappa \rightarrow \infty$) and 1 for weak screening ($\kappa \rightarrow 0$). For more details, see exercises 13.3 and 13.4. The expression of $\delta\nu^H(\epsilon)$ is proportional to (13.29) so that the total correction to the density of states per spin direction writes

$$\delta\nu(\epsilon) = -\frac{\lambda_\nu}{2\pi} \int_0^\infty \frac{\pi T t}{\sinh \pi T t} Z(t) \cos \epsilon t dt \quad (13.35)$$

where the interaction is described by the parameter λ_ν which here takes the value $\lambda_\nu = 1 - 2F$. This value corresponds to a static interaction and is different from the value obtained for a dynamically screened interaction $U(\mathbf{q}, \omega)$ (p. 504). Moreover, λ_ν depends on the range of the interaction through the parameter F . Since the latter varies between 0 and 1, the sign of the density of states correction seems to depend on the nature of the screened interaction. If the exchange term is larger than the Hartree term ($F \ll 1$), that is for a potential whose range is larger than the Fermi wavelength, the density of states correction is negative as observed experimentally. However, it seems that $\delta\nu(\epsilon)$ might become positive for a very short range interaction. For example in copper where $F \simeq 0.6$, a positive correction might be expected. We shall see in section 13.4.3 that this result is an *artefact* and that taking into account the dynamical character of the screened interaction modifies the amplitude of the correction that stays always negative.

The density of states correction (13.35) reflects obviously the diffusive motion of the electrons and thus depends on space dimensionality⁷ through the probability $Z(t)$ which, in free space, is given by (5.24). For example, in a

⁷Recall that the diffusive motion depends on the dimensionality d , but the Coulomb interaction is always three-dimensional if all the dimensions of the sample are larger than κ^{-1} . In a semiconductor where κ^{-1} is large, the nature of the interaction may change in the presence of gates, due to the existence of image charges [315].

quasi-1d system of volume Ω at zero temperature, we have $Z(t) = \Omega/\sqrt{4\pi Dt}$. Thus, from (13.35) and (15.82),

$$\delta\rho(\epsilon) = -\frac{\lambda_\nu}{4\pi\sqrt{2}} \frac{1}{\sqrt{D\epsilon}} \quad (13.36)$$

is obtained *per spin direction and unit volume*. Similarly, in two dimensions and in the limit $\epsilon\tau_e \ll 1$,

$$\delta\rho(\epsilon) = \frac{\lambda_\nu}{8\pi^2 D} \ln \epsilon\tau_e \quad , \quad (13.37)$$

is obtained from (15.83), and in three dimensions, (15.84) gives

$$\delta\rho(\epsilon) = \frac{\lambda_\nu}{8\pi^2\sqrt{2}D} \left(\sqrt{\frac{\epsilon}{D}} - C \right) \quad (13.38)$$

where C is a constant independent of ϵ .

The dependence of the density of states correction on the return probability exhibits the same functional dependence than the weak localization correction $\Delta\sigma$ (see 7.56). Formally we have

$$\frac{\delta\rho}{\rho_0} \propto \frac{\Delta\sigma}{\sigma_0} (L_\phi = L_\epsilon) \quad , \quad (13.39)$$

where $L_\epsilon = \sqrt{D/\epsilon}$. Here the long time cutoff is determined by the energy ϵ instead of the phase coherence time $1/\tau_\phi$. We summarize the above relations for the density of states correction, also called *density of states anomaly* by

$$\delta\rho(\epsilon) \propto -\frac{1}{D} \begin{cases} L_\epsilon - l_e & d = 1 \\ \ln \frac{L_\epsilon}{l_e} & d = 2 \\ \frac{1}{l_e} - \frac{1}{L_\epsilon} & d = 3 \end{cases} \quad (13.40)$$

Notice that the amplitude of the relative correction is of order $1/g$, where g is the dimensionless conductance.

Exercise 13.1. Show that

$$\operatorname{Re}P(\mathbf{r}, \mathbf{r}', \epsilon) = \operatorname{Im} \int_{\epsilon}^{\infty} d\omega \int P(\mathbf{r}, \mathbf{r}'', \omega) P(\mathbf{r}'', \mathbf{r}', \omega) d\mathbf{r}''$$

and more generally that

$$\int_{-\infty}^{\infty} \operatorname{Re}f'(\epsilon - \omega) P(\mathbf{r}, \mathbf{r}', \omega) d\omega = -\operatorname{Im} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int P(\mathbf{r}, \mathbf{r}'', \omega) P(\mathbf{r}'', \mathbf{r}', \omega) d\mathbf{r}''$$

or

$$\int_{-\infty}^{\infty} f'(\epsilon - \omega) \operatorname{Re} P(\mathbf{q}, \omega) d\omega = - \int_{-\infty}^{\infty} f(\epsilon - \omega) \operatorname{Im} P^2(\mathbf{q}, \omega) d\omega \quad (13.41)$$

where P is either the Diffuson P_d , or the Cooperon P_c .

Exercise 13.2. Show that the density of states correction (13.35) can be also written in the equivalent forms

$$\delta\nu(\epsilon) = -\frac{1-2F}{2\pi} \int_{-\infty}^{\infty} d\omega f(\epsilon - \omega) \sum_q \operatorname{Im} P_d^2(\mathbf{q}, \omega) \quad (13.42)$$

or

$$\delta\nu(\epsilon) = -\frac{1-2F}{4\pi} \int_0^{\infty} d\omega \left[\tanh \frac{\epsilon + \omega}{2T} + \tanh \frac{\omega - \epsilon}{2T} \right] \sum_q \operatorname{Im} P_d^2(\mathbf{q}, \omega) \quad (13.43)$$

Exercise 13.3. The parameter F represents the ratio of the Hartree and Fock (exchange) contributions. By calculating the Hartree contribution (13.32) in reciprocal space, show that F can be also written in the form [315]

$$F = \frac{\langle U(\mathbf{p} - \mathbf{p}') \rangle}{U} \quad (13.44)$$

where $U(\mathbf{p} - \mathbf{p}')$ is the Fourier transform of the interaction $U(\mathbf{r})$ and the average is made upon the momenta \mathbf{p} and \mathbf{p}' taken on the Fermi surface. Check directly that

$$\langle U(\mathbf{p} - \mathbf{p}') \rangle = \int a(\mathbf{q}) U(\mathbf{q}) d\mathbf{q} = \int g^2(\mathbf{R}) U(\mathbf{R}) d\mathbf{R} \quad (13.45)$$

where $a(\mathbf{q})$ is the Fourier transform of $g^2(\mathbf{R})$ defined in (3.98).

Exercise 13.4. Calculation of F in the Thomas-Fermi approximation

In three dimensions, using the expression $U(\mathbf{R}) = \frac{e^2}{R} e^{-\kappa R}$ and the relation (3.98) for $g(\mathbf{R})$, show that in the limit $\kappa l_e \gg 1$,

$$F = \frac{\kappa^2}{4k_F^2} \ln\left(1 + \frac{4k_F^2}{\kappa^2}\right) \quad (13.46)$$

F reaches the value 1 for a perfectly screened interaction (see Figure 13.1). For copper where $\kappa/k_F \simeq 1.51$, F is of the order of 0.6.

In two dimensions, show that in the limit $\kappa l_e \gg 1$,

$$F = \frac{2}{\pi} \frac{\kappa}{\sqrt{\kappa^2 - 4k_F^2}} \arctan \frac{\sqrt{\kappa^2 - 4k_F^2}}{2k_F} \quad (13.47)$$

or

$$F = \int \frac{d\theta}{2\pi} \frac{1}{1 + (2k_F/\kappa) \sin \theta/2} \quad (13.48)$$

Show first that in two dimensions $U(\mathbf{q}) = 2\pi e^2/(q + \kappa)$ and use the corresponding expression (3.102) of $a(\mathbf{q})$.

13.4.2 Tunnel conductance and density of states anomaly

The change in the density of states in the vicinity of the Fermi level can be observed experimentally by tunnel conductance measurements. The experiment consists of connecting the metal we want to study to another metal whose density of states is known. One of the two metals is oxidized before growing a layer of the second metal, thus creating an oxide barrier between the two conductors. The width of the oxide layer can be controlled and constitutes a tunnel barrier. The tunnel current is proportional to the density of states of the two metals, and its measurement gives access to the density of states anomaly. The dip in the tunnel conductance is not specific to the weak disorder regime considered here, but rather is a general characteristic of the Coulomb interaction which subsists even for strong disorder (near the metal-insulator transition) or for semiconductors.

Let us recall the measurement principle. The tunnel current $I(V)$, for a voltage $V > 0$ applied between two metals a and b , depends on the tunnel probability to transfer electrons between the two metals. The tunnel rate between an initial state i of metal a and a final state f of metal b is given by the Fermi golden rule

$$\Gamma_{i \rightarrow f}(V) = \frac{2\pi}{\hbar} |t_{if}|^2 \delta(E_i - E_f + eV) \quad , \quad (13.49)$$

where t_{if} is a matrix element which describes the coupling between the two states and which depends on the geometry of the junction. The tunnel rate between metal a and metal b depends on the occupation numbers of the initial and final states. It is given by

$$\Gamma_{ab}(V) = \frac{2\pi}{\hbar} \sum_{i,f} |t_{if}|^2 f(E_i) [1 - f(E_f)] \delta(E_i - E_f + eV) \quad (13.50)$$

where $f(E)$ is the Fermi distribution. At finite temperature, there is also a finite transition probability $\Gamma_{ba}(V)$ from b to a , so that the tunnel current between a and b is

$$I = e(\Gamma_{ab} - \Gamma_{ba}) = 2\pi \frac{e}{\hbar} \sum_{i,f} |t_{if}|^2 [f(E_i) - f(E_f)] \delta(E_i - E_f + eV) \quad . \quad (13.51)$$

Assuming that the tunnel matrix element depends only weakly on the energy and that the voltage V and the temperature T are small compared to the Fermi energy and the height of the tunnel barrier, the sums can be replaced by integrals, and we can introduce the respective density of states $\rho_a(\epsilon)$ and $\rho(\epsilon)$ of the reference electrode and of the conductor being studied. We then get

$$I(V) = 2\pi \frac{e}{\hbar} |t|^2 \int_{-\infty}^{\infty} \rho_a(\epsilon) \rho(\epsilon + eV) [f(\epsilon) - f(\epsilon + eV)] d\epsilon . \quad (13.52)$$

If the densities of states vary only weakly near the Fermi level (they are denoted ρ_a and ρ_0), the integral takes a very simple form, and for a small voltage gives a tunnel current $I(V)$ proportional to V , and thus a linear characteristic which defines the tunnel conductance G_t

$$G_t = 2\pi \frac{e^2}{\hbar} |t|^2 \rho_a \rho_0 . \quad (13.53)$$

Assuming that the density of states of the reference electrode is energy independent, a variation $\delta\rho(\epsilon)$ of the density of states of the metal being studied leads to a variation $\delta I(V)$ of the current and thus to a variation $\delta G_t(V)$ of the tunnel conductance given by

$$\delta G_t(V) = \frac{d\delta I}{dV} = -2\pi \frac{e^2}{\hbar} \rho_a |t|^2 \int_{-\infty}^{\infty} d\epsilon \delta\rho(\epsilon) f'(\epsilon - eV) \quad (13.54)$$

so that, at zero temperature

$$\frac{\delta G_t(V)}{G_t} = \frac{\delta\rho(eV)}{\rho_0} . \quad (13.55)$$

The reduction of the tunnel conductance is thus a direct measurement of the variation of the density of states due to Coulomb interaction.

The first experimental evidence for the variations of the tunnel conductance as given by (13.40) is shown in Figure 13.2. It displays the dependence of the tunnel conductance G_t as a function of the voltage V for a tunnel contact *InO_x-insulator-Pb* with indium oxide films of different thickness. When the film thickness increases, we see a crossover from a two-dimensional behavior (logarithmic) of the density of states anomaly towards a three-dimensional behavior proportional to \sqrt{V} [318].

The one-dimensional behavior has also been observed in aluminium wires [319]. The behavior (13.36) in $1/\sqrt{V}$ is visible ⁸ in Figure 13.3.

Exercise 13.5. Tunnel conductance at finite temperature

Using (13.35) and (13.54), show that, at finite temperature, the relative correction to the tunnel conductance writes (exchange term) :

$$\frac{\delta G_t(V, T)}{G_t} = -\frac{1}{2\pi\nu_0} \int_0^{\infty} Z(t) R^2(t) \cos eVt dt \quad (13.56)$$

where the function $R(t)$ is given by $R(t) = \pi T t / \sinh(\pi T t)$.

⁸A quantitative comparison with the theoretical prediction (13.36) is more complicated [319]. It takes into account other effects neglected in our calculation, such as the influence of the electromagnetic environment on the conductance as well as finite geometry effects. It is, however, interesting that these additional effects leave the dependence of the tunnel conductance as the function of the voltage unchanged, and give simply a renormalization of the diffusion constant which appears in the function $Z(t)$ (see Exercise 13.9).

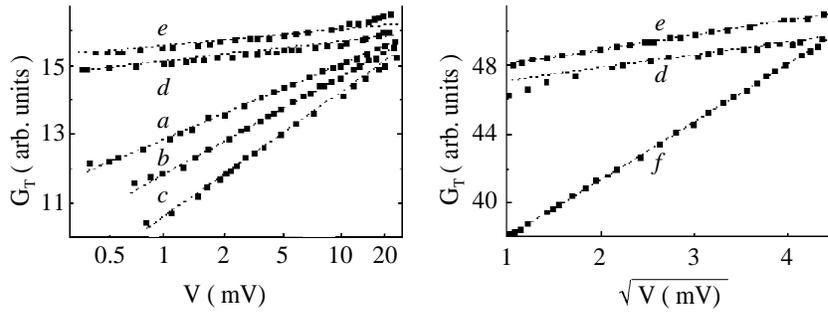


Figure 13.2: Tunnel conductance versus $\ln V$ (left) and \sqrt{V} (right), for a junction InO_x -insulator-lead. The different curves are obtained by varying the thickness of the indium oxide film : (a) $a = 160\text{\AA}$; (b) $a = 190\text{\AA}$; (c) $a = 210\text{\AA}$; (d) $a = 310\text{\AA}$; (e) $a = 460\text{\AA}$; (f) $a = 2600\text{\AA}$ [318].

13.4.3 Dynamically screened interaction

Since it does not account for the dynamical screening of the interaction, that is, its frequency dependence, the above derivation of the density of states is an approximation. The systematic calculation [316] of the density of states correction in the Hartree-Fock approximation using the expression (13.6) of the effective potential is obtained by evaluating the two diagrams of Figure 13.4 (see also Figure 13.5). The two upper diagrams are the usual representations of the Hartree and exchange (Fock) corrections in the many-body formalism [320]. The two lower diagrams propose a topologically equivalent representation which exhibits more explicitly the role of the structure factor Γ describing the diffusive nature of the electronic motion. The exchange (Fock) term contributing to the variation δG^F of the Green's function due to interactions [320] is obtained by a separation of slow and rapid spatial variations and, using Table 3.8, we find

$$\delta G^F(\mathbf{r}_0, \mathbf{r}_0) = - \int_{-\infty}^{\infty} \frac{d\omega}{2i\pi} f(\epsilon - \omega) \frac{f^{2,1}}{\gamma_e} \int \Gamma_{\omega}(\mathbf{r}_0, \mathbf{r}) \frac{1}{\gamma_e^2} U_{\omega}(\mathbf{r}, \mathbf{r}') \Gamma_{\omega}(\mathbf{r}', \mathbf{r}_0) d\mathbf{r} d\mathbf{r}' . \quad (13.57)$$

Using (4.37) and the expression of $f^{2,1}$ given in Table 3.1, we deduce

$$\delta G^F(\mathbf{r}_0, \mathbf{r}_0) = \rho_0 \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int P_d(\mathbf{r}_0, \mathbf{r}, \omega) U_{\omega}(\mathbf{r}, \mathbf{r}') P_d(\mathbf{r}', \mathbf{r}_0, \omega) d\mathbf{r} d\mathbf{r}' . \quad (13.58)$$

$U_{\omega}(\mathbf{r})$ is the Fourier transform of $U(\mathbf{q}, \omega)$ given by (13.12). The corresponding correction to the local density of states is related by (3.25) to the imaginary part of $\delta G^F(\mathbf{r}_0, \mathbf{r}_0)$, so that

$$\delta \rho^F(\mathbf{r}_0, \mathbf{r}_0) = -\frac{\rho_0}{\pi} \text{Im} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int P_d(\mathbf{r}_0, \mathbf{r}, \omega) U_{\omega}(\mathbf{r}, \mathbf{r}') P_d(\mathbf{r}', \mathbf{r}_0, \omega) d\mathbf{r} d\mathbf{r}' . \quad (13.59)$$

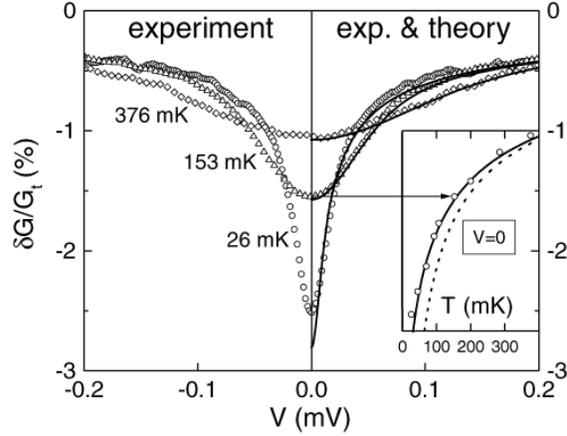


Figure 13.3: Measurement of the tunnel conductance tunnel on aluminium wires as a function of the voltage and for different temperatures. The continuous lines present the theoretical prediction (13.36), with a diffusion coefficient D^* renormalized by the geometry (Exercise 13.9) [319].

The correction to the total density of states is obtained by integrating over r_0 , and it writes ⁹

$$\delta\nu^F = -\frac{\rho_0}{\pi} \text{Im} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \sum_{\mathbf{q}} U(\mathbf{q}, \omega) P_d^2(\mathbf{q}, \omega) . \quad (13.61)$$

In the limit of a static interaction $U(\mathbf{q})$, the result (13.29) is recovered (see Exercise 13.2). On the other hand, for a dynamically screened interaction and in the limit $q \ll \kappa$, we have

$$U(\mathbf{q}, \omega) \simeq U \frac{-i\omega + Dq^2}{Dq^2} . \quad (13.62)$$

In three dimensions, the result (13.28) obtained for a static interaction is recovered, up to a factor 2 (see Exercise 13.6).

We now consider the Hartree term, which is obtained from the corresponding diagram of the Figure 13.4 and gives a contribution δG^H to the local Green's

⁹A tunnel conductance measurement allows access to the *local* density of states $\delta\rho(\mathbf{r}_0, \mathbf{r}_0)$. In a non translation invariant system, the density of states anomaly depends on the measurement point and is proportional to the return probability to \mathbf{r}_0

$$\delta\rho(\epsilon, \mathbf{r}_0) \propto \int_0^{\infty} \frac{\pi T t}{\sinh \pi T t} P(\mathbf{r}_0, \mathbf{r}_0, t) \cos \epsilon t dt . \quad (13.60)$$

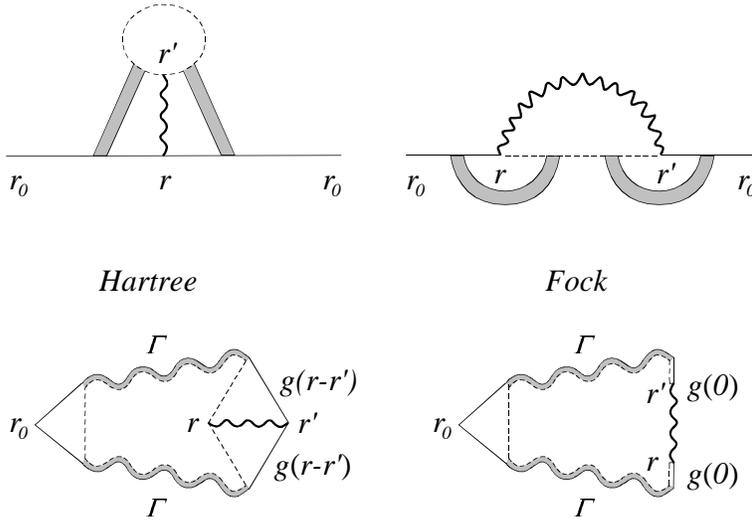


Figure 13.4: *Hartree and exchange (Fock) diagrams for the local density of states anomaly. The Hartree diagram contains the function $g^2(\mathbf{r} - \mathbf{r}')$ and the Fock diagram contains the factor $g^2(0) = 1$.*

function of the form ¹⁰

$$\delta G^H(\mathbf{r}_0, \mathbf{r}_0) = -2\rho_0 \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int P_d(\mathbf{r}_0, \mathbf{r}, \omega) g^2(\mathbf{r} - \mathbf{r}') U(\mathbf{r}, \mathbf{r}') P_d(\mathbf{r}', \mathbf{r}_0, \omega) d\mathbf{r} d\mathbf{r}' . \quad (13.63)$$

For this term, the energy exchange during the interaction is zero, and the interaction $U(\mathbf{r}, \mathbf{r}')$ remains static. Therefore the local density of states correction is

$$\delta \rho^H(\mathbf{r}_0, \mathbf{r}_0) = -2 \frac{\rho_0}{\pi} \text{Im} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \int P_d(\mathbf{r}_0, \mathbf{r}, \omega) g^2(\mathbf{r} - \mathbf{r}') U(\mathbf{r}, \mathbf{r}') P_d(\mathbf{r}', \mathbf{r}_0, \omega) d\mathbf{r} d\mathbf{r}' . \quad (13.64)$$

Integrating over \mathbf{r}_0 and using (13.45), we obtain

$$\delta \nu^H = 2F \frac{U \rho_0}{\pi} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \sum_{\mathbf{q}} \text{Im} P_d^2(\mathbf{q}, \omega) . \quad (13.65)$$

For the Hartree term, the interaction *is not* dynamically screened and the static result (13.33) is recovered. Finally, for a dynamically screened interaction, the correction to the density of states is obtained by adding the two contributions, so that

¹⁰The Hartree (13.63) and Fock (13.58) contributions differ in sign and by a factor 2 which originate respectively from the exchange and the spin.

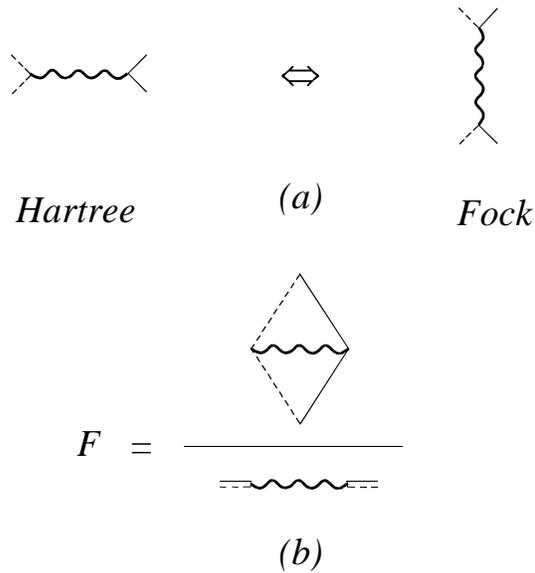


Figure 13.5: The two diagrams, Hartree and Fock, are identical given the correspondence (a). (b) Representation of the parameter F (13.34) as the ratio of two diagrams.

$$\delta\nu = -\frac{\rho_0}{\pi} \int_{-\infty}^{\infty} f(\epsilon - \omega) d\omega \sum_{\mathbf{q}} \text{Im} [(U(\mathbf{q}, \omega) - 2FU)P_d^2(\mathbf{q}, \omega)] \quad (13.66)$$

In three dimensions, Exercise 13.6 shows that, taking into account the dynamically screened interaction, the exchange term is multiplied by a factor 2. The parameter λ_ν , which gives the strength of the density of states anomaly (13.35) thus becomes $\lambda_\nu = 2 - 2F$. Moreover the perturbative calculation which leads to (13.35) and (13.66) is valid only for $F \ll 1$. It can be shown that the prefactor of the exchange term is 2 only to first order in perturbation. To the next order there is a term linear in F (equal to $F/2$) so that the prefactor λ_ν of the Hartree-Fock correction is indeed [315, 321] :

$$\lambda_\nu = 2 - \frac{3F}{2} \quad , \quad d = 3 \quad . \quad (13.67)$$

By definition (13.34), $0 < F < 1$, and the correction is thus always *negative*. For the case of other dimensionalities, see reference [315]¹¹.

¹¹The results presented here are obtained in the perturbative limit, that is when $\delta\nu/\nu \ll 1$. A non-perturbative expression has been obtained in the reference [322].

Remark

Another contribution to the density of states anomaly is also obtained by replacing the Diffusons by Cooperons in the diagrams of Figure 13.4. In this case it can be shown that truncating the perturbation series to lowest order is meaningless, and we must take into account the infinite series (called *Cooper channel renormalization*) of diagrams built with interaction lines and Cooperons. This gives an additional contribution to the density of states anomaly proportional to the Diffuson contribution up to a small multiplicative factor $1/\ln(T_c/\epsilon)$ (or $\ln(\ln T_c\tau_e/\ln T_c/\epsilon)$ in two dimensions). For the Coulomb interaction, T_c is a characteristic energy of the order of the bandwidth. For an attractive interaction, T_c is the superconducting temperature [315].

Exercise 13.6. Show that, in three dimensions, the contribution of the exchange term is multiplied by a factor 2 when the dynamical character of the screening is taken into account.

For a static interaction $U(\mathbf{q}, \omega) = U$, relation (13.61) shows that the exchange term contains the integral

$$U \operatorname{Im} \int_0^\infty \frac{1}{(-i\omega + Dq^2)^2} q^2 dq .$$

If the interaction is dynamically screened, that is for $U(\mathbf{q}, \omega) = U(-i\omega + Dq^2)/Dq^2$, the integral becomes

$$\frac{U}{D} \operatorname{Im} \int_0^\infty \frac{1}{-i\omega + Dq^2} dq .$$

An integration by parts shows that the dynamical screening of the interaction multiplies the result of the static case by a factor 2. For similar calculations in other dimensionalities, see reference [315].

Exercise 13.7. From relations (13.61) and (15.114), show that the exchange term contribution to the density of states anomaly can also be written in the form

$$\delta\rho^F = -\frac{\rho_0}{2\pi\Omega} \int_0^\infty d\omega \left[\tanh \frac{\omega + \epsilon}{2T} + \tanh \frac{\omega - \epsilon}{2T} \right] \sum_{\mathbf{q}} \operatorname{Im}[U(\mathbf{q}, \omega) P_d^2(\mathbf{q}, \omega)] . \quad (13.68)$$

Exercise 13.8. Show that the relations (13.61) and (13.29) are equivalent for a static interaction $U(\mathbf{r} - \mathbf{r}')$.

13.4.4 Capacitive effects

In this section we show how the geometry of the system can play an important role in determining the density of states anomaly. Consider for example a quasi-one-dimensional wire of length L and of squared section W^2 , placed on a metallic electrode and separated from this electrode by a tunnel junction of thickness a . In this case, the Coulomb interaction depends on the capacitance of the junction, and the uniform component of the interaction has the form

$$U(\mathbf{q} = 0, \omega) = \frac{e^2}{C} , \quad (13.69)$$

where C is the capacitance per unit length of the junction. At short distance, that is for large wavevectors (but still in the diffusive limit), the dynamically

screened interaction keeps the form (13.12) provided we replace ρ_0 by the one-dimensional density of states $\rho_{1d} = \rho_0 W^2$:

$$U(\mathbf{q}, \omega) = \frac{1}{2\rho_{1d}} \frac{-i\omega + Dq^2}{Dq^2} , \quad (13.70)$$

In order to account for both behaviors, it is shown in Appendix A13.1 that the interaction has to be written in the form [319, 323, 324]

$$U(\mathbf{q}, \omega) = \frac{-i\omega + Dq^2}{2\rho_{1d}Dq^2 - i\omega C/e^2} \quad (13.71)$$

or equivalently

$$U(\mathbf{q}, \omega) = \frac{1}{2\rho_{1d}} \frac{D^*}{D} \frac{-i\omega + Dq^2}{-i\omega + D^*q^2} , \quad (13.72)$$

where the coefficient $D^* = 2\rho_0 W^2 e^2 D/C \gg D$ can be interpreted as an effective diffusion coefficient which describes the propagation of the electromagnetic field in the junction ¹². Since the conductance of the wire is $\sigma = 2e^2 D \rho_0$, this diffusion coefficient can be rewritten as $D^* = 1/RC$ where $R = 1/(\sigma W^2)$ is the resistance of the wire per unit length and C is the capacitance of the junction per unit length (eq. 13.209).

Exercise 13.9. Calculate the density of states anomaly for a wire placed in the vicinity of a metallic electrode [319, 323].

By inserting interaction (13.72) in the expression (13.66) of the density of states correction and after integrating over \mathbf{q} and ω , we obtain :

$$\delta\rho(\epsilon) = -\frac{1}{2\pi\sqrt{2}} \frac{1}{\sqrt{D\epsilon}} \frac{D^*/D}{1 + \sqrt{D^*/D}} \quad (13.73)$$

for the exchange term at zero temperature. Show that for a wire whose thickness is larger than the screening length, the ratio $D^*/D \propto (\kappa W)^2$ is large. Thus the contribution of the exchange term to the density of states correction (13.36) must be multiplied by the factor $\sqrt{D^*/D}$. Show that the Hartree term is not modified.

Remark : dynamical Coulomb blockade

Another way to describe the tunnel conductance is to relate the density of states anomaly to the impedance of the environment of the system being studied. The relevant quantity is then the probability that an electron crossing the tunnel barrier transfers a given energy to its environment [323, 325, 326]. For a geometry where this environment is the conductor itself, this formulation is equivalent to the approach presented in this chapter.

¹²This effective diffusion coefficient of the electromagnetic field should not be confused with the diffusion coefficient $D^* = v_F l^*/d$ introduced in (4.171) to describe anisotropic collisions.

13.5 Correction to the conductivity

Taking into account the interaction between electrons also leads to a reduction of the conductivity. Without going into the details of the calculations, we can argue that the reduction is a consequence of the correction to the density of states. Both effects result from the scattering of an electron by the charge fluctuations induced by disorder. The temperature dependence of the conductivity $\sigma(T)$ is related to its energy dependence at $T = 0K$ by relation (7.125). Since the conductivity is proportional to the density of states (Einstein relation), we expect that the density of states anomaly leads to a correction of the conductivity given by

$$\frac{\delta\sigma(T)}{\sigma_0} = \int d\epsilon \left(-\frac{\partial f}{\partial \epsilon} \right) \frac{\delta\nu(\epsilon)}{\nu_0} , \quad (13.74)$$

where σ_0 is the Drude conductivity (7.14). For a static interaction, the density of states correction is given by (13.35), and using (15.109) we have ¹³

$$\delta\sigma(T) = -\lambda_\sigma \left(\frac{e^2 D}{\pi \Omega} \right) \int_0^\infty \left(\frac{\pi T t}{\sinh \pi T t} \right)^2 Z(t) dt \quad (13.75)$$

with $\lambda_\sigma = 1 - 2F$. As for the density of states correction, this value corresponds to a static interaction. For a quasi-1d system, $Z(t) = \Omega/\sqrt{4\pi Dt}$, so that with the help of (15.89), we obtain

$$\delta\sigma(T) = -\lambda_\sigma \frac{e^2}{\pi^2} \frac{3}{8} \sqrt{\frac{\pi}{2}} \zeta\left(\frac{3}{2}\right) \left(\frac{D}{T}\right)^{1/2} . \quad (13.76)$$

In two dimensions $Z(t) = \Omega/4\pi Dt$, and in the limit $T\tau_e \ll 1$, we obtain, using (15.90),

$$\delta\sigma(T) = -\lambda_\sigma \frac{e^2}{4\pi^2} \ln \frac{e^\gamma}{2\pi T \tau_e} \quad (13.77)$$

where $\gamma \simeq 0.577$ is the Euler constant. In three dimensions, $Z(t) = \Omega/(4\pi Dt)^{3/2}$ and (15.93) lead to

$$\delta\sigma(T) = -\lambda_\sigma \frac{e^2}{\pi^2} \frac{\sqrt{\pi}}{8\sqrt{2}} \zeta\left(\frac{1}{2}\right) \left(\frac{T}{D}\right)^{1/2} \quad (13.78)$$

up to a subtractive constant.

Like for the density of states anomaly, the results differ slightly when the dynamic character of the interaction is taken into account. A full-fledged treatment shows that the results (13.76, 13.77, 13.78) are still valid provided λ_σ is given by (see Exercise 13.11) :

$$\lambda_\sigma = \frac{4}{d} - \frac{3F}{2} . \quad (13.79)$$

¹³In order to compare this relation and the following ones with the weak localization in chapter 7, we have to reintroduce \hbar . Result (13.75) can also be obtained using another method called "quasi-classical" and is presented in this form in reference [327].

Remark

Like for the density of states anomaly, another contribution to $\delta\sigma(T)$ is obtained by replacing Diffusons by Cooperons (see remark p. 505). This contribution is proportional to the Diffuson contribution times a reduction term $1/\ln(T_c/T)$ (or $\ln(\ln T_c\tau_e/\ln T_c/T)$ in two dimensions). For the Coulomb interaction, T_c is a characteristic energy of the order of the bandwidth so that $T \ll T_c$. The correction is thus negative. For an attractive interaction, T_c is the superconducting temperature, which leads, even for $T \gg T_c$, to an increase in conductivity. There are two other classes of diagrams. The so-called Maki-Thomson diagrams describe the diffusion of electrons by superconducting fluctuations and give a correction proportional to the weak localization correction [329, 330]. The so-called Aslamasov-Larkin correction is related to the Cooper pair fluctuations and is important only in the vicinity of T_c . For a review of theoretical and experimental results on this subject, see [315, 331].

It is interesting to compare the correction (13.75) to the conductivity with the weak localization correction (7.53). Both are of the same order. The physical mechanisms at the origin of these corrections are different, but both are related to the integrated return probability. The temperature dependencies, however, are different. In the case of the weak localization correction, the temperature enters only through the phase coherent time $\tau_\phi(T) \propto T^{-p}$ (section 7.4.3), leading to different temperature dependencies in $d = 1$ and in $d = 3$, whereas they are both logarithmic in two dimensions (compare (7.66) with (13.76, 13.77) and (13.78)). The usual way to extract experimentally the correction due to electron-electron interactions is to apply a magnetic field in order to suppress the weak localization correction.

Exercise 13.10. Show that

$$\int_{-\infty}^{\infty} [f(\epsilon - \omega) - f(\epsilon + \omega)] \left(-\frac{\partial f}{\partial \epsilon} \right) d\epsilon = \frac{\partial}{\partial \omega} \left(\omega \coth \frac{\beta\omega}{2} \right) \quad (13.80)$$

Exercise 13.11. Show that the correction to the conductivity for a static interaction $U = 1/(2\rho_0)$ can be written in the form

$$\frac{\delta\sigma}{\sigma_0} = -\frac{1-2F}{4\pi\nu_0} \int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial \omega} \left(\omega \coth \frac{\beta\omega}{2} \right) \sum_{\mathbf{q}} \text{Im} P_d^2(\mathbf{q}, \omega) . \quad (13.81)$$

To that purpose, start from the relation (13.42) for the density of states correction. For an interaction $U(\mathbf{q}, \omega)$, it is shown in reference [328] that the correction to the exchange term writes (see next exercise) :

$$\frac{\delta\sigma}{\sigma_0} = -\frac{2}{\pi d\Omega} \int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial \omega} \left(\omega \coth \frac{\beta\omega}{2} \right) \sum_{\mathbf{q}} Dq^2 \text{Im}[U(\mathbf{q}, \omega) P_d^3(\mathbf{q}, \omega)] . \quad (13.82)$$

Show that for the dynamically screened interaction $U(\mathbf{q}, \omega) = U \frac{-i\omega + Dq^2}{Dq^2}$, the correction due to exchange is :

$$\frac{\delta\sigma}{\sigma_0} = -\frac{1}{\pi\nu_0 d} \int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial \omega} \left(\omega \coth \frac{\beta\omega}{2} \right) \sum_{\mathbf{q}} \text{Im} P_d^2(\mathbf{q}, \omega) \quad (13.83)$$

and differs from the exchange term in (13.81) only by a factor $4/d$.

Exercise 13.12. Correction to the conductivity

Reference [328] presents the calculation of the interaction contribution to the conductivity. For the exchange term, this calculation involves the three diagrams of Figure 13.6. Show that the sum of these three diagrams is zero.

To prove this, it is useful to redraw the diagrams in a different way so that the long range and short range parts are more explicitly separated (Figure 13.6). This representation involves Hikami boxes whose structure is similar to the boxes used in diagrams for conduction fluctuations. It can be shown that their sum is zero, by using the results of section 11.2.1. From (11.17), the sum of diagrams (a) + (b) is proportional to :

$$2 \frac{k_F^2}{d} (H^{(A)} + H^{(B)}) = 2 \frac{k_F^2}{d} \tilde{H}' = 2 \frac{k_F^2}{d} 2\pi\rho_0\tau_e^3 . \quad (13.84)$$

The factor 2 accounts for diagrams similar to (a) and (b) but where the retarded and advanced parts have been exchanged. Diagram (c) is proportional to

$$-\frac{k_F^2}{d} H^{(A)} = -\frac{k_F^2}{d} 4\pi\rho_0\tau_e^3 . \quad (13.85)$$

The $-$ sign comes from the average of incoming momenta which are opposite. The sum of the three diagrams is thus zero.

The non-vanishing diagrams contributing to the conductivity correction are represented in Figure 13.7. They are built of two retarded (or two advanced) Green's functions. From relation 7.1, we know that they are smaller than the above diagrams by a factor $1/k_F l_e$. However, they are divergent. Using the expansion $\overline{G}(\mathbf{k} - \mathbf{q}) = \overline{G}(\mathbf{k}) - \mathbf{v} \cdot \mathbf{q} \overline{G}(\mathbf{k})^2$, show that each "triangle" is proportional to the wavevector and that both diagrams are thus proportional to :

$$\sum_{\mathbf{q}} q_x^2 \text{Im} P_d^3(\mathbf{q}, \omega) U(\mathbf{q}, \omega) . \quad (13.86)$$

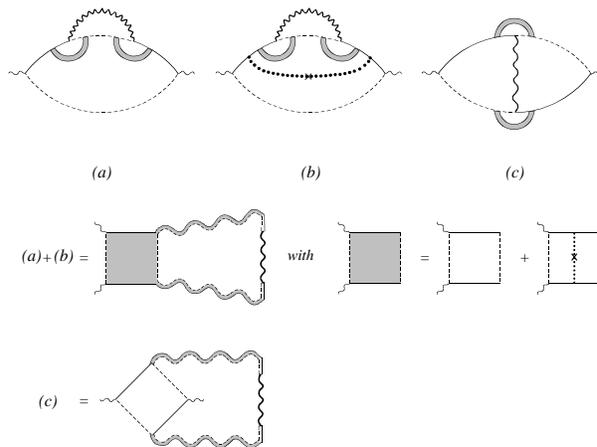


Figure 13.6: *Diagrams for the exchange contribution to the conductivity. The topologically equivalent representation drawn below makes it clear that their sum is zero. The Hartree terms, whose sum is also zero, are obtained by the transformation displayed in Figure 13.5.*

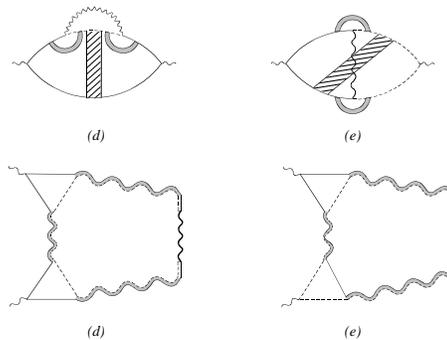


Figure 13.7: *Conductivity diagrams (exchange). The two upper diagrams are drawn in the usual representation, whereas the two lower diagrams provide a topologically equivalent representation.*

13.6 Lifetime of quasiparticle

13.6.1 Introduction : Landau theory and disorder

The Coulomb interaction between two electrons is strongly screened by the presence of the other electrons (section 13.2). Each electron, “dressed” by the screening cloud, is called a *quasiparticle*. The Landau theory of “Fermi liquids” formalizes this concept of quasiparticle and shows that their properties are essentially the same as those of non-interacting electrons, provided a renormalization of physical parameters such as the electron mass [332].

In fact, the quasiparticles weakly interact via the screened interaction, also

called residual interaction, and, because of this interaction, a quasiparticle acquires a finite lifetime. The Landau theory relies on the fact that this quasiparticle lifetime diverges near the Fermi level.

In this section, we study the quasiparticle lifetime and show that the disorder plays an essential role. The determination of this lifetime is crucial to determining whether the low energy properties, *i.e.*, close to the Fermi level, can still be described within the framework of the Fermi liquid theory, that is with non interacting quasiparticles. This is also important in order to understand what limits phase coherence in the interacting electron gas, a central question in mesoscopic physics.

In the absence of residual interaction, a quasiparticle has an infinite lifetime. With the interaction, the probability $\mathcal{P}(t)$ that a quasiparticle stays in its initial state has the form [332] :

$$\mathcal{P}(t) = e^{-t/\tau_{ee}(\epsilon)} \quad , \quad (13.87)$$

where $\tau_{ee}(\epsilon, T)$ defines the lifetime of the quasiparticle. This lifetime depends on the energy ϵ measured with respect to the Fermi level, as well as on the temperature T . In the absence of disorder, Landau has shown that, in three dimensions, the lifetime τ_{ee} of a quasiparticle is given by

$$\frac{1}{\tau_{ee}(\epsilon, T)} \simeq \max\left(\frac{\epsilon^2}{\epsilon_F}, \frac{T^2}{\epsilon_F}\right) \quad . \quad (13.88)$$

Near the Fermi level, the quasiparticle is well defined since the width $1/\tau_{ee}(\epsilon)$ of a state goes to zero more rapidly than its energy ϵ when approaching the Fermi level (see Appendix A13.2).

13.6.2 Lifetime at zero temperature

In this section, we show that in the presence of disorder, multiple scattering increases the probability that two quasiparticles interact and thus reduces the electronic lifetime. More precisely, in the diffusion approximation and at zero temperature, the lifetime can be written [315] ¹⁴

$$\frac{1}{\tau_{ee}(\epsilon)} \simeq \Delta \left(\frac{\epsilon}{E_c}\right)^{d/2} \quad \epsilon \gg E_c \quad . \quad (13.89)$$

The dependence of this power law on space dimensionality d is the signature of the diffusive regime. E_c is the Thouless energy and $\Delta = 1/\nu_0$ is the average level spacing at the Fermi energy in the absence of interaction. This expression is limited to the case where the excitation energy ϵ is larger than E_c . In the opposite limit, the lifetime varies as [333]

$$\frac{1}{\tau_{ee}(\epsilon)} \simeq \Delta \left(\frac{\epsilon}{E_c}\right)^2 \quad \epsilon \ll E_c \quad . \quad (13.90)$$

¹⁴One might expect that the temperature dependence of $1/\tau_{ee}(T)$ could be obtained by replacing ϵ by T as in (13.88) for the ballistic case, thus leading to $1/\tau_{ee}(T) \propto T^{d/2}$. In section 13.6.3, we show that this is not correct for $d \leq 2$.

In this section, we derive these two limiting behaviors. In order to evaluate the electronic lifetime, we consider an eigenstate $|\alpha\rangle$ of the non-interacting disordered Hamiltonian, whose energy ϵ_α is above the Fermi level¹⁵. This state interacts with a filled state $|\gamma\rangle$ of energy ϵ_γ (Figure 13.8). The lifetime of the state $|\alpha\rangle$ is given by the Fermi golden rule

$$\frac{1}{\tau_\alpha} = 4\pi \sum_{\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon_\alpha + \epsilon_\gamma - \epsilon_\beta - \epsilon_\delta) . \quad (13.91)$$

A factor 2 accounts for the spin degeneracy of the state $|\gamma\rangle$. The matrix element $\langle\alpha\gamma|U|\beta\delta\rangle$ describes the interaction between the two states $|\alpha\rangle$ and $|\gamma\rangle$ which evolve into the two final states $|\beta\rangle$ and $|\delta\rangle$. Let us notice that $\epsilon_\gamma < 0$ and that the final states must be empty so that their energies obey the constraints $\epsilon_\beta > 0$ and $\epsilon_\delta > 0$.

Exercise 13.13. Check that, for a constant matrix element, the lifetime $\tau_{ee}(\epsilon)$ given by (13.91) varies as $1/\epsilon^2$.

The lifetime has the form

$$\frac{1}{\tau_{ee}(\epsilon)} = 2\pi U^2 \nu_f(\epsilon) ,$$

where $\nu_f(\epsilon) = 2 \sum'_{\beta\gamma\delta} \delta(\epsilon + \epsilon_\gamma - \epsilon_\beta - \epsilon_\delta)$ is the density of final states. The sum \sum' is limited to states such that $\epsilon_\gamma < 0$, $\epsilon_\beta > 0$ and $\epsilon_\delta > 0$. Replacing this sum by integrals and using the density of states ν_0 , we obtain

$$\nu_f(\epsilon) = 2\nu_0^3 \int_{-\infty}^0 d\epsilon_\gamma \int_0^\infty d\epsilon_\beta \int_0^\infty d\epsilon_\delta \delta(\epsilon + \epsilon_\gamma - \epsilon_\beta - \epsilon_\delta) = \nu_0^3 \epsilon^2 .$$

If the matrix elements do not depend on the energy, the lifetime varies as $1/\epsilon^2$. This energy dependence is simply related to the density of final states, whence the dependence (13.88) obtained by Landau [332] (see also Appendix A13.2).

In order not to single out a given state, we must average the lifetime over all states having the same energy ϵ . Thus, we calculate

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{4\pi}{\nu_0} \sum_{\alpha\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon_\alpha + \epsilon_\gamma - \epsilon_\beta - \epsilon_\delta) \delta(\epsilon - \epsilon_\alpha) . \quad (13.92)$$

Denoting by ϵ' the energy of the states $|\gamma\rangle$, energy conservation implies that the final states $|\beta\rangle$ and $|\delta\rangle$ have energies $\epsilon - \omega$ and $\epsilon' + \omega$, where ω is the energy transfer due to the interaction (Figure 13.8). The inverse lifetime becomes

$$\begin{aligned} \frac{1}{\tau_{ee}(\epsilon)} &= \frac{4\pi}{\nu_0} \int_0^\epsilon d\omega \int_{-\omega}^0 d\epsilon' \sum_{\alpha\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \\ &\times \delta(\epsilon - \epsilon_\alpha) \delta(\epsilon' - \epsilon_\gamma) \delta(\epsilon - \omega - \epsilon_\beta) \delta(\epsilon' + \omega - \epsilon_\delta) . \end{aligned} \quad (13.93)$$

¹⁵More precisely, we consider the non-interacting quasiparticle states, whose spectrum is assumed to have the same statistical properties as the non-interacting electrons.

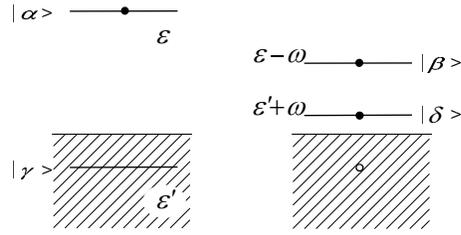


Figure 13.8: A quasiparticle in a state $|\alpha\rangle$ of energy $\epsilon_\alpha = \epsilon$ interacts with another quasiparticle $|\gamma\rangle$ of energy $\epsilon_\gamma = \epsilon'$ in the Fermi sea. The final state is made of two quasiparticles above the Fermi sea and one hole.

Upon averaging over disorder, we obtain for the lifetime

$$\boxed{\frac{1}{\tau_{ee}(\epsilon)} = 4\pi\nu_0^3 \int_0^\epsilon \omega W^2(\omega) d\omega} \quad (13.94)$$

with

$$W^2(\omega) = \frac{1}{\nu_0^4} \sum_{\alpha\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon - \epsilon_\alpha) \delta(\epsilon' - \epsilon_\gamma) \delta(\epsilon - \omega - \epsilon_\beta) \delta(\epsilon' + \omega - \epsilon_\delta) . \quad (13.95)$$

We will see that the characteristic matrix element $W(\omega)$ depends only on the energy transfer ω ¹⁶. In the literature, the interaction “kernel”, defined by $K(\omega) = 4\pi\nu_0^3 W^2(\omega)$, is frequently used, for instance to rewrite the inverse lifetime as $1/\tau_{ee}(\epsilon) = \int_0^\epsilon \omega K(\omega) d\omega$.

We now calculate $1/\tau_{ee}$ at the diffusion approximation. The matrix element $\langle\alpha\gamma|U|\beta\delta\rangle$, calculated in the basis of the eigenfunctions $\phi_i(\mathbf{r})$ of the Hamiltonian (2.1), is

$$\langle\alpha\gamma|U|\beta\delta\rangle = \int d\mathbf{r}_1 d\mathbf{r}_2 \phi_\alpha^*(\mathbf{r}_1) \phi_\gamma^*(\mathbf{r}_2) \phi_\beta(\mathbf{r}_1) \phi_\delta(\mathbf{r}_2) U_\omega(\mathbf{r}_1 - \mathbf{r}_2) , \quad (13.96)$$

where $U_\omega(\mathbf{r})$ is the dynamically screened potential. Making use of (3.26) which relates the wave functions to the non-local density of states $\rho_\epsilon(\mathbf{r}, \mathbf{r}')$, $W^2(\omega)$ can be rewritten as

$$W^2(\omega) = \frac{1}{\nu_0^4} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}'_1 d\mathbf{r}'_2 U_\omega(\mathbf{r}_1 - \mathbf{r}_2) U_\omega(\mathbf{r}'_1 - \mathbf{r}'_2) \\ \times \frac{\rho_\epsilon(\mathbf{r}_1, \mathbf{r}'_1) \rho_{\epsilon-\omega}(\mathbf{r}'_1, \mathbf{r}_1)}{\rho_{\epsilon'}(\mathbf{r}_2, \mathbf{r}'_2) \rho_{\epsilon'+\omega}(\mathbf{r}'_2, \mathbf{r}_2)} , \quad (13.97)$$

where the disorder average of the product of four Green's functions has been decoupled into a product of two averages. To evaluate $W^2(\omega)$, we use (4.207) and (3.99) so that¹⁷

¹⁶For that reason, the integral over ϵ' in (13.93) provides simply a factor ω .

¹⁷The Cooperon gives a negligible contribution.

$$\overline{\rho_\epsilon(\mathbf{r}, \mathbf{r}')\rho_{\epsilon-\omega}(\mathbf{r}', \mathbf{r})} = \frac{\rho_0}{\pi} \text{Re}P_d(\mathbf{r}, \mathbf{r}', \omega) + \rho_0^2 g^2(\mathbf{R}) \quad , \quad (13.98)$$

whose Fourier transform is given by

$$\frac{\rho_0}{\pi} \left(\text{Re}P_d(\mathbf{q}, \omega) + \frac{\pi}{2|q|v_F} \theta(2k_f - |q|) \right) \quad . \quad (13.99)$$

The second term is independent of disorder and gives exactly the Landau contribution (13.216) coming from large values of q . For small q ($ql_e \ll 1$), the contribution of the Diffuson dominates, so that (13.97) becomes

$$\begin{aligned} W^2(\omega) &= \frac{1}{\pi^2 \nu_0^2 \Omega^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}'_1 d\mathbf{r}'_2 U_\omega(\mathbf{r}_1 - \mathbf{r}_2) U_\omega(\mathbf{r}'_1 - \mathbf{r}'_2) \\ &\times \text{Re}P_d(\mathbf{r}_1, \mathbf{r}'_1, \omega) \text{Re}P_d(\mathbf{r}_2, \mathbf{r}'_2, -\omega) \quad , \end{aligned} \quad (13.100)$$

or, after a Fourier transform

$$W^2(\omega) = \frac{1}{\pi^2 \nu_0^2 \Omega^2} \sum_{\mathbf{q} \neq 0} |U(\mathbf{q}, \omega)|^2 [\text{Re}P_d(\mathbf{q}, \omega)]^2 \quad . \quad (13.101)$$

A diagrammatic representation of this quantity is shown on Figure 13.9. In the

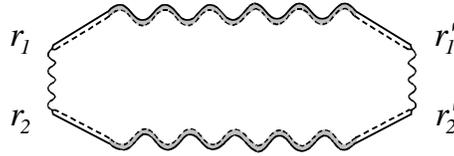


Figure 13.9: Diagrammatic representation of $W^2(\omega)$ as given by relation (13.100). Notice that this structure is quite similar to the one of the density of states correlation function $K_\rho(\omega)$ (Figure 10.11).

diffusion approximation, the dynamically screened potential is given by (13.12) and therefore¹⁸

$$W^2(\omega) = \frac{1}{4\pi^2 \nu_0^4} \sum_{\mathbf{q} \neq 0} \frac{1}{\omega^2 + D^2 q^4} \quad , \quad (13.104)$$

¹⁸An equivalent expression of $W^2(\omega)$ is

$$W^2(\omega) = \frac{1}{4\pi^2 \nu_0^4} \sum_{\mathbf{q} \neq 0} \frac{1}{Dq^2} \text{Re}P_d(\mathbf{q}, \omega) = \frac{1}{4\pi^2 \nu_0^4} \sum_{\mathbf{q} \neq 0} \frac{1}{\omega} \text{Im}P_d(\mathbf{q}, \omega) \quad , \quad (13.102)$$

which yields for the lifetime :

$$\frac{1}{\tau_{e\epsilon}(\epsilon)} = \frac{1}{\pi \nu_0} \int_0^\epsilon \omega d\omega \sum_{\mathbf{q} \neq 0} \frac{1}{Dq^2} \text{Re}P_d(\mathbf{q}, \omega) = \frac{1}{\pi \nu_0} \int_0^\epsilon d\omega \sum_{\mathbf{q} \neq 0} \text{Im}P_d(\mathbf{q}, \omega) \quad . \quad (13.103)$$

which depends solely on ω and can be expressed in terms of the integrated return probability $Z(t)$

$$W^2(\omega) = \frac{1}{4\pi^2\nu_0^4} \frac{1}{\omega} \int_0^\infty Z(t) \sin \omega t \, dt . \quad (13.105)$$

Finally, the electronic lifetime (13.94) writes

$$\boxed{\frac{1}{\tau_{ee}(\epsilon)} = \frac{2}{\pi\nu_0} \int_0^\infty \frac{Z(t)}{t} \sin^2 \frac{\epsilon t}{2} dt} \quad (13.106)$$

For a metal of volume Ω , we can identify two different regimes :

- For $t \ll \tau_D$, where τ_D is the Thouless time (5.34), an electron described as a diffusive wave packet is insensitive to the boundaries and behaves as in an infinite medium where, according to (5.24), $Z(t) = \Omega/(4\pi Dt)^{d/2}$. From (15.86), we obtain for the integral (13.105)¹⁹

$$W^2(\omega) = \frac{dc_d}{16} \frac{1}{\nu_0^4 \omega^2} \left(\frac{\omega}{E_c} \right)^{d/2} \quad (13.107)$$

so that the electronic lifetime is equal to

$$\boxed{\frac{1}{\tau_{ee}(\epsilon)} = \frac{\pi}{2} c_d \Delta \left(\frac{\epsilon}{E_c} \right)^{d/2}} \quad (\epsilon \gg E_c) \quad (13.108)$$

where c_d is a constant defined in (10.63), with $c_1 = \sqrt{2}/\pi^2$, $c_2 = 1/4\pi^2$, $c_3 = \sqrt{2}/6\pi^3$. Such a behavior has been indeed observed in silver wires ($d = 1$) for which $W^2(\omega) \propto \omega^{-3/2}$ and $1/\tau_{ee}(\epsilon) \propto \epsilon^{1/2}$, although the measured prefactor came out to be larger than the value predicted here [323, 334]. Other behaviors in disagreement with those results have been observed in gold and copper wires and have been attributed to other relaxation mechanisms such as the coupling to two-levels systems [334] or magnetic impurities [335].

¹⁹The small time regime corresponds to the case where the excitation energy ϵ is much larger than the Thouless energy, $\epsilon \gg E_c$. In this case, the sum (13.104) on wavevectors can be replaced by an integral and we recover (13.107).

Remarks**• Screening effect**

The behavior of the lifetime depends only weakly on the exact nature of the screened potential. Indeed, assuming a static potential, that is $U(\mathbf{q}, \omega = 0) = \Omega/2\nu_0$ instead of a dynamically screened potential, the sum (13.104) becomes

$$W^2(\omega) = \frac{1}{4\pi^2\nu_0^4} \sum_{\mathbf{q} \neq 0} \frac{D^2 q^4}{(\omega^2 + D^2 q^4)^2} . \quad (13.109)$$

A high frequencies $\omega \gg E_c$, the sum can be replaced by an integral and we recover a power law similar to (13.107) :

$$W^2(\omega) \propto \frac{1}{\nu_0^4 \omega^2} \left(\frac{\omega}{E_c} \right)^{d/2} , \quad (13.110)$$

where only the prefactor has been modified.

• Lifetime and spectral rigidity

It is interesting to compare the expressions of $1/\tau_{ee}(\epsilon)$ and of the variance $\Sigma^2(E)$ of the distribution of energy levels in the diffusive regime (relation 10.52). We find

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{\pi\Delta}{2} [\Sigma^2(\epsilon) - \Sigma_0^2(\epsilon)] \quad (13.111)$$

where the contribution $\Sigma_0^2(\epsilon)$ of the zero mode has been subtracted. This relation can also be understood from the similarity of the diagrams for the correlation function $K(\omega)$ and for $W^2(\omega)$ shown respectively on Figures 10.11 and 13.9.

• The limit $t \gg \tau_D$ corresponds to the ergodic regime in which the diffusive electronic wave packet explores all the accessible volume Ω . Thus we would expect $Z(t)$ to be driven only by the zero mode. This is not so, because in expression (13.104) this mode has been removed in order to ensure electronic neutrality. The excitation energy ϵ is smaller than E_c and it is not possible to replace the sum (13.104) by an integral. In this limit, we obtain

$$W^2(\omega) = \frac{a_d}{4\pi^6} \frac{\Delta^4}{E_c^2} \propto \frac{\Delta^2}{g^2} , \quad (13.112)$$

where the coefficient a_d is defined by the series

$$a_d = \sum_{n_x, n_y, n_z} \frac{1}{(n_x^2 + n_y^2 + n_z^2)^2} . \quad (13.113)$$

The ratio E_c/Δ is the dimensionless conductance g defined by (7.25). For $\omega \ll E_c$, the characteristic matrix element of the interaction is thus energy independent and of order Δ/g . The inverse lifetime in this case [333] is

$$\boxed{\frac{1}{\tau_{ee}(\epsilon)} = \frac{a_d}{2\pi^5} \Delta \left(\frac{\epsilon}{E_c} \right)^2} \quad (\epsilon \ll E_c) \quad (13.114)$$

It is noteworthy that this lifetime depends on boundary conditions through the coefficient a_d . For example, if the sample is connected to reservoirs only in the

Ox direction, the boundary conditions $n_x \in \mathbb{N}^*$, $n_y \in \mathbb{N}$, $n_z \in \mathbb{N}$ are the same as in the sum (11.32) for conductance fluctuations, *i.e.*, $a_d = b_d$. For an isolated sample, only the mode $n_x = n_y = n_z = 0$ is excluded, so that $a_1 = b_1 = \pi^4/90$, $a_2 = b_1 + b_2 = 2.59$ and $a_3 = b_1 + b_2 + b_3 = 5.11$.

Exercise 13.14. Show that there is an additional contribution to (13.97) which involves the product $\overline{\rho_\epsilon(\mathbf{r}_1, \mathbf{r}'_1) \rho_{\epsilon'+\omega}(\mathbf{r}'_2, \mathbf{r}_2)} \overline{\rho_{\epsilon'}(\mathbf{r}_2, \mathbf{r}'_2) \rho_{\epsilon-\omega}(\mathbf{r}'_1, \mathbf{r}_1)}$. Draw the corresponding diagram with the help of figures 13.9 and 13.5. Show that this contribution is of order F^2 [315]. Note that this term does not depend only on ω as in (13.97), but also on $\epsilon - \epsilon' - \omega$.

Remark : lifetime and dielectric function

The previous expressions for the electronic lifetime can be reformulated to introduce the dielectric function $\epsilon(\mathbf{q}, \omega)$. To do this, we show from (13.10) that the following identity holds :

$$\frac{1}{2\rho_0} \text{Im} \left[\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right] = \omega \frac{4\pi e^2}{q^2} \frac{\text{Re}P_d(\mathbf{q}, \omega)}{|\epsilon(\mathbf{q}, \omega)|^2} , \quad (13.115)$$

so that the combination $|U(\mathbf{q}, \omega)|^2 [\text{Re}P_d(\mathbf{q}, \omega)]^2$ which enters expression (13.101) for $W^2(\omega)$ fulfills :

$$\omega |U(\mathbf{q}, \omega)|^2 [\text{Re}P_d(\mathbf{q}, \omega)]^2 = \frac{4\pi e^2}{2\rho_0 q^2} \text{Im} \left[-\frac{1}{\epsilon(\mathbf{q}, \omega)} \right] \text{Re}P_d(\mathbf{q}, \omega) . \quad (13.116)$$

For $W^2(\omega)$, this leads to

$$W^2(\omega) = \frac{1}{2\pi^2 \nu_0^3 \Omega} \sum_{\mathbf{q}} \frac{4\pi e^2}{q^2 \omega} \text{Im} \left[\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right] \text{Re}P_d(\mathbf{q}, \omega) , \quad (13.117)$$

and for the lifetime (13.94) :

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{2}{\pi \Omega} \int_0^\epsilon d\omega \sum_{\mathbf{q}} \frac{4\pi e^2}{q^2} \text{Im} \left[\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right] \text{Re}P_d(\mathbf{q}, \omega) . \quad (13.118)$$

13.6.3 Quasiparticle lifetime at finite temperature

The time $\tau_{ee}(\epsilon)$ represents the lifetime of a quasiparticle injected above the Fermi sea at $T = 0K$. At finite temperature, the probability for a quasiparticle to stay in its initial state is assumed to keep the form [332]

$$\mathcal{P}(t, \epsilon, T) = e^{-t/\tau_{ee}(\epsilon, T)} , \quad (13.119)$$

where $\tau_{ee}(\epsilon, T)$ is the quasiparticle lifetime at finite energy and finite temperature. To calculate this lifetime, we just need to include the Fermi factors in (13.94) [315]²⁰ :

²⁰Notice that we consider here the temperature effect coming from the Fermi statistics and not from the coupling to other degrees of freedom such as phonons.

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\epsilon' F(\epsilon, \epsilon', \omega) W^2(\omega) \quad (13.120)$$

where $F(\epsilon, \epsilon', \omega)$ is a combination of Fermi factors $f_\epsilon = 1/(e^{\beta\epsilon} + 1)$:

$$F(\epsilon, \epsilon', \omega) = f_{\epsilon'}(1 - f_{\epsilon-\omega})(1 - f_{\epsilon'+\omega}) + (1 - f_{\epsilon'})f_{\epsilon-\omega}f_{\epsilon'+\omega} . \quad (13.121)$$

The first term in this expression is larger when $\epsilon > 0$. It describes the decay of an electron-like state above the Fermi level. The second term dominates when $\epsilon < 0$ and describes the decay of a hole-like state into the Fermi sea. For $\epsilon = 0$, both terms are equal. Integrating upon ϵ' (relation 15.115), we obtain

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega \omega W^2(\omega) f_{\epsilon-\omega} \frac{e^{\beta\epsilon} + 1}{e^{\beta\omega} - 1} . \quad (13.122)$$

This lifetime can also be obtained from the imaginary part of the self-energy of a quasiparticle in the presence of a screened interaction [317]²¹. At zero temperature, we recover the result (13.108).

Remark : relaxation towards equilibrium

The time $\tau_{ee}(\epsilon, T)$ can be also interpreted as the relaxation time towards Fermi equilibrium distribution. This relaxation is defined from the Boltzmann equation [315, 336]

$$\begin{aligned} \frac{\partial n_\epsilon}{\partial t} &= -4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega W^2(\omega) \\ &\times \int_{-\infty}^{\infty} d\epsilon' [n_\epsilon n_{\epsilon'}(1 - n_{\epsilon-\omega})(1 - n_{\epsilon'+\omega}) - n_{\epsilon-\omega} n_{\epsilon'+\omega}(1 - n_\epsilon)(1 - n_{\epsilon'})] . \end{aligned} \quad (13.123)$$

The relaxation term contains two contributions which describes respectively the quasiparticles leaving a given quantum state (“out” contribution) and reaching this state (“in” contribution). At equilibrium, n_ϵ is equal to the Fermi factor $f_\epsilon = 1/(e^{\beta\epsilon} + 1)$ and the term in brackets is zero. By linearizing around the equilibrium distribution $n_\epsilon = f_\epsilon + \delta n_\epsilon$, we obtain the equation

$$\begin{aligned} \frac{\partial \delta n_\epsilon}{\partial t} &= -4\delta n_\epsilon \pi\nu_0^3 \int_{-\infty}^{\infty} d\omega W^2(\omega) \\ &\times \int_{-\infty}^{\infty} d\epsilon' [f_{\epsilon'}(1 - f_{\epsilon-\omega})(1 - f_{\epsilon'+\omega}) + (1 - f_{\epsilon'})f_{\epsilon-\omega}f_{\epsilon'+\omega}] \end{aligned} \quad (13.124)$$

The r.h.s. term is of the form $-\delta n_\epsilon/\tau_{ee}(\epsilon, T)$. Thus $\tau_{ee}(\epsilon, T)$ can be interpreted as the relaxation time towards equilibrium distribution.

²¹Using (15.117), the relation (13.122) can also be written in the form [317] :

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 2\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega W^2(\omega) \omega \left(\coth \frac{\beta\omega}{2} + \tanh \frac{\beta}{2}(\epsilon - \omega) \right) .$$

13.6.4 Quasiparticle lifetime at the Fermi level

We now consider more specifically the lifetime of a quasiparticle *near the Fermi level* ($\epsilon = 0$) and *at finite temperature*. Physical properties such as conductance are expressed in terms of *single particle* states at the Fermi level. It is thus essential to understand the range of validity of the description in terms of independent quasiparticles. In the following, we denote by

$$\tau_{in}(T) = \tau_{ee}(\epsilon = 0, T) \quad (13.125)$$

the relaxation time of a quasiparticle at the Fermi level²². From relation (13.122), we have [337]²³

$$\frac{1}{\tau_{in}(T)} = \frac{1}{\tau_{ee}(0, T)} = 8\pi\nu_0^3 \int_0^\infty d\omega W^2(\omega) \frac{\omega}{\sinh \beta\omega} . \quad (13.127)$$

From relations (13.127) and (13.105), it is also possible to obtain an expression of the lifetime as a function of the return probability $Z(t)$:

$$\frac{1}{\tau_{in}(T)} = \frac{2}{\pi\nu_0} \int_0^\infty \frac{d\omega}{\sinh \beta\omega} \int_0^\infty dt Z(t) \sin \omega t \quad (13.128)$$

or, using (15.116) :

$$\frac{1}{\tau_{in}(T)} = \frac{T}{2\nu_0} \int_0^\infty Z(t) \tanh \frac{\pi T t}{2} dt . \quad (13.129)$$

For the diffusion in free space, $Z(t)$ is given by (5.24), so that the time integral (13.128) is proportional to $\omega^{d/2-1}$ (relation 15.28)

$$\frac{1}{\tau_{in}(T)} = \frac{\pi d c_d}{2\nu_0} \int_0^\infty \frac{d\omega}{\omega \sinh \beta\omega} \left(\frac{\omega}{E_c} \right)^{d/2} . \quad (13.130)$$

Therefore, in three dimensions, we have

$$\frac{1}{\tau_{in}(T)} = \frac{\sqrt{2}}{4\pi^2\nu_0} \int_0^\infty \frac{d\omega}{\omega \sinh \beta\omega} \left(\frac{\omega}{E_c} \right)^{3/2} \simeq \frac{T}{\nu_0} \int_0^T \frac{d\omega}{\omega^2} \left(\frac{\omega}{E_c} \right)^{3/2} \quad (13.131)$$

so that

²²We should not confuse the lifetime of a quasiparticle at the Fermi level and at finite temperature, $\tau_{in}(T) = \tau_{ee}(\epsilon = 0, T)$, with the time $\tau_{ee}(\epsilon = T, T = 0)$ sometimes introduced in the literature by means of the substitution $\epsilon \rightarrow T$ in the expression of the zero temperature relaxation time. This second time has no physical significance.

²³From relations (13.127) and (13.117), we can express the quasiparticle relaxation time in terms of the dielectric function

$$\frac{1}{\tau_{in}(T)} = \frac{4}{\pi\Omega} \int_0^\infty \frac{d\omega}{\sinh \beta\omega} \sum_{\mathbf{q}} \frac{4\pi\epsilon^2}{q^2} \text{Im} \left(\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right) \text{Re} P_d(\mathbf{q}, \omega) . \quad (13.126)$$

$$\boxed{\frac{1}{\tau_{in}(T)} \simeq \Delta \left(\frac{T}{E_c} \right)^{3/2} \quad (d = 3)} \quad (13.132)$$

up to a numerical factor. Note that the exponent of the power law is the same as the exponent for the energy dependence of the lifetime at zero temperature (13.108). This result follows at once if we notice that relevant processes in the quasiparticle relaxation described by $\omega W^2(\omega)$ are those for which the energy transfer ω is of order T .

It would be tempting to generalize this result to any dimension and to conclude that $1/\tau_{in}(T) \propto T^{d/2}$. This is not correct for $d \leq 2$. In this case, the contribution of e-e processes with *low energy transfer* $\omega \simeq 0$ dominates and leads to a divergence in the integral (13.130). In order to cure this divergence, it is worth noticing that $\tau_{in}(T)$ represents precisely the lifetime of an eigenstate, so that the energy transfer ω cannot be defined with an accuracy better than $1/\tau_{in}$. Consequently, there is no energy transfer smaller than $1/\tau_{in}(T)$, so that the integral (13.130) needs to be cut off self-consistently for ω smaller than $1/\tau_{in}(T)$. For $d \leq 2$, we thus obtain a self-consistent relation for τ_{in} :

$$\frac{1}{\tau_{in}(T)} \simeq \frac{1}{\nu_0} \int_{1/\tau_{in}}^{\infty} \frac{d\omega}{\omega \sinh \beta\omega} \left(\frac{\omega}{E_c} \right)^{d/2} \simeq \frac{T}{\nu_0} \int_{1/\tau_{in}}^T \frac{d\omega}{\omega^2} \left(\frac{\omega}{E_c} \right)^{d/2} \quad (13.133)$$

where the thermal factor has been replaced by a cutoff at $\omega \sim T$. In two dimensions, $1/\tau_{in}(T)$ is proportional to the temperature (within logarithmic corrections) :

$$\frac{1}{\tau_{in}(T)} \simeq \Delta \frac{T}{E_c} \ln \frac{E_c}{\Delta} \quad (d = 2) \quad . \quad (13.134)$$

In one dimension, and since $T\tau_{in} \gg 1$, the integral becomes proportional to $\sqrt{\tau_{in}}$ so that the self-consistent relation leads to

$$\boxed{\frac{1}{\tau_{in}(T)} \simeq \Delta \left(\frac{E_c}{\Delta} \right)^{1/3} \left(\frac{T}{E_c} \right)^{2/3} \quad (d = 1)} \quad (13.135)$$

Remark : non-exponential relaxation of quasiparticles in dimension $d \leq 2$

The introduction of the low-energy cutoff may appear as a handwaving and artificial way to handle the low energy divergence in (13.130). The profound reason for this divergence is that, for $d \leq 2$, *the relaxation of quasiparticles is not exponential* [338]. The relaxation rate $-d \ln \mathcal{P}/dt$ is no longer constant as was assumed in (13.119). Indeed, we know from the Fermi golden rule that, after a time t , energy must be conserved within $1/t$. Thus the energy transfer ω cannot be defined with a precision better than $1/t$ and we have to cut off contributions of energies smaller than $1/t$ [338]. Thus (13.126) becomes

$$\ln \mathcal{P} = -\frac{4t}{\pi\Omega} \int_{1/t}^{\infty} \frac{d\omega}{\sinh \beta\omega} \sum_{\mathbf{q}} \frac{4\pi e^2}{q^2} \text{Im} \left(\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right) \text{Re} P_d(\mathbf{q}, \omega) , \quad (13.136)$$

that is

$$\ln \mathcal{P} = -\frac{\pi d c_d}{2\nu_0} t \int_{1/t}^{\infty} \frac{d\omega}{\omega \sinh \beta\omega} \left(\frac{\omega}{E_c} \right)^{d/2} . \quad (13.137)$$

The lower cutoff does not affect the relaxation in dimension $d = 3$, since the integral converges at low frequency. For $d \leq 2$, however, the low frequency behavior drives the relaxation. Consider the case $d = 1$. We obtain, for times $t \gg 1/T$:

$$\ln \mathcal{P} \simeq -\frac{\sqrt{2}}{2\pi\nu_0\sqrt{E_c}} T t \int_{1/t}^T \frac{d\omega}{\omega^{3/2}} \simeq -\frac{\sqrt{2}T}{\pi\nu_0\sqrt{E_c}} t^{3/2} \quad (13.138)$$

which leads to an non-exponential behavior for the quasiparticles relaxation :

$$\mathcal{P}(t, T) \sim e^{-[t/\tau_{in}(T)]^{3/2}} \quad d = 1 \quad (13.139)$$

with

$$\frac{1}{\tau_{in}(T)} \sim \left(\frac{\Delta T}{E_c^{1/2}} \right)^{2/3} . \quad (13.140)$$

This argument shows that the low frequency divergence is indeed the signature of a non-exponential behavior. Moreover, we recover the characteristic time obtained in (13.135). In dimension $d = 2$, we have

$$\mathcal{P}(t, T) \sim e^{-t/(\tau_{in} \ln Tt)} . \quad (13.141)$$

Remark : validity of the Fermi liquid description

The relaxation rate $1/\tau_{in}(T)$ stays smaller than the temperature T . With the help of (13.132) and (13.134), we check that it is always the case for $d \geq 2$. In one dimension, $1/\tau_{in}$ decreases more slowly than temperature. We might wonder whether quasiparticles are still well defined at low temperature and question the validity of the Fermi liquid description. However, $1/\tau_{in}$ becomes of order T at extremely low temperature of order Δ/g , with $g \sim E_c/\Delta \gg 1$, which so far is not accessible and which is zero in the thermodynamic limit.

13.7 Phase coherence

13.7.1 Introduction

In the preceding section, we have studied the relaxation of a quasiparticle at the Fermi level and at finite temperature. This relaxation is characterized by

the time $\tau_{in}(T)$, and in dimension $d \leq 2$ it is no longer exponential.

In this section, we wish to study the nature of the processes which limit the phase coherence and therefore the observation of interference effects such as weak localization. We shall denote τ_{ϕ}^{ee} the characteristic time associated to the loss of phase coherence.

- A first simple approach is to consider that phase coherence is limited by the lifetime of quasiparticles. Since the multiple scattering trajectories that are paired in the Cooperon are defined for a given energy state, they cannot interfere for times larger than $\tau_{in}(T)$. This results in an irreversible dephasing between the trajectories and thus a loss of phase coherence. It is therefore appealing to assume that

$$\tau_{\phi}^{ee}(T) = \tau_{in}(T) = \tau_{ee}(\epsilon = 0, T) . \quad (13.142)$$

We have also shown that quasiparticle relaxation is not exponential for $d \leq 2$ and we might ask whether phase relaxation is also non exponential.

- A second approach consists in calculating *directly* the dephasing $\langle e^{i\Phi(t)} \rangle$ resulting from electron-electron interaction and accumulated between time reversed conjugated multiple scattering sequences. To that purpose, we replace the interaction between electrons by an effective interaction which describes the coupling of a single electron to the electromagnetic field created by the other electrons [339, 340, 341]. This electric noise is called Nyquist noise.

- In developing this second approach, we shall see not only that these two characteristic times τ_{in} and τ_{ϕ}^{ee} are equal, but also that the two processes, *quasiparticles relaxation* and *phase relaxation*, are very similar so that the loss of phase coherence, described by the average $\langle e^{i\Phi(t)} \rangle$, behaves like the probability $\mathcal{P}(t, \epsilon = 0, T)$.

Remark : definition of the phase coherence time

This definition is not unique. It depends on the physical quantity we consider. For states which contribute to electronic transport and which are close to the Fermi level, the definition (13.142) is quite natural. However, at finite temperature T , states which contribute to transport are located in an energy interval of width T around the Fermi level and the dephasing time $\tau_\phi^{ee}(\epsilon, T)$ depends in principle on the energy. Thus it would be equally natural to consider an average of $\tau_\phi^{ee}(\epsilon, T)$ over this energy range. Consider for example the weak localization correction given by

$$\delta\sigma(T) = \int \delta\sigma(\epsilon) \left(\frac{-\partial f}{\partial \epsilon} \right) d\epsilon \quad , \quad (13.143)$$

where $\delta\sigma(\epsilon)$ is the correction for a given energy ϵ . As an example, in dimension $d = 2$, we have

$$\delta\sigma(\epsilon) = -\frac{e^2}{\pi h} \ln \frac{\tau_\phi^{ee}(\epsilon, T)}{\tau_e} \quad , \quad (13.144)$$

so that the weak localization correction $\delta\sigma(T)$ can be written in the form

$$\delta\sigma(T) = -\frac{e^2}{\pi h} \ln \frac{\tau_\phi^{ee}(T)}{\tau_e} \quad (13.145)$$

and the phase coherence time $\tau_\phi^{ee}(T)$ is then defined by the average [357]

$$\ln \tau_\phi^{ee}(T) = \int \left(\frac{-\partial f}{\partial \epsilon} \right) \ln \tau_\phi^{ee}(\epsilon, T) d\epsilon \quad . \quad (13.146)$$

We may check that $\tau_\phi^{ee}(T)$ does not significantly depend on the way used to perform the energy average, so that we shall keep the definition (13.142) in the following.

13.7.2 Phase coherence in a fluctuating electric field

We now want to determine how the electron-electron interaction leads to a dephasing between time reversed trajectories. To that purpose, we assume that the total electric field acting on a given electron and resulting from all other electrons can be replaced by an effective fluctuating electric field whose characteristics are imposed by the fluctuation-dissipation theorem [339].

To proceed further, we consider the contribution of the Cooperon to the return probability $P_c(\mathbf{r}, \mathbf{r}, t)$ in a time dependent electric potential $V(\mathbf{r}, t)$. In Appendix A6.3, we have shown that this contribution can be written in the form (6.247) :

$$P_c(\mathbf{r}, \mathbf{r}, t) = P_c^{(0)}(\mathbf{r}, \mathbf{r}, t) \langle e^{i\Phi} \rangle_C \quad , \quad (13.147)$$

where $P_c^{(0)}$ is the probability in the absence of the fluctuating potential, and Φ is the relative phase accumulated along a pair of time reversed trajectories after a time t (relations 6.245 and 6.246) :

$$\Phi = \frac{e}{\hbar} \int_0^t [V(\mathbf{r}(\tau), \tau) - V(\mathbf{r}(\tau), \bar{\tau})] d\tau \quad (13.148)$$

where $\bar{\tau} = t - \tau$ and $\langle \dots \rangle_C$ is the average taken over the distribution of diffusion paths.

We also have to average the thermal fluctuations of the electric potential and we denote this average by $\langle \dots \rangle_T$. These fluctuations being Gaussian ²⁴, the average $\langle e^{i\Phi} \rangle_T$ is given by

$$\langle e^{i\Phi} \rangle_T = e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} . \quad (13.149)$$

We now need to determine the average $\langle e^{i\Phi} \rangle_{T,c}$ on *both* diffusion paths and thermal fluctuations. We will start with the calculation of $\langle \Phi^2 \rangle_T$. From (13.148), we have :

$$\langle \Phi^2 \rangle_T = \frac{e^2}{\hbar^2} \int_0^t \int_0^t \langle [V(\tau_1) - V(\bar{\tau}_1)][V(\tau_2) - V(\bar{\tau}_2)] \rangle_T d\tau_1 d\tau_2 \quad (13.150)$$

where $V(\tau) = V(\mathbf{r}(\tau), \tau)$ and $V(\bar{\tau}) = V(\mathbf{r}(\tau), \bar{\tau})$. We define the correlator $\langle VV \rangle_T(\mathbf{q}, \omega)$ by the Fourier transform

$$\langle V(\mathbf{r}, \tau)V(\mathbf{r}', \tau') \rangle_T = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{d\omega}{2\pi} \langle VV \rangle_T(\mathbf{q}, \omega) e^{i[\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}') - \omega(\tau - \tau')]} . \quad (13.151)$$

Its thermal average is related by the fluctuation-dissipation theorem to the dielectric function [332, 343, 342] ²⁵ :

$$e^2 \langle VV \rangle_T(\mathbf{q}, \omega) = \frac{4\pi e^2}{q^2} \text{Im} \left[\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right] \frac{2}{1 - e^{-\beta\omega}} \quad (13.152)$$

where, according to (13.11), $\text{Im}(-1/\epsilon(\mathbf{q}, \omega)) = \omega/4\pi\sigma_0$. The processes that contribute to the dephasing have an energy $|\omega|$ lower than temperature. Indeed, because of the Pauli principle, an electron at the Fermi level cannot exchange an energy larger than T with its environment. Then for $|\omega| < T$, we replace the thermal function by its high temperature limit :

$$\boxed{e^2 \langle VV \rangle_T(\mathbf{q}, \omega) = \frac{2e^2 T}{\sigma_0 q^2}} \quad (13.153)$$

The integrand in (13.151) no longer depends on frequency, so that integrating upon ω gives ²⁶

$$\langle V(\mathbf{r}, \tau)V(\mathbf{r}', \tau') \rangle_T = \frac{\delta(\tau - \tau')}{(2\pi)^d} \frac{2T}{\sigma_0} \int \frac{d\mathbf{q}}{q^2} e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \quad (13.154)$$

²⁴The modes of the electromagnetic field are quadratic and their fluctuations are thus Gaussian.

²⁵Here we consider only the longitudinal fluctuations of the electromagnetic field. The transverse fluctuations are screened by the skin effect [341, 340, 339] and can be neglected, except in confined geometries.

²⁶As a result of the cutoff T on energy transfer ω , the δ function that shows up in (13.154) is in fact a strongly peaked function of width $1/T$. But since relevant times including the quasiparticle relaxation time, are much larger than $1/T$ (remark p. 521) this function can be safely replaced by a δ function [344, 341].

and by inserting this expression in (13.150) :

$$\langle \Phi^2 \rangle_T = \frac{4e^2 T}{\sigma_0 \hbar^2} \int_0^t d\tau \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2} [1 - \cos \mathbf{q} \cdot (\mathbf{r}(\tau) - \mathbf{r}(\bar{\tau}))] \quad (13.155)$$

This quantity depends on paths $\mathbf{r}(\tau)$. We now have to calculate the average

$$\left\langle e^{-\frac{1}{2} \langle \Phi^2 \rangle_T} \right\rangle_c \quad (13.156)$$

over the distribution of closed diffusive paths which contribute to the Cooperon $P_c(\mathbf{r}, \mathbf{r}, t)$. This average is obtained following the functional integral approach developed in reference [339]. The Cooperon can be written as

$$\begin{aligned} P_c(\mathbf{r}, \mathbf{r}, t) &= \int_{\mathbf{r}(0)=\mathbf{r}}^{\mathbf{r}(t)=\mathbf{r}} \mathcal{D}\{\mathbf{r}\} \exp \left(- \int_0^t \frac{\dot{\mathbf{r}}^2(\tau)}{4D} d\tau - \frac{1}{2} \langle \Phi^2 \rangle_T \right) \\ &= P_c^{(0)}(\mathbf{r}, \mathbf{r}, t) \cdot \left\langle e^{-\frac{1}{2} \langle \Phi^2 \rangle_T} \right\rangle_c, \end{aligned} \quad (13.157)$$

where $P_c^{(0)}(\mathbf{r}, \mathbf{r}, t)$ is the return probability in the absence of a fluctuating electric field. In order to decouple the paths $\mathbf{r}(\tau)$ and $\mathbf{r}(\bar{\tau})$ which enter in $\langle \Phi^2 \rangle_T$, we first use the semi-group relation (5.7) in the form

$$P_c(\mathbf{r}, \mathbf{r}, 0, t) = \int d\mathbf{R} P_c(\mathbf{r}, \mathbf{R}, 0, t/2) P_c(\mathbf{R}, \mathbf{r}, t/2, t). \quad (13.158)$$

We have denoted explicitly the initial and final times. Then, we perform the change of variables $\mathbf{R}(\tau) = [\mathbf{r}(\tau) + \mathbf{r}(\bar{\tau})]/\sqrt{2}$ and $\boldsymbol{\rho}(\tau) = [\mathbf{r}(\tau) - \mathbf{r}(\bar{\tau})]/\sqrt{2}$. After integration on \mathbf{R} , $P_c(\mathbf{r}, \mathbf{r}, t) = C(\boldsymbol{\rho} = 0, \boldsymbol{\rho} = 0, t/2)$ with [339] :

$$C(\boldsymbol{\rho} = 0, \boldsymbol{\rho} = 0, \frac{t}{2}) = \frac{1}{\sqrt{2}} \int_{\boldsymbol{\rho}(0)=0}^{\boldsymbol{\rho}(t/2)=0} \mathcal{D}\{\boldsymbol{\rho}\} \exp \left(- \int_0^{t/2} \left[\frac{\dot{\boldsymbol{\rho}}^2(\tau)}{4D} + U(\boldsymbol{\rho}) \right] d\tau \right) \quad (13.159)$$

where we have introduced the effective potential $U(\boldsymbol{\rho})$ defined by

$$U(\boldsymbol{\rho}) = \frac{4e^2 T}{\sigma_0 \hbar^2} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2} [1 - \cos \mathbf{q} \cdot \boldsymbol{\rho} \sqrt{2}]. \quad (13.160)$$

The integral $C(0, 0, t)$ obeys the differential equation ²⁷ :

²⁷We should remember that the functional integral

$$F(\mathbf{r}, \mathbf{r}', t) = \int_{\mathbf{r}(0)=\mathbf{r}}^{\mathbf{r}(t)=\mathbf{r}'} \mathcal{D}\{\mathbf{r}\} \exp \left(- \int_0^t \left[\frac{\dot{\mathbf{r}}^2(\tau)}{4D} + U(\mathbf{r}) \right] d\tau \right) \quad (13.161)$$

obeys the differential equation :

$$\left[\frac{\partial}{\partial t} - D \Delta_{\mathbf{r}'} + U(\mathbf{r}') \right] F(\mathbf{r}, \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}') \delta(t) \quad (13.162)$$

$$\left[\frac{\partial}{\partial t} - D\Delta_{\rho} + U(\rho) \right] C(0, \rho, t) = \frac{1}{\sqrt{2}} \delta(\rho) \delta(t) . \quad (13.163)$$

We now derive an expression of the dephasing from the solution of this differential equation.

Exercise 13.15. Nyquist noise

Using the fluctuation-dissipation theorem written in the form (13.153), recover the Nyquist expression for the voltage noise in a conductor (restoring the Boltzmann constant k_B) [345, 346] :

$$\langle V^2 \rangle_T(\omega) = 2k_B T R \quad (13.164)$$

for ω both positive *and* negative. $V = V(L) - V(0)$ is the voltage drop at the edge of a wire of resistance R , of length L and of section S .

Hint : from (13.151) and (13.153), calculate the thermal fluctuations of the voltage for the wire geometry (using Ohm's law $R = L/(S\sigma_0)$) :

$$\langle V(r)V(r') \rangle_T(\omega) = \int \frac{dq}{2\pi} \frac{2Rk_B T}{q^2 L} e^{iq(r-r')} . \quad (13.165)$$

By fixing $r = 0$ and $r' = L$ at the edges of the samples and using the integral (15.97), relation (13.164) is obtained.

13.7.3 Phase coherence time in dimension $d = 1$

We consider now the case $d = 1$, for which we have seen that the quasiparticle relaxation presents peculiar characteristics due to the diverging contribution of energy exchanges with small energy transfer (section 13.6.3). Analogous features are thus expected for the phase relaxation. We consider a quasi-one-dimensional wire of section S . Using (15.97), the phase fluctuation (13.155) can be written :

$$\langle \Phi^2 \rangle_T = \frac{2e^2 T}{\hbar^2 \sigma_0 S} \int_0^t |r(\tau) - r(\bar{\tau})| d\tau , \quad (13.166)$$

so that

$$U(\rho) = \frac{2\sqrt{2}e^2 T}{\hbar^2 \sigma_0 S} |\rho| . \quad (13.167)$$

The differential equation (13.163) for C becomes

$$\left[\frac{\partial}{\partial t} - D \frac{\partial^2}{\partial \rho^2} + \frac{2\sqrt{2}e^2 T}{\hbar^2 \sigma_0 S} |\rho| \right] C(0, \rho, t) = \frac{1}{\sqrt{2}} \delta(\rho) \delta(t) \quad (13.168)$$

and its solution gives $C(0, 0, t)$ and therefore $P_c(r, r, t)$, from which we obtain the time dependence of the function $\left\langle e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} \right\rangle_C$. The calculation of this

function is done in Exercise 13.16, and we consider instead the weak localization correction given by relation (7.53) :

$$\Delta\sigma = -\frac{se^2D}{\pi\hbar} \int_0^\infty dt P_c(r, r, t) e^{-\gamma t} , \quad (13.169)$$

with $\gamma = 1/\tau_\gamma$, where τ_γ accounts for any dephasing time due to processes other than electron-electron interaction and assumed to induce an exponential relaxation of the phase. This weak localization correction is nothing but the Laplace transform of the probability $P_c(r, r, t)$:

$$\begin{aligned} P_\gamma(r, r) &= \int_0^\infty P_c(r, r, t) e^{-\gamma t} dt \\ &= \int_0^\infty C(0, 0, t/2) e^{-\gamma t} dt \\ &= 2C_{2\gamma}(0, 0) , \end{aligned} \quad (13.170)$$

where $C_\gamma(\rho, \rho')$, Laplace transform of $C(\rho, \rho', t)$, enters directly expression (13.169) :

$$\Delta\sigma = -\frac{2se^2D}{\pi\hbar} C_{2\gamma}(0, 0) \quad (13.171)$$

It obeys the differential equation :

$$\left[-D \frac{\partial^2}{\partial \rho^2} + \frac{2\sqrt{2}e^2T}{\hbar^2\sigma_0S} |\rho| + 2\gamma \right] C_{2\gamma}(0, \rho) = \frac{1}{\sqrt{2}} \delta(\rho) . \quad (13.172)$$

Introducing the characteristic time known as Nyquist time in the literature ²⁸ [345] :

$$\tau_N = \left(\frac{\hbar^2\sigma_0S}{e^2T\sqrt{D}} \right)^{2/3} \quad (13.173)$$

we obtain the dimensionless differential equation for $x = \rho\sqrt{2/D\tau_N}$:

$$\left[-\frac{\partial^2}{\partial x^2} + |x| + \frac{\tau_N}{\tau_\gamma} \right] C_{2\gamma}(0, x) = \frac{1}{2} \sqrt{\frac{\tau_N}{D}} \delta(x) . \quad (13.174)$$

With the help of (15.101, 15.102), we obtain :

$$P_\gamma(r, r) = 2C_{2\gamma}(0, 0) = -\frac{1}{2} \sqrt{\frac{\tau_N}{D}} \frac{\text{Ai}(\tau_N/\tau_\gamma)}{\text{Ai}'(\tau_N/\tau_\gamma)} , \quad (13.175)$$

²⁸This time is proportional to the quasiparticle relaxation time τ_{in} given by (13.135). We return to this remark later.

where Ai and Ai' are respectively the Airy function and its derivative [347]. From relation (13.171), the weak localization correction becomes

$$\Delta\sigma = s \frac{e^2}{hS} \sqrt{D\tau_N} \frac{\text{Ai}(\tau_N/\tau_\gamma)}{\text{Ai}'(\tau_N/\tau_\gamma)} \quad (13.176)$$

instead of

$$\Delta\sigma = -s \frac{e^2}{hS} \sqrt{D\tau_\gamma} \quad (13.177)$$

in the absence of electron-electron interaction (see 7.56). In the limit $\tau_\gamma \ll \tau_N$, we obtain

$$\Delta\sigma = -s \frac{e^2}{hS} \sqrt{D\tau_\gamma} \left[1 - \frac{1}{4} \left(\frac{\tau_\gamma}{\tau_N} \right)^{3/2} \right]. \quad (13.178)$$

Inversely, for $\tau_N \ll \tau_\gamma$

$$\Delta\sigma = -1.372 s \frac{e^2}{hS} \sqrt{D\tau_N}, \quad (13.179)$$

where we have used the asymptotic forms (15.103, 15.104). Finally, using the approximation (15.105) for the ratio Ai/Ai' , we obtain a very good approximation for the weak localization correction as a function of the times τ_γ and τ_N [348] :

$$\Delta\sigma \simeq -s \frac{e^2}{hS} \sqrt{D} \left(\frac{1}{2\tau_N} + \frac{1}{\tau_\gamma} \right)^{-1/2} \quad (13.180)$$

This approximation amounts to assuming an exponential relaxation of the phase, $\langle e^{i\Phi} \rangle_{T,C} \simeq e^{-2t/\tau_N}$. We give in Exercise 13.16 and in Figure 13.10 the exact form of this relaxation which differs from an exponential. However, at this approximation, it is found that, in order to estimate the phase relaxation time in the presence of other dephasing processes, it is thus sufficient to add their inverse dephasing times. The phase coherence time τ_ϕ^{ee} – or phase relaxation time – in the presence of electron-electron interaction is thus given by :

$$\tau_\phi^{ee} = 2\tau_N = 2 \left(\frac{\hbar^2 \sigma_0 S}{e^2 T \sqrt{D}} \right)^{2/3} \quad (13.181)$$

Exercise 13.16. Relaxation of the phase in a quasi-1d conductor

- Find the time dependence of the phase factor $\langle e^{i\Phi} \rangle_{T,C} = f(t/\tau_N)$. Using the Laplace transform of $P_c(r, r, t)$, show that relation (13.175) can be written as

$$\int_0^\infty \frac{1}{\sqrt{4\pi Dt}} f\left(\frac{t}{\tau_N}\right) e^{-t/\tau_N} dt = -\frac{1}{2} \sqrt{\frac{\tau_N}{D}} \frac{\text{Ai}(\tau_N/\tau_N)}{\text{Ai}'(\tau_N/\tau_N)}. \quad (13.182)$$

Noticing that the zeros of the Airy function and of its derivative are distributed on the negative real axis, calculate the integral in the complex plane by using $\text{Ai}''(x) = x\text{Ai}(x)$ and show that [338]

$$\langle e^{i\Phi} \rangle_{T,C} = \sqrt{\frac{\pi t}{\tau_N}} \sum_{n=1}^{\infty} \frac{e^{-|u_n|t/\tau_N}}{|u_n|}, \quad (13.183)$$

where the u_n 's are the zeroes of the function Ai' . For large n , they are well approximated by $|u_n| = (\frac{3\pi}{2}(n - \frac{3}{4}))^{2/3}$ [347]. Show that for $t < \tau_N$:

$$\langle e^{i\Phi} \rangle_{T,C} \simeq e^{-\sqrt{\pi}/4(t/\tau_N)^{3/2}}. \quad (13.184)$$

We check that the short time behavior of the phase relaxation is identical to the relaxation of a quasiparticle given by (13.139). As displayed in Figure 13.10, the function $f(t/\tau_N)$ is not an exponential.

- Calculate explicitly the average $\langle \Phi^2 \rangle_{T,C}$ over closed Brownian trajectories of duration t . Show first that $\langle |\mathbf{r}(\tau)| \rangle = \sqrt{\frac{4D}{\pi t}} \tau(t - \tau)$, and then use this result to obtain an approximation of $\langle e^{i\Phi} \rangle_{T,C}$:

$$\langle e^{i\Phi} \rangle_{T,C} = \langle e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} \rangle_C \simeq e^{-\frac{1}{2}\langle \Phi^2 \rangle_T} = e^{-\frac{\sqrt{\pi}}{4}(t/\tau_N)^{3/2}}. \quad (13.185)$$

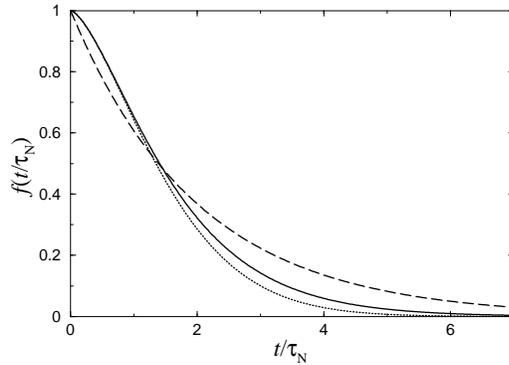


Figure 13.10: *Relaxation of the phase.* The continuous line represents the function $\langle e^{i\Phi} \rangle_{T,C} = f(t/\tau_N)$ obtained from equation (13.182). The dotted line corresponds to the approximation $e^{-\frac{\sqrt{\pi}}{4}(t/\tau_N)^{3/2}}$ obtained from the small time expansion (13.184). The phase relaxation is clearly non exponential. The approximation $e^{-t/2\tau_N}$ (dashed line) allows to identify the characteristic time $\tau_\phi^{ee} = 2\tau_N$.

Remark : Dephasing and geometry : qualitative arguments

The main results of this section devoted to phase relaxation in quasi-1d wires can be recovered with a simple argument. From (13.148), it is easy to find that for a δ -correlated potential of the form (13.154), the time dependence of the dephasing is given by

$$\frac{d\langle\Phi^2(t)\rangle}{dt} = \frac{e^2}{2\hbar^2}\langle V_t^2 \rangle, \quad (13.186)$$

where $\langle V_t^2 \rangle$ represents the voltage fluctuation on a time scale t . These fluctuations are related through Nyquist theorem (13.164), $\langle V_t^2 \rangle = 2k_B T R_t$, to the resistance $R_t = \sigma_0 r_t / S$ of a wire of typical length r_t which is a typical distance reached after a time t by diffusive trajectories. Therefore,

$$\frac{d\langle\Phi^2(t)\rangle}{dt} = \frac{e^2 k_B T}{\sigma_0 S \hbar^2} r_t \quad (13.187)$$

In an infinite wire, the typical length varies as $r_t \sim \sqrt{Dt}$, so that the variance of the phase, $\langle\Phi^2(t)\rangle$, varies non linearly with time :

$$\langle\Phi^2(t)\rangle \sim \left(\frac{t}{\tau_N}\right)^{3/2}$$

with the characteristic Nyquist time τ_N given by (13.173).

It is also interesting to consider the case of a *finite* wire of length L . For times t larger than the Thouless time $\tau_D = L^2/D$, an electron has explored completely the system. This is the ergodic regime, defined in section 5.5.3. In this regime the typical distance r_t is no longer time dependent and it is set by the length L of the system. Consequently, the time dependence of the dephasing is linear :

$$\langle\Phi^2(t)\rangle \sim \frac{t}{\tau_c}$$

where τ_c is size dependent and is given by

$$\frac{\hbar}{\tau_c} = \frac{e^2}{\hbar} \frac{k_B T L}{\sigma_0 S} = \frac{e^2}{\hbar} R k_B T. \quad (13.188)$$

The two times τ_N and τ_c are related by $\tau_N^3 = \tau_c^2 \tau_D$. They have a different temperature dependence. This difference has been stressed recently and should show up in Aharonov-Bohm experiments on rings [349, 350].

13.7.4 Phase coherence and quasiparticle relaxation

It is very interesting to compare the results obtained for the *phase relaxation* with those obtained in section 13.7.4 to describe the *quasiparticle relaxation*. We notice first that the phase coherence time $\tau_\phi^{ee}(T)$ obtained in dimension $d = 1$ is parametrically identical to the quasiparticle relaxation time $\tau_{in}(T)$ given by (13.135).

In order to understand this similarity, we return in more detail to the structure of the expressions which describe the quasiparticle and the phase relax-

ations. The quasiparticle relaxation is given by (13.136)²⁹ :

$$-\ln \mathcal{P}(t) = \frac{4}{\pi \hbar^2} t \int_{1/t}^{\infty} \frac{d\omega}{\sinh \beta \omega} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{4\pi e^2}{q^2} \text{Im} \left(\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right) \text{Re} P_d(\mathbf{q}, \omega) \quad (13.190)$$

On the other hand, we have directly calculated the dephasing induced by fluctuations of the electric field on the Cooperon. This dephasing is characterized by the average values (13.150) and (13.151). Using the relation (5.22), $\langle e^{i\mathbf{q}\cdot\mathbf{r}(\tau)} \rangle_C = e^{-Dq^2\tau}$, we obtain the phase fluctuation in the form³⁰

$$\begin{aligned} \frac{1}{2} \langle \Phi^2 \rangle_{T,C} &= \frac{1}{\pi \hbar^2} \int_0^t \int_0^t d\tau_1 d\tau_2 \int \frac{d\mathbf{q}}{(2\pi)^d} \int_{|\omega| < T} d\omega \frac{4\pi e^2}{q^2} \text{Im} \left(\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right) \frac{1}{1 - e^{-\beta\omega}} \\ &\times -e^{-Dq^2|\tau_1 - \tau_2|} \text{Re} \left(e^{-i\omega(\tau_1 - \tau_2)} - e^{i\omega(\tau_1 - \bar{\tau}_2)} \right) , \end{aligned} \quad (13.191)$$

where $\bar{\tau} = t - \tau$. The integration over ω is up to T , a consequence of the Pauli principle discussed on p. 524. In order to show the equivalence of the approaches leading to relations (13.190) and (13.191), consider this latter expression.

Instead of integrating on the frequency as it was done to obtain (13.154), let us first integrate on time. After a tedious calculation, we obtain³¹ :

$$\begin{aligned} \frac{1}{2} \langle \Phi^2 \rangle_{T,C} &\simeq \frac{2}{\pi \hbar^2} t \int_{|\omega| < T} \frac{d\omega}{1 - e^{-\beta\omega}} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{4\pi e^2}{q^2} \\ &\times \text{Im} \left(\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right) \text{Re} P_d(\mathbf{q}, \omega) \left(1 - \frac{\sin \omega t}{\omega t} \right) . \end{aligned} \quad (13.193)$$

Quite remarkably, we notice that the compensation between the two correlators $\langle V(\tau_1)V(\tau_2) \rangle_T$ and $\langle V(\tau_1)V(\bar{\tau}_2) \rangle_T$ introduced in (13.150) provides naturally,

²⁹Expressions (13.190) and (13.193) can be greatly simplified by noticing that

$$\frac{4\pi e^2}{q^2} \text{Im} \left(\frac{-1}{\epsilon} \right) \text{Re} P_d(\mathbf{q}, \omega) = \frac{1}{2\rho_0} \text{Im} P_d(\mathbf{q}, \omega) . \quad (13.189)$$

We have chosen to keep the expressions where the dielectric function appears explicitly.

³⁰More precisely, it can be shown that $\langle e^{i\mathbf{q}\cdot\mathbf{r}(\tau)} \rangle_C = e^{-Dq^2\tau(t-\tau)/t}$. But the approximation used here is sufficient in the large time limit [340].

³¹We notice that, for an odd function $F(\omega)$,

$$\int_{-\infty}^{\infty} \frac{F(\omega)d\omega}{1 - e^{-\beta\omega}} = \int_0^{\infty} F(\omega) \coth \frac{\beta\omega}{2} d\omega .$$

We can then rewrite (13.193) in the equivalent form

$$\begin{aligned} \frac{1}{2} \langle \Phi^2 \rangle_{T,C} &\simeq \frac{1}{\pi \hbar^2} t \int_0^T d\omega \coth \frac{\beta\omega}{2} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{4\pi e^2}{q^2} \\ &\times \text{Im} \left(\frac{-1}{\epsilon(\mathbf{q}, \omega)} \right) \text{Re} P_d(\mathbf{q}, \omega) \left(1 - \frac{\sin \omega t}{\omega t} \right) . \end{aligned} \quad (13.192)$$

through the term $(1 - \sin \omega t / \omega t)$, a low frequency cutoff of order $1/t$, and leads to a time dependence $\langle \Phi^2 \rangle_{T,C}$ going as $t^{3/2}$ ³².

Relations (13.190) and (13.193) show that the relaxations of a quasiparticle state and of the phase are identical, provided that the following remarks hold true :

- Although it is natural to expect that phase coherence is limited by the lifetime of quasiparticles, it is far from being *a priori* obvious that the two relaxation processes are identical. In particular the frequency transfer ω has a different meaning in each case. For quasiparticle relaxation, it means an energy transfer between quantum states, while for the phase relaxation, it means the frequency of the fluctuating modes of the electric field. The correspondence between the two mechanisms is provided by the fluctuation-dissipation theorem.

- In (13.193), the lower cutoff on energies ω results quite naturally from the difference between the two correlators and thus from the structure of the Cooperon which couples two time-reversed trajectories. But in (13.190), it results from the Fermi golden rule. We emphasize the essential role played by the correlator $\langle V(\tau_1)V(\overline{\tau_2}) \rangle_T$ which describes the potential correlation between time reversed trajectories.

- The thermal function $\omega / \sinh \beta \omega$ which shows up in the Fermi golden rule derivation of (13.190) can be rewritten in the form

$$\frac{\omega}{\sinh \beta \omega} = \frac{2\omega}{1 - e^{-\beta \omega}} [1 - f(-\omega)] . \quad (13.194)$$

We recognize the thermal factor $\omega / (1 - e^{-\beta \omega})$ that characterizes thermal fluctuations of the electromagnetic field. It is multiplied by a Fermi factor $1 - f(-\omega) = f(\omega)$ which cuts the contribution of energy exchanges up to a value of order T . This term expresses the constraint due to the Pauli principle, which is explicitly taken into account by the Fermi factors of relation (13.121). By contrast, in the calculation of the phase relaxation leading to equation (13.193), only the first term appears $\omega / (1 - e^{-\beta \omega})$, because it describes the interaction of a single electron with a fluctuating electromagnetic field.

However, an additional constraint follows from the existence of the Fermi sea, namely that a quasiparticle of energy ϵ can relax only to available empty state $\epsilon - \omega$, whence the Fermi factor $1 - f(\epsilon - \omega)$. For quasiparticles at the Fermi level, this leads to the result (13.194). This multiplicative factor leads to the cutoff $\omega < T$ in the integral (13.193). The energy exchanged with the fluctuating field due to the other electrons cannot be larger than T . With this precaution, it appears that the relaxation of quasiparticles and of the phase are equivalent mechanisms [344] ³³.

³²In the calculation leading to the relation (13.193), we have supposed that $Dq^2t \gg 1$. This hypothesis, also considered in reference [340], amounts to neglecting other terms which have the same time dependence.

³³This subtlety was at the origin of a debate [351, 352] triggered by experimental results

*Quasiparticle and phase relaxations
are driven by the same time scale :*

$$\tau_{\phi}^{ee}(T) = \tau_{in}(T)$$

13.7.5 Phase coherence time in dimensions $d = 2$ and $d = 3$

In the quasi-1d case, the relaxation of the phase is mainly driven by the low frequency fluctuations. It is instructive to reconsider the results obtained in section 13.7.2, for dimensionalities 2 and 3. As for $d = 1$, the potential $U(\boldsymbol{\rho})$ could be determined, and we could solve the differential equation (13.162). Here we limit ourselves to more qualitative considerations by evaluating the fluctuations $\langle \Phi^2 \rangle_T$ of the phase.

For $d = 2$, the integral (13.155) on the wavevector diverges and must be cut off at $Dq^2 \simeq T$ since there is no energy exchange larger than T with the fluctuations of the electric field. By using (15.98), we obtain

$$\frac{1}{2} \langle \Phi^2 \rangle_T = \frac{e^2 T}{\pi \sigma_0 \hbar^2 a} \int_0^t d\tau \log \frac{2}{q_c |\mathbf{r}(\tau) - \mathbf{r}(\bar{\tau})|} \sim \frac{e^2 T}{\pi \sigma_0 \hbar^2 a} t \ln \frac{1}{Tt} \quad (13.195)$$

where $q_c \simeq \sqrt{T/D}$ and a is the film thickness. Therefore, in dimension $d = 2$, the relaxation of the phase is exponential with a logarithmic correction. The phase coherence rate, defined by $\frac{1}{2} \langle \Phi^2 \rangle_T \simeq 1$, varies linearly with the temperature [339] :

$$\frac{1}{\tau_N} \simeq \frac{e^2 T}{2\pi \sigma_0 \hbar^2 a} \ln \frac{2\pi \sigma_0 \hbar a}{e^2} \quad (13.196)$$

and we notice that this time is the same as the quasiparticle relaxation time (13.134).

In dimension $d = 3$, the integral (13.155) diverges for the large wavevectors. As for $d = 2$, it must be cut off for $q_c \sim \sqrt{T/D}$. We thus obtain an exponential relaxation with the characteristic time τ_N given by

$$\frac{1}{\tau_N} = \frac{e^2 T^{3/2}}{\pi^2 \sigma_0 \hbar^2 \sqrt{D}} \quad (13.197)$$

and we recover a result similar to (13.132).

We conclude by noticing that, in dimensions 2 and 3, the phase relaxation and the quasiparticle relaxation are driven by energy transfers of the order of temperature T .

[353] which seemed to exhibit a saturation of the phase coherence time at low temperature. Assuming that the energy an electron can exchange with its electric environment is not limited by the temperature but by the elastic collision rate leads to a saturation of the phase coherence time at low temperature (Within this assumption, the divergence which appears at high energy is not cut off by temperature but by the inverse collision rate.).

13.7.6 Measurements of the phase coherence time τ_ϕ^{ee}

The usual method of determining experimentally the phase coherence time consists in measuring the magnetoresistance and deducing the weak localization correction (section 7.5). The latter is sensitive to phase coherence and is suppressed by different dephasing mechanisms, which result from a magnetic field, spin-orbit coupling (τ_{so}), coupling to magnetic impurities (τ_m), electron-phonon interaction (τ_{ph}) or from the electron-electron interaction (τ_ϕ^{ee}). Let us also recall that the weak localization correction does not involve any intrinsic temperature dependence, in contrast for example to the conductance fluctuations : it depends solely on L_ϕ but not on L_T (see 11.49). Some of these mechanisms are well controlled : the magnetic field is an external parameter and by measuring the magnetoresistance, it is possible to determine the various characteristic times. The strength of the spin-orbit coupling can be tuned by a change in the concentration of heavy substitution atoms. The electron-phonon coupling decreases rapidly at low temperature $1/\tau_{ph} \propto T^3$ and can then be neglected. What remains at low temperature is the dephasing due to magnetic impurities and to electron-electron interaction.

In practice, fitting the experimental curves by the theoretical expression of the magnetoconductance allows to extract the different characteristic times. For example, in $d = 2$, the magnetoconductance can be fitted with expression (7.82). In $d = 1$, expression (7.56) gives the magnetoresistance of a wire of length L , in a perpendicular magnetic field (Exercise 7.3 and eq. 7.84) :

$$\frac{\Delta R}{R} = s \frac{e^2}{hRL} \left(\frac{3}{2} L_{trip} - \frac{1}{2} L_{sing} \right) . \quad (13.198)$$

The triplet and singlet contributions are respectively given by (eq. 7.75)

$$\frac{1}{L_{trip.}^2} = \frac{1}{D} \left(\frac{1}{\tau_\phi^{ee}} + \frac{4}{3\tau_{so}} + \frac{2}{3\tau_m} \right) + \frac{W^2}{12L_B^4} \quad (13.199)$$

and

$$\frac{1}{L_{sing.}^2} = \frac{1}{D} \left(\frac{1}{\tau_\phi^{ee}} + \frac{2}{\tau_m} \right) + \frac{W^2}{12L_B^4} , \quad (13.200)$$

where $L_B = \sqrt{\hbar/2eB}$ is the magnetic length.

Figure 13.11 presents the temperature dependence of the phase coherence time measured in a gold wire. The phase coherence time is found to vary as $T^{-2/3}$ in good agreement with (13.181). Figure 13.12 shows results obtained on metallic wires made of gold, silver and copper. The $T^{-2/3}$ dependence is also observed on gold and silver wires. At higher temperature, the time τ_ϕ deviates from the behavior predicted by relation (13.181). This indicates the existence of an additional dephasing mechanism due to the coupling to other degrees of freedom, such as the electron-phonon coupling [340], the latter giving a contribution proportional to T^{-3} which is not calculated here. The experimental results obtained in gold and silver wires are well described by the

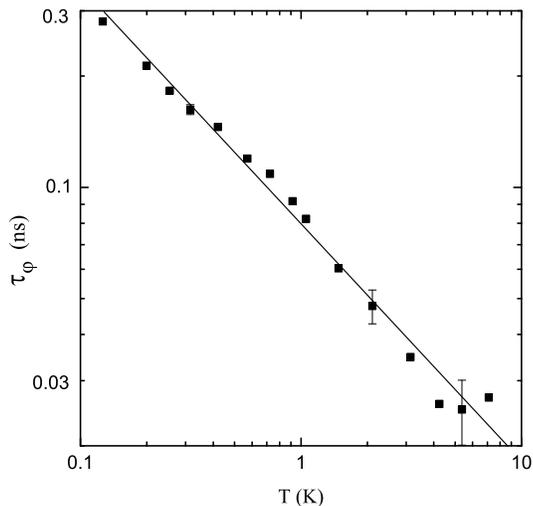


Figure 13.11: Phase coherence time τ_ϕ^{ee} as a function of temperature in a gold wire. The straight line corresponds to $\tau_\phi \propto T^{-0.64}$ (P.M. Echternach et al., *Phys. Rev. B* **48**, 11516 (1993)).

relation

$$\frac{1}{\tau_\phi} = AT^{2/3} + BT^3 \quad (13.201)$$

where A and B are fitting parameters.

We may conclude by saying that the low temperature saturation of the phase coherence time results from the coupling to degrees of freedom such as magnetic impurities [335, 348, 354] or two-level systems [348] and not from electron-electron interaction. This conclusion is reinforced by Figure 13.12 which shows that the saturation in the case of silver wires depends on the purity of the sample ³⁴.

³⁴Recent experimental studies confirm that the low temperature dependence of τ_ϕ is driven by magnetic impurities and may be non monotonous : it first saturates and then it further increases at lower temperature [348, 354]. This behavior has been attributed to the Kondo effect which tends to screen the magnetic impurities at low T (see remark p. 234). At lower temperature the RKKY coupling between impurities may be also relevant [355].

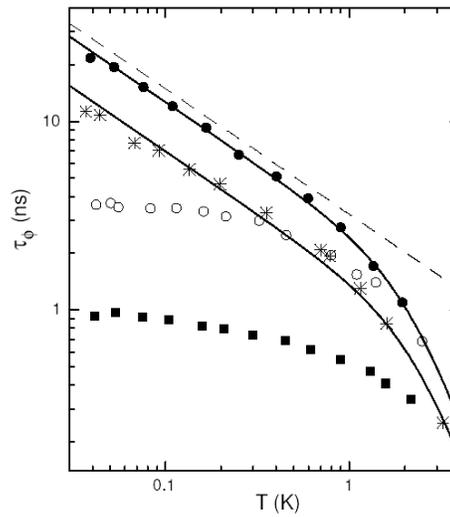


Figure 13.12: Temperature dependence of the phase coherence time τ_ϕ of four metallic wires (silver (● and ○), gold (*)) and copper (■)) obtained by magnetoresistance measurements [348]. For the purest samples (silver (●) and gold (*)), a dependence $\tau_\phi^{-1}(T) = AT^{2/3} + BT^3$ is observed (solid curves). The dashed line represents the contribution $AT^{2/3}$ for the silver samples. For less pure silver (○) and for copper (■), this power law dependence is no longer observed. The low temperature saturation has been attributed to other dephasing mechanisms.

13.1 Appendix A13.1 Screened Coulomb potential in confined geometry

In section 13.2, we have described the screened potential for a three-dimensional sample. More generally, the form of the interaction depends on space dimensionality and on the environment of the sample. For an isolated sample, the Fourier transform of the Coulomb potential e^2/R depends on the dimensionality d as

$$U_0(\mathbf{q}) = \begin{cases} \frac{4\pi e^2}{q^2} & (d = 3) \\ \frac{2\pi e^2}{q} & (d = 2) \\ 2e^2 \ln \frac{1}{qW} & (d = 1) \end{cases} \quad (13.202)$$

The last expression corresponds to a quasi-one-dimensional wire of section W^2 and is valid for distances larger than W , that is, for $qW \ll 1$ ³⁵. From (13.6), the static screened potential ($\omega = 0$) is given by the relation $U(\mathbf{q}) = U_0(\mathbf{q})/[1 + 2\rho_0 U_0(\mathbf{q})]$ where ρ_0 is the density of states. We thus obtain :

$$U(\mathbf{q}) = \begin{cases} \frac{4\pi e^2}{q^2 + \kappa_3^2} & (d = 3) \\ \frac{2\pi e^2}{q + \kappa_2} & (d = 2) \\ \frac{2e^2}{\ln^{-1} \frac{1}{qW} + 4e^2 \rho_{1d}} & (d = 1) \end{cases} \quad (13.203)$$

For $d = 3$, the screening vector κ_3 is given by $\kappa_3^2 = \kappa^2 = 8\pi e^2 \rho_{3d}$ where ρ_{3d} is the density of states per unit volume and per spin direction in $d = 3$. In $d = 2$, the screening vector is given by $\kappa_2 = 4\pi e^2 \rho_{2d}$ where ρ_{2d} is the density of states in $d = 2$. For example in a quasi-two-dimensional sample of thickness W , $\rho_{2d} = \rho_{3d}W$, so that $\kappa_{2d} = \kappa_{3d}^2 W/2$. The dynamically screened interaction is given by (13.6), that is,

³⁵For the case $d = 1$, the cutoff in the logarithm results from the finite thickness W of the wire and it is given up to a numerical factor.

$$U(\mathbf{q}, \omega) = \begin{cases} \frac{4\pi e^2}{q^2 + \kappa_3^2 \frac{Dq^2}{-i\omega + Dq^2}} & (d = 3) \\ \frac{2\pi e^2}{q + \kappa_2 \frac{Dq^2}{-i\omega + Dq^2}} & (d = 2) \\ \frac{2e^2}{\ln^{-1} \frac{1}{qW} + 4e^2 \rho_{1d} \frac{Dq^2}{-i\omega + Dq^2}} & (d = 1) \end{cases} . \quad (13.204)$$

In any dimension and in the diffusive limit $ql_e \ll 1$, the screened Coulomb interaction retains the unique form

$$U(\mathbf{q}, \omega) \xrightarrow{q \rightarrow 0} \frac{1}{2\rho_d} \frac{-i\omega + Dq^2}{Dq^2} . \quad (13.205)$$

However, in $d = 1$, this expression of the screened potential may lead to divergences (*e.g.* for the calculation of the density of states correction in eq. (13.66)) and the full form of the interaction must be kept.

In a confined geometry, the expression of the screened interaction depends on the nature of the environment (see section 13.4.4 and reference [315]). For instance, if we consider a two-dimensional sample placed at a distance a from a metallic gate, the image charges induced by the gate modify the static interaction which becomes

$$U_0(\mathbf{q}) = \frac{2\pi e^2}{q} (1 - e^{-2qa}) , \quad (13.206)$$

so that at a distance smaller than a , that is, for $qa \gg 1$, we recover the $2d$ static Coulomb interaction. At large distance, that is for $qa \ll 1$, the interaction becomes

$$U_0(q \rightarrow 0) = 4\pi e^2 a . \quad (13.207)$$

and it can be cast in the form $U_0(\mathbf{q} \rightarrow 0) = e^2/C$, where the capacitance per unit surface is defined as $C = 1/(4\pi a)$.

Similarly, for a wire of section $W \times W$ at a distance a from a gate, the static interaction is

$$U_0(\mathbf{q}) = 2e^2 \left[\ln \frac{e^{-\gamma}}{qW} - K_0(2qa) \right] , \quad (13.208)$$

where $\gamma \simeq 0.577$ is the Euler constant and K_0 is a modified Bessel function [347] (15.76). In the limit $qa \ll 1$, we recover the one-dimensional result ³⁶. At a large distance, that is for $qa \ll 1$, we obtain

$$U_0(q \rightarrow 0) = 2e^2 \ln \frac{a}{W} \quad (13.209)$$

³⁶See remark 35.

which is again of the form $U_0(q \rightarrow 0) = e^2/C$, where we define the capacitance per unit length $C = 1/(2 \ln a/W)$.

Once the static interaction $U_0(\mathbf{q})$ is obtained for these specific geometries, we deduce the dynamically screened interaction from the relation (13.6) :

$$U(\mathbf{q}, \omega) = \frac{1}{U_0^{-1}(q) + 2\rho_d \frac{Dq^2}{-i\omega + Dq^2}} \quad (13.210)$$

and in the diffusive limit where $U_0(q \rightarrow 0, \omega) = e^2/C$,

$$U(q \rightarrow 0) \rightarrow \frac{-i\omega + Dq^2}{2\rho_d Dq^2 - i\omega C/e^2} \quad (13.211)$$

which is the expression (13.71).

Lastly, it is important to notice that the form of the screened interaction depends on the length scale under consideration. For a wire and a length scale smaller than its width W , the interaction retains its three-dimensional form, whereas it is of a $1d$ type for larger values of the length.

The expression of the screened interaction depends on space dimensionality and the energy scale ϵ can be used to monitor the crossover between dimensionalities $d = 1$ and $d = 3$ through the comparison between the energy dependent length $L_\epsilon = \sqrt{D/\epsilon}$ and the thickness W .

13.2 Appendix A13.2 Lifetime in the absence of disorder

This appendix recalls the main steps of the calculation of the quasiparticle lifetime for an interacting electron gas in the absence of disorder. The result is at the basis of the Landau theory of Fermi liquids.

In the absence of disorder, the eigenstates are plane waves indexed by their momentum, that is $|\alpha\rangle = |\mathbf{p}\rangle$, $|\gamma\rangle = |\mathbf{p}'\rangle$, $|\beta\rangle = |\mathbf{p} - \mathbf{q}\rangle$ and $|\delta\rangle = |\mathbf{p}' + \mathbf{q}\rangle$. Therefore, the relation (13.92) becomes :

$$\begin{aligned} \frac{1}{\tau_{ee}(\epsilon)} &= \frac{4\pi}{\nu_0} \int_0^\epsilon d\omega \int_{-\omega}^0 d\epsilon' \sum_{\mathbf{p}\mathbf{p}'\mathbf{q}} \frac{|U_{\mathbf{q}}|^2}{\Omega^2} \\ &\times \delta(\tilde{\epsilon} - \epsilon_{\mathbf{p}}) \delta(\tilde{\epsilon}' - \epsilon_{\mathbf{p}'}) \delta(\tilde{\epsilon} - \omega - \epsilon_{\mathbf{p}-\mathbf{q}}) \delta(\tilde{\epsilon}' + \omega - \epsilon_{\mathbf{p}'+\mathbf{q}}) \end{aligned} \quad (13.212)$$

where $\tilde{\epsilon} = \epsilon + \epsilon_F$ and $\tilde{\epsilon}' = \epsilon' + \epsilon_F$. We denote by $U_{\mathbf{q}}$ the Fourier transform of the screened interaction potential, as given by the relation (13.2), in the Thomas-Fermi approximation. The quadratic dispersion relation for electrons implies that $\epsilon_{\mathbf{p}-\mathbf{q}} = \epsilon_{\mathbf{p}} - \mathbf{v}_F \cdot \mathbf{q} + q^2/2m$ and $\sum_{\mathbf{p}} = \nu_0 \int d\epsilon_{\mathbf{p}} d\varpi$. The integration over the momenta \mathbf{p} and \mathbf{p}' leads to

$$\frac{1}{\tau_{ee}(\epsilon)} = 2\pi\nu_0 \int_0^\epsilon \omega d\omega \sum \frac{|U_{\mathbf{q}}|^2}{\Omega^2} \int \delta(\omega - \mathbf{v} \cdot \mathbf{q} + \frac{q^2}{2m}) \delta(\omega - \mathbf{v}' \cdot \mathbf{q} - \frac{q^2}{2m}) d\varpi d\varpi' \quad (13.213)$$

where ϖ (resp. ϖ') is the solid angle (\mathbf{v}, \mathbf{q}) (resp. $(\mathbf{v}', \mathbf{q})$). Upon angular integration and in the limit $\omega \ll \epsilon_F$, we obtain

$$\frac{1}{\tau_{ee}(\epsilon)} = 2\pi\nu_0^3 \epsilon^2 \langle |U|^2 \rangle, \quad (13.214)$$

where the interaction parameter is

$$\langle |U|^2 \rangle = \frac{1}{4\nu_0^2 \Omega^2} \sum_{|\mathbf{q}| < 2k_F} \frac{|U_{\mathbf{q}}|^2}{(v_F q)^2}. \quad (13.215)$$

In dimension $d = 3$, from the expression (13.2) of the screened potential and in the limit $\kappa \ll k_F$ [332], we deduce

$$\boxed{\frac{1}{\tau_{ee}} = \frac{\pi^2}{64} \frac{\kappa}{k_F} \frac{\epsilon^2}{\epsilon_F}} \quad (13.216)$$

where κ is the inverse screening length (13.4). We notice that the strength of the interaction is of order

$$\langle |U|^2 \rangle \propto \frac{\Delta^3}{\epsilon_F}. \quad (13.217)$$

For any κ and k_F , we obtain

$$\begin{aligned}
\frac{1}{\tau_{ee}} &= \frac{\pi \epsilon^2}{16} \frac{\kappa^4}{k_F \epsilon_F} \int_0^{2k_F} \frac{dq}{(q^2 + \kappa^2)^2} \\
&= \frac{\kappa}{k_F} \frac{\pi \epsilon^2}{16 \epsilon_F} \left(\frac{k_F \kappa}{\kappa^2 + 4k_F^2} + \frac{1}{2} \arctan \frac{2k_F}{\kappa} \right) . \quad (13.218)
\end{aligned}$$

which indeed reproduces (13.216) in the limit $\kappa \ll k_F$, whereas for $\kappa \gg k_F$ it becomes

$$\frac{1}{\tau_{ee}} = \frac{\pi}{8} \frac{\epsilon^2}{\epsilon_F} . \quad (13.219)$$