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# Self-trapped leaky waves and their interactions

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We present soleakon: nonlinear self-trapped leaky modes displaying particlelike features. A "soleakon" forms when a wave function induces a *potential barrier*, whose *resonant state (leaky mode)* corresponds to the wave function itself. We show that, for a proper set of parameters, soleakons are robust and propagate while maintaining their envelope almost indefinitely. However, they eventually disintegrate abruptly. These entities exhibit particlelike interactions behavior, which is nevertheless profoundly different from soliton collisions.

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Optical spatial solitons have been studied extensively: from the first observations of Kerr solitons in $CS_2$ [1] and glass slab waveguide [2], to observations of non-Kerr solitons supported by a variety of physical mechanisms [3]. However, in spite of the diversity of mechanisms supporting solitons, they all share universal features of particlelike behavior [4]. This is manifested in their interaction properties, including elastic collisions in integrable systems which conserve the number of solitons [5], as well as complex interactions in non-Kerr media [6] in which solitons exhibit fusion [7], fission, annihilation [8] and spiraling [9]. Solitons interact through their jointly induced refractive-index change. The interactions are short range in all nonlinear media with a local response [5–9], or long range, if the medium displays a highly nonlocal response [10], where the index change extends far away from the localized optical field. Another universal property of solitons is that their tails decay exponentially, which arises from the fact that solitons are guided modes (bound states) of their self-induced waveguides (potentials) [11]. This concept holds for any bright solitons, including temporal solitons, for which the self-trapping occurs in a moving time frame, and solitons in other systems beyond optics.

However, waveguiding does not necessarily imply that the wave must be a guided mode. In fact, efficient linear waveguiding can also be achieved by populating leaky modes (unbound states). A leaky mode is a superposition of radiation (continuum) states [12,13], forming a wavepacket that is highly localized at the vicinity of the structure, but oscillatory outside the waveguide and diverges exponentially far away from it. As such, leaky modes behave fundamentally different from guided modes. The propagation constant of a leaky mode is complex [13], with the imaginary part associated with unidirectional power flow from the localized section to the radiative part. The power in the localized section decays with propagation. However, the decay rate can be extremely small, yielding long-lived localized modes.

Here, we take the key concept of solitons: nonlinear modes of their own self-induced potential and generalize it to *leaky* modes. We find a type of a self-trapped wave—*the soleakon:* a wavepacket that induces a potential and at the same time populates its leaky mode, self-consistently. Like solitons, soleakons are elementary entities that should appear in a large number of nonlinear systems. Soleakons share some of the properties of solitons: they are robust, maintain their shape, and conserve most of their power for very long (yet finite) propagation distances: orders of magnitude larger than the "diffraction length" of the same wavepacket in a linear medium. Soleakon interactions exhibit repulsion at small angles, and then fusion and annihilation at larger angles (unlike bright solitons in non-Kerr media, which fuse only at small angles). At the same time, soleakons display properties unique to leaky modes, interacting with each other through radiation (continuum) waves, giving rise to a *very long-range, position-dependent interaction.* Furthermore, because soleakons decay into specific spatial frequencies, we find that they can be resonantly amplified by a plane wave.

Soleakons are nonlinear entities associated with linear leaky modes of their self-induced waveguide. Let us discuss leaky modes first. Leaky modes are solutions of the propagation equation when applying outgoing boundary conditions. A leaky mode radiates power to infinity, and the imaginary part of its propagation constant reflects this decay. Interestingly, the real part of the propagation constant is smaller than the wavenumber $k_0 n_{cl}$, where $n_{cl}$ is the refractive index far away from the waveguide. As such, the spatial spectrum of a leaky mode belongs entirely to radiation modes. In order to excite a leaky mode, one has to excite properly its localized section, which resembles a bound state. Because a leaky mode is not a true eignemode, the radiation modes comprising it dephase, hence radiation is constantly emitted away at a distinct angle.

Let us introduce nonlinearity and seek self-trapped leaky modes. We seek a wavepacket launched into a homogeneous nonlinear medium [14] and inducing an index change, for which the wave function corresponds to the leaky mode (at the vicinity of the waveguide). This leads to "self-confinement," with small power leakage corresponding to the imaginary part of the propagation constant (which now varies with propagation). We find the wave functions and propagation constants of soleakons numerically, through a modified self-consistency method [11]. To apply outgoing boundary conditions, we add a complex absorbing potential (CAP) [15] to the self-consistency method. Then, in each iteration, instead of choosing the bound states (which would lead to solitons) we choose a leaky mode–a localized wave function having a complex eigenvalue. The calculation converges after a few iterations. An example of a soleakon is shown in Fig. 1(a). It has one pronounced peak with oscillatory tails, self-trapped in the index structure shown below it.

Having found soleakons, it is natural to ask whether they are robust to noise and can maintain their entity for large

 

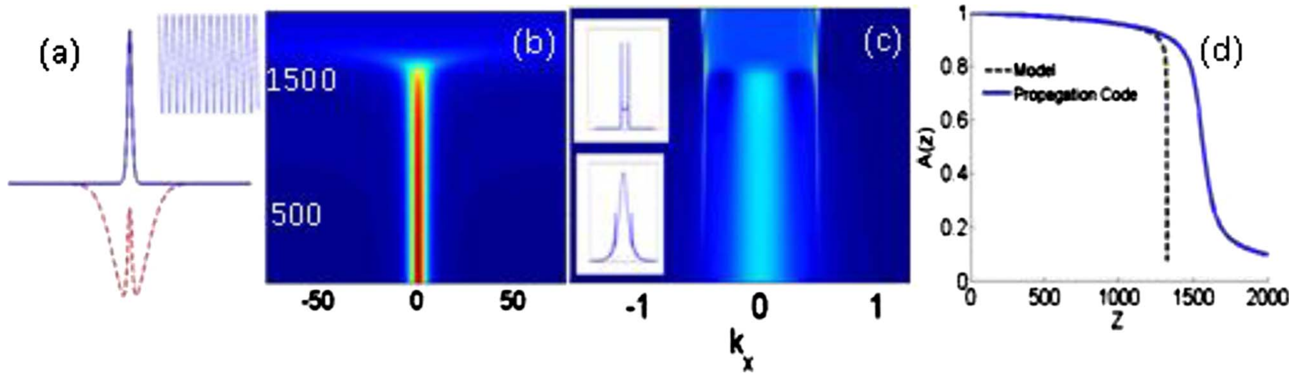


FIG. 1. (Color online) Soleakon properties. (a) Wave function (blue solid line) and induced refractive index (red dashed line). Inset: the oscillating field far from the waveguide. (b) Nonlinear propagation in real space. (c) $K$-space nonlinear propagation. The insets show the spatial Fourier transform of the wave function at the early stages of propagation and after the soleakon disintegrates. (d) The projection $A(z)$ from the simulation (solid line) and the model (dashed line).

propagation distance. We study soleakons in a medium in which a "double-barrier" W structure [Fig. 1(a)] can be realized. Linear leaky modes of such structures can be long lived [16]. Inducing a W-shaped index change can be realized in any medium containing both positive and negative nonlinearities, with different functionalities with respect to the intensity. The negative nonlinearity should have a wider profile, which could arise, for example, from a saturable [17] or from a nonlocal [10] nonlinearity. The positive (self-focusing) nonlinearity could have any other form that creates a narrower profile, to achieve an index change as in Fig. 1(a). The joint effect of the negative-wide and positive-narrow nonlinearities, results in the desired double-barrier potential. This is just one realization, and one can implement many others, e.g., index change similar to that of hollow waveguides, etc. The case depicted in Fig. 1(a) can be realized for example in glasses, polymers etc. (which exhibit both a negative nonlocal nonlinearity and the self-focusing optical Kerr effect), or in photorefractive crystals (with a saturable self-defocusing screening nonlinearity [17] and the self-focusing optical Kerr effect). Beyond optics, Bose-Einstein condensates display a nonlocal nonlinearity due to the dipole effects of the atoms [18]. Here we demonstrate the effects by using the nonlocal negative nonlinearity and a positive Kerr nonlinearity, although we stress that we investigated the existence of soleakons also for saturable nonlinearity combined with the Kerr effect, and did not find any fundamental difference. The paraxial system is described by the nonlinear Schroedinger-like equation

$$i\frac{\partial\Psi}{\partial z}+\frac{\partial^2\Psi}{\partial x^2}+n_1\left\{\int_{-\infty}^{\infty}|\Psi_{(\xi-x)}|^2e^{-(\xi/\sigma)^2}d\xi\right\}\Psi+n_2|\Psi|^2\Psi=0$$

where $\Psi$ is the slowly varying envelope of the electric field, $n_1$ is the strength of the nonlocal nonlinearity ($n_1<0$), $\sigma$ is the nonlocality range, and $n_2$ is the Kerr coefficient.

We use the modified self-consistency method as described above, and find the soleakon wave function and refractive-index change [Fig. 1(a)]. In contrast to solitons, here $\Psi$ does not decay exponentially away from the center, but is instead oscillating with constant amplitude [inset in Fig. 1(a)]. Having found the soleakon, we study its propagation by using it as an initial condition and simulating Eq. (1). First, we examine linear propagation (both $n_1,n_2=0$), and find that the beam broadens within a distance smaller than 10 in normalized units. However, under the proper nonlinear conditions (for which $\Psi$ was calculated), the beam is propagating almost without change to a distance of $\sim 1500$, and then it is abruptly terminated, due to the continuous losses of this self-trapped leaky mode [Fig. 1(b)]. It is instructive to examine the spatial power spectrum of the soleakon during propagation [Fig. 1(c)]. Expectedly, the power spectrum is almost propagation invariant (until the soleakon abruptly disintegrates), comprising of a central lobe and two side lobes associated with the radiation into a narrow region in $k$ space (lower insert). As the soleakon disintegrates (upper insert), the side lobes become pronounced and the spectrum between the side lobes becomes flat. The position of the side lobes can be calculated from the propagation constant of the soleakon. We define a decay rate of the soleakon as

$$\gamma(z)=\frac{\partial}{\partial z}\log\langle\Psi(x,z)|\Psi(x,z=0)\rangle,$$

where the brackets denote integration over all $x$, and $\Psi(x,z=0)$ is localized in the induced-waveguide area. In a linear waveguide, the decay rate of a leaky mode is constant. Since the soleakon is a leaky mode of its own induced waveguide, *its decay rate is increasing with propagation*. Near $z=0$, the decay rate is equal to the rate in a fixed waveguide. As $z$ increases, radiation is emitted from the soleakon, hence the waveguide slowly changes its shape and the decay rate increases. As the soleakon disintegrates, the decay rate sharply increases, reflecting the strong loss from the localized section, in both the spatial and spectral domains. We find strong dependence on the nonlocality range $\sigma$: 20% change in $\sigma$ increases the propagation distance by an order of magnitude.

The dynamics of soleakons is slow (compared to diffraction broadening) and adiabatic: the soleakon slowly loses power, hence the induced waveguide varies, but the wave function self-adjusts-until disintegration point, where transi-

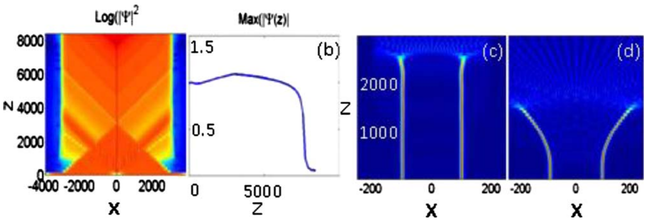

FIG. 2. (Color online) Soleakon pumping and interactions. (a) Natural logarithm of the absolute value of the field envelope. The soleakon appears as a thin line in the middle, and two plane waves appear on either side of the soleakon; the pumping ends at $\sim z=3000$. (b) The maximum absolute value of the soleakon amplitude. In the absence of pumping, the soleakon disintegrates at $z\sim 1500$ [Fig. 1(d)]. (c) and (d) Soleakon collision for two distances, where in (c) there are regions of mutual attraction, and the lifetime is increased due to interaction. The difference between (c) and (d) results from a tiny change in the initial distance between the soleakons.

tion is abrupt. During the adiabatic evolution, the decay rate $\gamma(z)$ at plane $z$ corresponds to the decay rate of the *exact linear leaky mode* of the *induced waveguide* at the same plane $z$. We construct an analytic model for the particular case where the wave function is described by hyperbolic secant, with its functional width dependence, and a radiation term of amplitude $c(z)$,

$$\Psi(x,z) = \varphi(z)\text{sech}\left[\frac{\varphi(z)}{\varphi(0)}\frac{x}{w_0}\right] + c(z)$$

where $w_0$ is the initial width of $\Psi$ (which differs from the width of a Kerr soliton, due to the potential barrier), $\varphi(z)$ is the amplitude of the hyperbolic secant, and $c(z)$ determines the radiation amplitude in the vicinity of the waveguide. One can then define a projection parameter $A(z)=\langle\Psi(x,0)|\Psi(x,z)\rangle$, which is assumed (due to adiabaticity) to obey

$$\frac{da}{dz} = -\gamma(z)A(z). \quad (1)$$

Using complex scaling [14], we find the momentary decay rate for each waveguide realization $[\gamma(A)]$. The radiation amplitude $c(z)$ is also transformed to $c(A)$ by evaluating the radiation amplitude in each $z$ according to the previous step ($c$ can be estimated from knowledge of $\varphi$, $\beta$ as $\sqrt{\frac{P\Gamma}{2\beta}}$, with $P$ being the power). Integrating Eq. (1) yields

$$z = -\int_{A(0)}^{A(z)} \frac{dA}{\Gamma(A)A}.$$

We then fit $\Gamma(A)$ to a polynomial function, and solve for $A(z)$. Figure 1(d) shows $A(z)$ from the simulations against $A(z)$ from the model. The correspondence is excellent until close to the collapse, where evolution is no longer adiabatic. This model describes well at least one family of soleakons, facilitating analytic predictions for its features.

The results in Fig. 1 prove that soleakons are indeed solitonlike entities. However, since the soleakon populates a *leaky mode* of the induced waveguide and *not a bound state*,

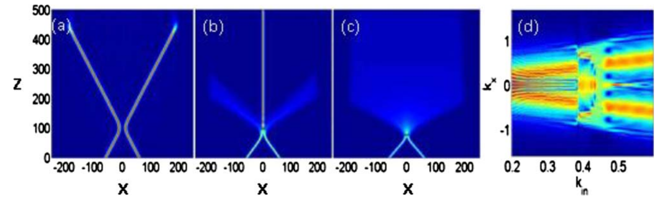

FIG. 3. (Color online) Soleakon collisions. (a) Soleakons repel at small incidence angles. (b) At larger angles soleakons fuse into one stronger soleakon, with larger lifetime. (c) Soleakon annihilation. (d) Scattering matrix constructed for the collision process, with the horizontal axis being the wave vector corresponding to the launch angle, and the vertical axis corresponding to the Fourier transform at a distance of $z=200$.

a plane wave can couple to the induced waveguide through resonant coupling. Such a situation has never been found with self-trapped bright beams. Utilizing such resonant coupling to the soleakon, one can either pump power into the soleakon or extract power from it, depending on the relative phase between the soleakon and the plane wave. Figure 2(a) shows a soleakon with the parameters of Fig. 1(e), amplified by two plane waves, phase matched to increase the soleakon power (we use two plane waves to avoid transfer of transverse momentum to the soleakon). The intensity of the waves is 100 times smaller than the soleakon intensity, hence the index change they induce is extremely small. Nevertheless, these waves amplify the soleakon efficiently when they are at the resonant angle [Fig. 2(b)]: the maximum intensity of the soleakon increases until $z=3000$, where the pumping waves cease to pump the soleakon (due to their finite transverse width). Pumping increases the soleakon lifetime from $\sim 1500$ [Fig. 1(d)] to $\sim 7500$ [Fig. 2(b)].

The profound difference between soleakons and solitons is also manifested in their interactions. Solitons interact through the index change induced by their decaying tails [4], whereas soleakons interact through the index induced by the interference between their radiation fields and through resonant coupling of radiation between soleakons. As such, soleakons display interaction from very large distances even when the nonlinearity is spatially local. For example, soleakons can form in media exhibiting a saturable self-defocusing (photovoltaic or photorefractive) nonlinearity and the optical Kerr effect, with both nonlinearities being local in space. Nevertheless, such soleakons interact from very large distances, since the radiation from each soleakon travels at a specific angle, and eventually reaches the other soleakon (for a small enough decay rate). Consequently, interactions between soleakons are extremely long range, even if the underlying nonlinearities are fully local. This is fundamentally different from soliton interactions, for which interaction range is determined by the range of nonlocality [9].

Soleakon interactions are highly sensitive to their relative distances. Soleakons exhibit both attraction and repulsion, even for the same realizations: two soleakons can start with attraction, move toward one another [Fig. 2(c)], and then—once their relative distance has changed—the interaction becomes repulsive [Fig. 2(d)]. This is the outcome of interference between radiation terms residing outside the induced waveguide supporting each soleakon. Also, in the same vein as prolonging the soleakon lifetime by "pumping" it with a plane wave, *soleakons can influence the lifetime of each other through interaction*. See Fig. 2(c), where the soleakons are identical thus their resonant angle is equal and the radiation plays the role of a pumping wave, prolonging the lifetime of both.

When bringing two in-phase soleakons closer together, and launching them at small relative angles, the interaction is repulsion [Fig. 3(a)], due to the negative sign of the wide nonlinearity. As the relative angle is increased, when it reaches some threshold, the two soleakons merge to form a single soleakon, with a stronger peak intensity and longer lifetime [Fig. 3(b)]. Notice that, when the soleakons merge, radiation is always ejected (as expected from the law of brightness), in the form of two symmetric wings. Interestingly, we observe that such interactions can transform *two intense soleakons* into a *single soliton*. As we further increase the angle between the soleakons, there is a critical angle at which the soleakons annihilate each other into a continuum of spatial frequencies [Fig. 3(c)]. The "kinetic energy" at this critical angle roughly corresponds to the maximum of the potential barrier forming when the two soleakons constructively interfere. The annihilation of the soleakons occurs for some finite range of relative angles, and then gradually, as the angle is further increased, the soleakons pass through each other virtually unaffected. The interaction between soleakons can be summarized by a "scattering matrix," depicting the power spectrum of the light at the output as a function of the mean transverse wavenumber at incidence $k_{in}$ (incidence angle) [Fig. 3(d)]. At small $k_{in}$, the soleakons repel, and two soleakons emerge, each with $k_{out}=k_{in}$. As $k_{in}$ reaches the fusion region, the output power is localized around $k_{out}=0$, and some power is lost to radiation. As we further increase $k_{in}$, we reach the critical energy for annihilation, appearing as a continuum of frequencies. At even larger $k_{in}$, the spectrum is continuously transformed into two beams again, indicating the soleakons passing through each other. As we increase the soleakons' intensity, they transform into interacting solitons: the width of the fusion and annihilation regions becomes numerically irresolvable.

Before closing, we discuss the difference between soleakons and another form of nonlinear localized waves which decay during propagation: quasisolitons (QSs) [19]. QSs arise when higher-order dispersion is added to an equation which otherwise supports solitons, e.g., temporal solitons in the presence of higher-order dispersion. However, for QSs the underlying equation is inherently different, and it does not support any bound states. This gives rise to profound differences: the origin of the loss to radiation waves, the decay rate and the resonance frequency, all behave differently for soleakons and QSs.

To conclude, we presented soleakons: long-lived self-trapped leaky waves which eventually disintegrate abruptly. Soleakons are symbiosis between solitons and leaky waves; hence, they exhibit features of both solitons (self guiding and collision properties) and leaky modes (finite lifetime and resonant coupling). These features suggest ideas of interactions among self-localized waves. For example, two far-away soleakons interact strongly since they are at resonance, while they do not interact with other self-localized entities (solitons or soleakons) that are much closer. Other future ideas include exciting soleakons by plane waves as a reverse process to soleakon disintegration. This soleakon concept holds for various systems beyond optics, e.g., cold atoms displaying the nonlocal dipole interaction and more.

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