

# Causality effects on accelerating light pulses

Ido Kaminer,<sup>1</sup> Yaakov Lumer,<sup>1</sup> Mordechai Segev,<sup>1,\*</sup> and Demetrios N. Christodoulides<sup>2</sup>

<sup>1</sup>Physics Department and Solid State Institute, Technion, 2011, Haifa 32000, Israel

<sup>2</sup>CREOL - College of Optics & Photonics, University of Central Florida, Orlando, FL 32816, USA  
msegev@tx.technion.ac.il

**Abstract:** We study accelerating and decelerating shape-preserving temporal Airy wave-packets propagating in dispersive media. We explore the effects of causality, and find that, whereas decelerating pulses can asymptotically reach zero group velocity, pulses that accelerate towards infinite group velocity inevitably break up, after a specific critical point. The trajectories and the features of causal pulses are analyzed, along with the requirements for the existence of the critical point and experimental schemes for its observation. Finally, we show that causality imposes similar effects on accelerating pulses in the presence of local Kerr-like nonlinearities.

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## References and links

1. M. V. Berry and N. L. Balazs, "Nonspreading wave packets," *Am. J. Phys.* **47**(3), 264–267 (1979).
2. G. A. Siviloglou and D. N. Christodoulides, "Accelerating finite energy Airy beams," *Opt. Lett.* **32**(8), 979–981 (2007).
3. G. A. Siviloglou, J. Broky, A. Dogariu, and D. N. Christodoulides, "Observation of accelerating Airy beams," *Phys. Rev. Lett.* **99**(21), 213901 (2007).
4. P. Saari, "Laterally accelerating airy pulses," *Opt. Express* **16**(14), 10303–10308 (2008).
5. I. M. Besieris and A. M. Shaarawi, "Accelerating Airy wave packets in the presence of quadratic and cubic dispersion," *Phys. Rev. E Stat. Nonlin. Soft Matter Phys.* **78**(4), 046605 (2008).
6. K.-Y. Kim, C.-Y. Hwang, and B. Lee, "Slow non-dispersing wavepackets," *Opt. Express* **19**(3), 2286–2293 (2011).
7. A. Chong, W. H. Renninger, D. N. Christodoulides, and F. W. Wise, "Airy-Bessel wave packets as versatile linear light bullets," *Nat. Photonics* **4**(2), 103–106 (2010).
8. D. Abdollahpour, S. Suntsov, D. G. Papazoglou, and S. Tzortzakis, "Spatiotemporal airy light bullets in the linear and nonlinear regimes," *Phys. Rev. Lett.* **105**(25), 253901 (2010).
9. R. Y. Chiao and P. W. Milonni, "Fast Light, Slow Light," *Opt. Photonics News* **13**(6), 26–30 (2002).
10. I. Kaminer, M. Segev, and D. N. Christodoulides, "Self-accelerating self-trapped optical beams," *Phys. Rev. Lett.* **106**(21), 213903 (2011).
11. Y. Fattal, A. Rudnick, and D. M. Marom, "Soliton shedding from Airy pulses in Kerr media," *Opt. Express* **19**(18), 17298–17307 (2011).
12. G. P. Agrawal, *Nonlinear Fiber Optics, Optics and Photonics* (Academic Press, London, 2001).
13. E. Greenfield, M. Segev, W. Walasik, and O. Raz, "Accelerating light beams along arbitrary convex trajectories," *Phys. Rev. Lett.* **106**(21), 213902 (2011).

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## 1. Introduction

In an intriguing paper from 1979, Berry and Balazs found a shape-preserving self-accelerating solution to the free-particle Schrödinger equation, in the form of an Airy function [1]. Almost 30 years later, Siviloglou and Christodoulides used the analogy between the Schrödinger equation and the optical paraxial wave equation to predict and demonstrate Airy self-accelerating beams [2,3]. Near the end of their paper [2], they also discussed temporal accelerating wave-packets, utilizing the mathematical equivalence between pulse propagation in dispersive optical fibers and the paraxial wave equation describing the propagation of optical beams in homogenous media. However, there is an important physical difference between spatial and temporal accelerations: although both have the Airy shape, a spatial accelerating beam bends its trajectory in space, whereas only the temporal accelerating pulse truly changes its actual group velocity. Additionally, there is another fundamental difference

between the two cases of accelerating wave-packets: the temporally accelerating pulse must obey causality, whereas the spatial Airy beam can bend either way, depending only on its shape at the launch plane.

Many theoretical and experimental studies have followed reference [2,3], but only a handful dealt with acceleration in the temporal domain. For example, cases where the temporal acceleration is integrated into the self-bending Airy beam, to create lateral acceleration [4]. Another example describes a method to spatially modulate an input pulse, causing it to accelerate in time. A detailed study is provided in [5], where the accelerating Airy solution is found and generalized to systems with third order dispersion. In all previous studies on accelerating pulses, including in the presence of third-order dispersion, the wavepacket accelerates along a parabolic trajectory. In a related area, temporally-accelerating Airy solutions were predicted in plasmonic slab waveguides [6]. Interestingly, extending the concept of accelerating wavepackets into several dimensions has led to exciting experiments towards the generation of linear light bullets [7,8]. This ongoing research on accelerating pulses raises many questions: Does an accelerating pulse accelerate forever, until its group velocity diverges? Assuming that pulses can also be decelerated, how slow can they get [9]? Furthermore, following the recent prediction on the existence of self-accelerating self-trapped nonlinear beams [10], it is natural to ask what other physical phenomena will appear for Airy pulses in the presence of nonlinearity, as initial studies [10, 11] have begun to unravel? But perhaps the most intriguing question is what happens when the Airy pulse is designed to accelerate to a group velocity faster than light? Is this at all possible? If so, under what conditions, and at which point would this happen? What would be the physics when a pulse “attempts” to acquire an infinite speed? Undoubtedly, this region, “close to the boundaries of absurd”, holds most of the intriguing physics.

Here we study the propagation of accelerating and decelerating shape-preserving temporal wave-packets in linear media and in media exhibiting instantaneous and spatially-local Kerr-like nonlinearities. The requirements for the existence of such pulses are analyzed, and a setup with realistic parameters is proposed. We find two regimes. For decelerating pulses, we find that the velocity of the wavepacket begins with the group-velocity of the pulse, and decreases all the way to zero, which it would reach asymptotically after an infinite propagation distance. For accelerating pulses, we show that the velocity of the wavepacket begins with the group velocity of the pulse, and seems to diverge even after a finite propagation distance, where we find that the wavepacket breaks apart. This phenomenon can be observed even in the presence of high order dispersion terms, as it is caused solely because causality is imposed on the initial conditions. We investigate this point of “diverging speed” and find what happens to the pulse before and after the point. We conclude with applications arising from the abrupt breakup of the Airy pulse.

## 2. Derivation and trajectories

We begin with the general case that includes linear media and media exhibiting an instantaneous Kerr-like, spatially local, nonlinear response. In a one-dimensional (1D) dispersive medium, with second order dispersion of  $k''$ , and negligible higher dispersion, the slowly-varying envelope approximation gives

$$i\psi_z + i\psi_t/v_0 - \frac{1}{2}k''\psi_{tt} + \gamma\psi = 0 \quad (1)$$

where  $\psi = \psi(z,t)$  is the (complex) envelope of the wavepacket, and  $\gamma = \gamma(|\psi|^2)$  is the nonlinear response, which throughout this paper is assumed to be a function of the pulse intensity  $|\psi|^2$  only. From this point, we follow the derivation of our previous paper [10], which has predicted and analyzed nonlinear self-accelerating beams. As shown there, for a broad class of spatially-local nonlinearities, any localized shape-preserving accelerating solution must move along a parabolic trajectory. Temporally, this means that the nonlinearity must be instantaneous, and also that, whenever the pulse intensity drops to zero the nonlinear effect also disappears. Therefore, we can write the general **accelerating** solution of Eq. (1) as

$$\psi = A \left( \frac{t-t_0}{T} - \frac{z}{Tv_0} - \frac{k'' z^2}{4T^4} \right) e^{i\theta(z,t)} \quad (2)$$

where  $A$  is some function of  $t$  (see [10]), which in the linear regime coincides with the Airy function, and  $v_0$  is the group velocity evaluated at the carrier frequency of the pulse. On the other hand  $t_0$  is an arbitrary shift in time. The phase term is a real function, given as

$$\theta = z(t-t_0) \left( \frac{k''}{2T^3} \right) - z^2 \left( \frac{k''}{2T^3 v_0} \right) - z^3 \left( \frac{k''^3}{12T^6} \right) \quad (3)$$

where  $T$  is a scaling parameter (say, the width of the main “lobe” in the temporal “pulse”). Figure 1a presents examples of  $T < 0$  (inverting it right-left would make it represent  $T > 0$ ). Hence, for  $T < 0$ , generating the pulse means increasing the amplitude exponentially, then slowing the increase until the maximum is reached, and then decreasing it in an oscillatory manner – following the shape of the Airy function. As we have shown in our paper on nonlinear accelerating beams [10], this general trend occurs also for all nonlinear accelerating beams in Kerr and saturable Kerr-like media: one side has exponential decay and the other decays in an oscillatory manner. Truncating the exponential tail has virtually no effect on the beam, whereas truncating the oscillatory tail defines a finite distance during which the beam is able to accelerate, and after that the acceleration stops, and the beam experiences diffraction broadening. The same situation will occur for a temporal pulse.

With this in mind, the choice between  $T > 0$  and  $T < 0$  determines which comes first: the exponentially decaying tail ( $T < 0$ ) or the oscillatory tail ( $T > 0$ ). The major difference between these two choices is easily understood when deriving the group velocity of the beam:

$$v = \frac{dz}{dt} = \frac{1}{\frac{1}{v_0} + \frac{k'' z}{2T^3}} = \frac{v_0}{1 + \frac{k'' z}{2T^3} v_0} \quad (4)$$

For  $T > 0$ , the group velocity decreases with distance (hence also in time), therefore, the wavepacket decelerates to zero group velocity, which is reached after an infinite distance (Fig. 2a). The opposite case, of  $T < 0$ , has an increasing velocity (Fig. 2b). Here, the wavepacket accelerates, and surprisingly, should reach an infinite speed after a finite distance of  $z_{critical} = 2|T|^3/(k''^2 v_0)$ . We call this point the “critical point”. Note that the critical point also has a meaning in the decelerating case – it is the point where the pulse slowed to exactly half of its initial velocity.

We emphasize that, up to this point, the formulation is completely general, without any need to specify the nonlinearity, except for stating that the nonlinear refractive index change is a local function of the optical intensity  $\gamma = \gamma(|\psi|^2)$ .

These findings are a direct outcome of the exact solution. They immediately raise the question, how can that be that an exact solution to a physical propagation equation yields wavepackets that could accelerate to infinite velocities? One could argue that perhaps the problem is with the equation, which uses dispersion to second-order only, whereas in all physical systems the dispersion curve is some analytic function, and the second order is merely the leading term in a Taylor expansion [12]. In other words, would adding third-order dispersion necessarily leads to some bound on the maximum velocity of an accelerating pulse? What happens to causality, how does it enter the evolution equation and how does it affect the solutions? The purpose of this article is to address this kind of questions.

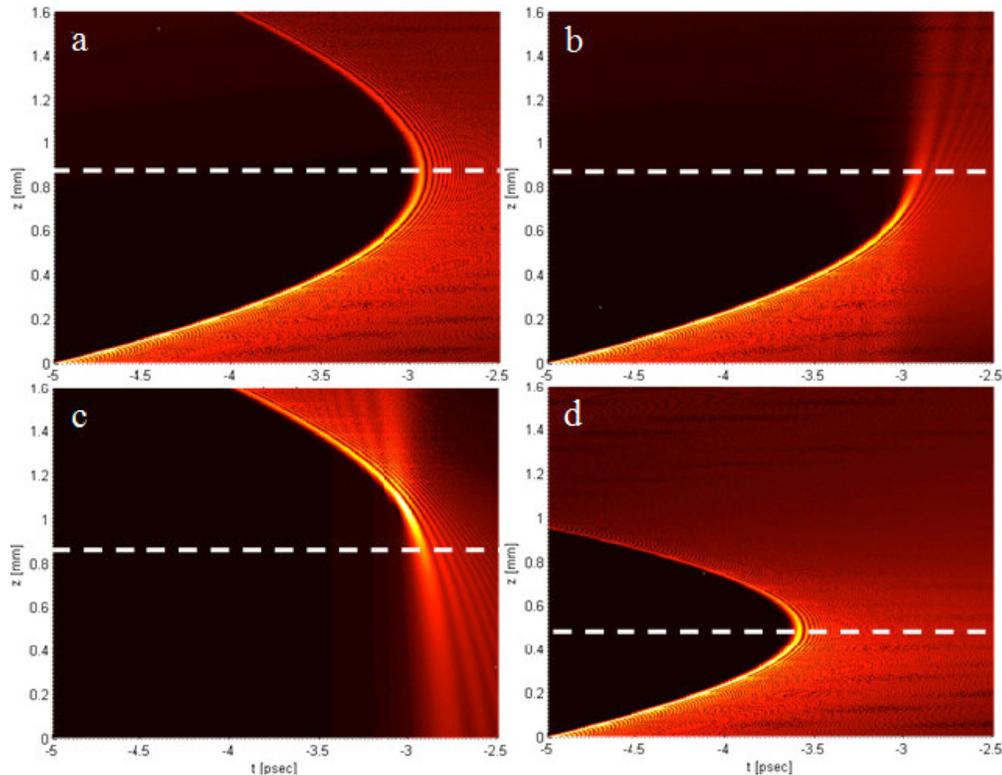


Fig. 1. Dynamics of an accelerating Airy wavepacket. (a) Without imposing causal initial conditions, the wavepacket comprises of two counter-propagating pulses, with positive and negative group velocities. (b) Physical propagation: frequencies with negative group velocities are eliminated. (c) Unphysical backward propagation: frequencies with positive group velocities are eliminated. (d) Propagation with third-order dispersion included, and without imposing causal initial conditions. The white dashed line marks the position of the critical point.

### 3. Solving the mystery: what actually happens at the vicinity of the critical point?

Let us first handle the linear case of Eq. (1). As shown above, the analytic solution for accelerating shape-preserving solutions of Eq. (1) yields a critical point. At such point, the trajectory of the main lobe (Fig. 1a) has infinite slope, which one may naively interpret as infinite velocity (dashed line). But even more bizarre is the fact that the trajectory continues after this critical point backwards in time (curve above the dashed line). It seems that not only the speed diverges, but also causality is violated. How can this be? What actually happens at the vicinity of the critical point?

The answer is actually intuitive. The analytic Airy solution of Eq. (1) does not represent one pulse accelerating towards the critical point, but rather two pulses. The pulses are “launched” from opposite sides of the  $z$ -axis, with opposite group velocities, and meet exactly at the critical point (see Figs. 1b,c, where the dashed white line marks the critical point). Figure 1b and 1c shows the propagation of the wavepacket comprised only of frequency components of the Airy pulse that have positive (negative) group velocity. What seems in Fig. 1a as causality breaking (above the dashed line) is simply the trajectory of the backward-moving pulse, because Fig. 1a is simply the superposition of Figs. 1b and 1c. The infinite slope of the trajectory at the critical point is observed as the interference between the tails of the two pulses when they reach their meeting point. Another intuitive explanation is that the wavepacket at any specific plane  $z$  is composed of both forward and backward propagating waves. Mathematically, this arises from Eq. (1) being a second-order differential equation.

Therefore, the seemingly causality-breaking is due to backward propagating waves that are assumed in the initial condition (as in Fig. 1c). This issue will be discussed in more details in the next section.

In experiments, one launches only one Airy-shaped pulse from only one side of the medium. i.e., the wavepacket at the initial plane  $z = 0$  has only forward propagating frequencies components (those with positive group velocity). Hence, the physical Airy pulse is actually a “half-Airy” pulse. This way, the pulse would not interfere with a backward propagating twin at the critical point. The actual dynamics is therefore not the propagation-invariant Airy solution of Eq. (1). This peculiarity – that a single Airy pulse does not have the full dynamics of an Airy solution – leads to an interesting question: What happens at the critical point when only one pulse arrives there? We expect the Airy pulse to propagate as a shape-preserving wave structure until the critical point, because it has no interaction with the non-existing twin. But then, exactly at the critical point and beyond it, the propagation-invariant Airy dynamics should break down and the pulse should suddenly change its shape.

To study the behavior at the vicinity of the critical point, we simulate the propagation of an Airy pulse launched from  $z = 0$ , after cutting out (in the Fourier space) the backward propagating frequencies. This yields the dynamics of the “half-Airy” pulse. The results are shown in Fig. 1b, and exhibit a rapid breakup of the pulse (within  $\sim 0.1mm$  in this example), that up to the critical point has been shape-preserving. The breakup occurs within a time frame  $T$  (hence the breakup distance is  $cT$ ), where  $T$ , as defined above, is approximately equal to one half of the width of the main “lobe” of the Airy pulse. That is to say that the breakup distance tends to be much shorter than the dispersion length of pulses of a spectral width that can be well described within the framework of Eq. (1). To put this breakup distance,  $(cT)$ , on quantitative grounds, it is instructive to compare it to the 'temporal Rayleigh length' near the critical point, which is approximately equal to the dispersion length of the first lobe ( $T^2/k$ ). Therefore, in order to call the breakup 'abrupt', the requirement is  $k''c \ll T$ . This condition is easily fulfilled when put together with the conditions in the next section.

The maximal intensity of the pulse remains constant during propagation until it gets close to the critical point, where the maximal intensity drops. The pulse continues to propagate after the critical point, but the pulse broadens and its maximum intensity decays. Figure 1c displays the dynamics of the opposite pulse, where the forward-propagating frequencies were cut out, hence simulating only the backward-moving part. The superposition of the two counterpropagating pulses is mathematically equivalent to Fig. 1a.

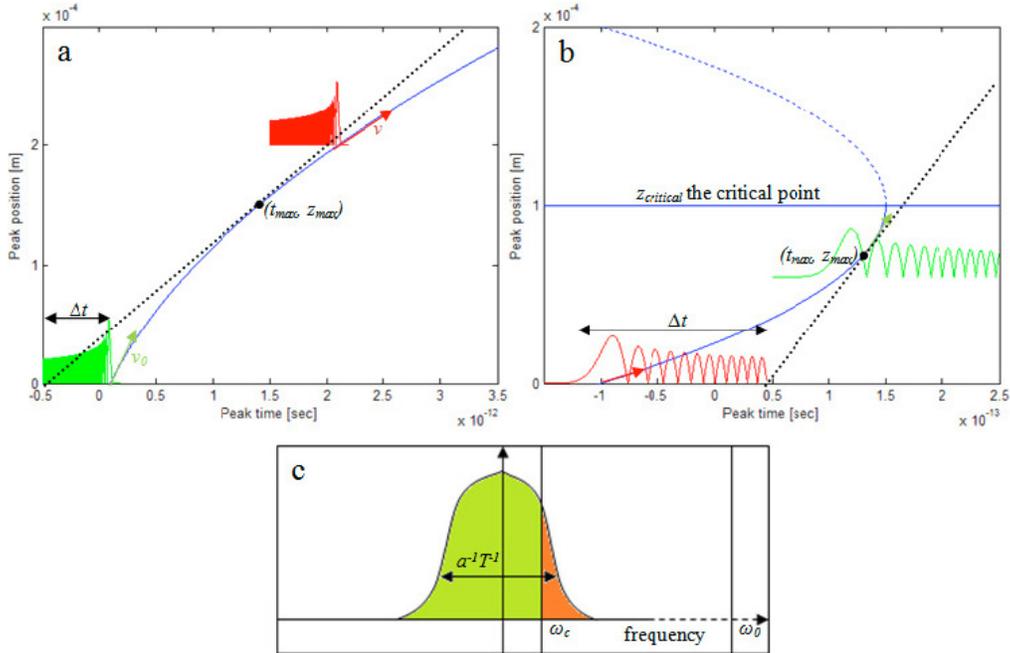


Fig. 2. Schematic trajectories of (a) a decelerating and (b) an accelerating Airy wavepacket. The dashed tangent lines mark the point where acceleration stops due to the finite extent of the pulse. (c) The spectrum of an Airy pulse truncated at the critical frequency, and compared to the carrier frequency.

#### 4. Critical point: existence conditions

One might be tempted to suggest that perhaps the critical point is non-physical, but rather an artifact of the approximations involved, such as the fact that the dispersion we use is second-order only. That is, one could justly question whether high-order dispersion could interfere with the breakup phenomenon and prevent it from happening. Moreover, in practical terms, the distance  $z_{critical}$  might be too far for a realistic experimental system, or require an extremely long pulse to observe the breakup. Also, the necessary frequency spectrum of the pulse might be too wide and come too close to the carrier frequency. This section comes to show that the critical point truly exists, and to investigate the necessary and sufficient conditions for the observation of the breakup and the critical point.

To handle the issue of higher-order dispersion, we simulate the propagation of the (non-causal) Airy solution of Eq. (1), in the presence of strong third-order dispersion. The result is presented in Fig. 1d. We find that adding the third-order dispersion indeed changes the propagation dynamics (compare Fig. 1d to Fig. 1a), but the rapid breakup behavior of the pulse is clearly apparent nevertheless.

As mentioned above, part of the spectrum of the Airy pulse gives rise to backward propagating waves, as shown in the example of Fig. 2c. Interestingly, the division between positive and negative group velocities is not symmetric. Let us explain the asymmetry. Denoting the transition point between the two parts of the spectrum as the “critical frequency”  $\omega_c$ , which is found by deriving the group velocity and setting it to zero. This procedure is simple when we neglect dispersion terms above the second order, and gives  $\omega_c = 1/(k''v_0)$ . To see the critical point, a non-negligible part of the power of the pulse should have its frequency above  $\omega_c$ . In addition, the carrier frequency  $\omega_0$  must be much larger than  $\omega_c$ , to allow the generation of such a pulse. From  $\omega_0 \gg \omega_c$ , we get a condition that does not depend on the pulse itself, but depends only on the medium in which the pulse is propagating, yielding

$$\omega_0 v_0 k'' \gg 1 \quad (5)$$

But this condition is insufficient to guarantee the phenomena associated with the rapid pulse breakup: the actual width of the Airy pulse ( $\sim$ number of lobes) must be also considered. Physically, the Airy wavepacket cannot have an infinitely-long tail. Hence, just like the Airy beams in space, which are always launched from a finite aperture and hence are diffraction-free only for a finite propagation distance, also for temporal pulses the tail must be truncated at some point. Hence, the propagation-invariant Airy dynamics of the temporal pulse lasts only for a finite distance. The concept of the critical point arises from causality, not from the truncation of the pulse. Hence, to observe the critical point, the truncation must be such that the propagation-invariant dynamics should last for a distance much larger than  $z_{critical}$ . Given a physical truncated pulse which accelerates and is shape-preserving for a finite distance  $z_{max}$ , let us compare  $z_{max}$  with  $z_{critical}$ . For convenience of the calculation, let us use the truncation method suggested in [2], where the initial Airy beam was multiplied by an exponential tail, to yield a “finite energy Airy pulse”. As in [2], using the parameter  $a$  for the decay rate and multiplying the wavepacket in the linear system by  $exp(a^2 t/T)$ , we get an effective Gaussian window in the Fourier plane  $exp(-a^2 \omega^2 T^2)$ . This Gaussian window is drawn in Fig. 2c, along with  $\omega_0$  and  $\omega_c$  which appear as vertical lines. To allow the generation of an accelerating beam which arrives at the critical point and breaks up there, the width of the spectrum,  $a^{-1} T^{-1}$ , should be larger than  $\omega_c$  and at the same time much smaller than  $\omega_0$ . The former condition is to ensure that part of the power will have a negative group velocity, and the latter is to ensure that the resolution of the temporal structure does not fall close to within a single oscillation period  $2\pi/\omega_0$ . Altogether, the ability to observe the critical point necessitates

$$T \omega_0 \gg \frac{1}{a} > \frac{T}{k'' v_0} \quad (6)$$

In the case of nonlinear propagation, the decay-rate parameter  $a$  loses its meaning, because the spectral comparison is not valid (and of course the pulse is not an Airy). To find the analogous constrains for the nonlinear case, we recall that  $a$  controls the distance of acceleration. The condition (6) arises from requiring that the finite acceleration distance will be larger than the critical distance. To find the acceleration distance in a nonlinear case, we qualitatively draw the line from the end of the tail tangent to the trajectory of the peak. Figures 2a and 2b present the trajectory of the wavepacket with the tangent (black dashed line) and the farthest point of acceleration/deceleration marked as  $(t_{max}, z_{max})$ . The point where the tangent crosses the  $t$  axis is determined by the width of the wavepacket, hence this temporal width,  $\Delta t$ , gives the distance of propagation: The ratio  $\Delta t/|T|$  is qualitatively the number of lobes that the wavepacket carries, and is equal to  $a^{-2}$ . Now, we compare this distance to the value of the critical point and derive the same inequality as in (6).

Note that, we find numerically that this “tangent” technique, whose logic is equivalent to caustics methods, works reasonably well also in nonlinear media, even though caustics assume linear rays. The underlying reason why this method works in the nonlinear case relates to the conservation of momentum in Eq. (1).

## 5. Physical realization

We would like to end this article with a specific design for experimental observation of the effects driven by causality on accelerating pulses. Namely, we ask what realistic system will exhibit the causality-driven pulse breakup? Substituting the wavelength of visible light, and group velocity close to  $c$  in expression (5) forces  $k''$  to be much larger than  $800 \text{ fsec}^2/\text{mm}$  (otherwise the fraction of power contained in the backward-propagating frequencies is negligible). Unfortunately, “conventional” optical materials (such as glass) have  $k''$  which is too small by an order of magnitude. Other commonly used materials (e.g., SF10 glass at short wavelengths) have their  $k''$  around a few hundreds  $\text{fsec}^2/\text{mm}$  - which is still not good enough, unless we work in the UV. Therefore, it would be necessary to use a medium with a tailored dispersion, where the second-order dispersion is high while the third-order dispersion is small.

Such propagation medium would be, for example, photonic crystal fibers with engineered dispersion properties.

For example, to fulfill the condition in (5), we look for a medium with  $k''=10,000 \text{ fsec}^2/\text{mm}$ , and use  $\lambda=400\text{nm}$ ,  $a=0.1$  and  $T=20\text{fsec}$ . Those parameters are used in the simulations creating Fig. 1a,b,c. In Fig. 1d we take  $k''=10,000 \text{ fsec}^3/\text{mm}$ , which we estimate from the relation between different orders of dispersion far from resonances. Higher values of the high-order dispersion only shift the critical point, without changing the actual phenomenon.

## 6. Discussion and conclusions

In this article, we have explored the behavior of accelerating and decelerating temporal wavepackets, and predicted the unexpected breakup of accelerating pulses. This general phenomenon is related to the violation of causality, and exists in both linear and nonlinear media. Moreover, it is also relevant when higher order dispersion is included, and also for pulses accelerating along non-parabolic trajectories – in the spirit of the general accelerating beams presented in [13]. Finally, although a clear experimental demonstration of the breakup and the associated phenomena would require specifically-designed medium with high second-order dispersion, we can envision possible applications making use of the rapid collapse, such as distance-controlled radiation, where a localized pulse emits a large amount of radiation after a prescribed distance. Last but not least, these ideas are not specific to optics: they occur in any dispersive waves systems. As such, these phenomena are relevant to sound waves, matter waves, etc.

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