Conformal properties of soft-operators : Use of Null-states

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- In this talk we will consider only massless particles, mostly in space-time dimensions $D = 1 + 5$ and $D = 1 + 3$.

- Now in D dimensional Minkowski space-time we parametrize the momentum $p^\mu(\omega, \vec{x})$ of a massless particle as,

$$p^\mu = \omega(1 + \vec{x}^2, 2\vec{x}, 1 - \vec{x}^2), \quad \vec{x} \in R^{D-2} = R^n \tag{1}$$

- The corresponding transformation of energy ω is given by,

$$\omega' = \left| \frac{\partial \vec{x}'}{\partial \vec{x}} \right|^{-\frac{1}{n}} \omega \tag{2}$$

Soft-theorem and Soft-operators

- Let ($A_a(\omega, x)$) $A_a^\dagger(\omega, x)$ be the (annihilation) creation operator of a photon in D dimensions with helicity a. a is a vector index of $SO(D-2)$ – the little group of massless particles in D dimensions.
- Let us now write Weinberg's soft-photon theorem, say, for an outgoing soft photon,

$$\begin{aligned}
&\langle \{\omega_i, x_i, q_i, out\}|\, A_a^{out}(\omega, x)\, |\{\omega_j, x_j, q_j, in\}\rangle \\
&= \Big[\frac{\gamma}{\omega}\Big(\sum_{i\in out} q_i \frac{2(x-x_i)_a}{(x-x_i)^2} - \sum_{j\in in} q_j \frac{2(x-x_j)_a}{(x-x_j)^2}\Big) \\
&\quad \times \langle \{\omega_i, x_i, q_i, out\}|\{\omega_j, x_j, q_j, in\}\rangle\Big] \\
&\quad + O(\omega^0) + O(\omega) + ...
\end{aligned} \tag{3}$$

where γ is a numerical constant. The incoming and outgoing states are charged scalars with charges q_i.

- Let us now define the soft-photon operator $S_a(x)$ as,

$$S_a(x) = \lim_{\omega \to 0} \omega A_a(\omega, x) \tag{4}$$

(Strominger; He et al. ; Kapec et al. ; Kapec et al. ; Campiglia et al. ; Kapec et al.)

- In terms of $S_a(x)$ the soft-theorem simplifies,

$$\begin{aligned} &\langle \{\omega_i, x_i, q_i, out\} |\, S_a^{out}(\omega, x)\, | \{\omega_j, x_j, q_j, in\} \rangle \\ &= \gamma \left(\sum_{i \in out} q_i \frac{2(x - x_i)_a}{(x - x_i)^2} - \sum_{j \in in} q_j \frac{2(x - x_j)_a}{(x - x_j)^2} \right) \\ &\times \langle \{\omega_i, x_i, q_i, out\} | \{\omega_j, x_j, q_j, in\} \rangle \end{aligned} \tag{5}$$

Lorentz transformation of soft-operators

- The Lorentz group $SO(D-1,1) = SO(n+1,1)$ acts on $\vec{x} \in R^n$ as conformal transformations.
- The crucial point is that under Lorentz transformation, $S_a(x)$ transforms like a primary operator of dimension 1 and spin 1, i.e,

$$S'_a(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{1}{D-2}} R_{ab}(x) S_b(x), \qquad x' = \Lambda x \tag{6}$$

  where $R_{ab}(x)$ is the local rotation matrix associated with the conformal transformation.
- This is quite general. In fact, if we formally write down a Laurent expansion in energy as,

$$A_{a_1 a_2 \dots a_l}(\omega, x) = \sum_{n=-\infty}^{\infty} \frac{S^n_{a_1 a_2 \dots a_l}(x)}{\omega^n} \tag{7}$$

  then $S^n_{a_1 a_2 \dots a_l}(x)$ is a conformal primary of dimension n and spin l.

Goal

- We want to study the properties of the conformal representation with the operators $S^n_{a_1 a_2 \ldots a_l}(x)$ as highest-weight vectors. The "theory" in which we do this has infinite-dimensional global symmetries. These symmetries impose constraints on the representation and vice versa. In this talk I will not explain the "vice versa" part.

- The constraints on the representation translate into constraints on the S-matrix elements with the insertion of the highest-weight vector, i.e, the soft-operator.

- Let me now say a few things about the symmetries.

Symmetry

- The symmetries are asymptotic symmetries in asymptotically flat space. The well known example is *BMS* group which acts on null-infinity. The standard *BMS* group is a semi-direct product of the Lorentz group $SO(D-1,1)$ and the infinite dimensional abelian group of super-translations. The supertranslations are generalisations of the D global space-time translations.
- For example, if we write the metric in Bondi coordinates $(r, u, \vec{x})$, then at null-infinity $(r \to \infty)$ supertranslation acts as,

$$\vec{x} \to \vec{x}, \quad u \to u + f(\vec{x}) \tag{8}$$

  where $f(\vec{x})$ is an arbitrary function on the celestial sphere S^{D-2}, with (stereographic) coordinates $\vec{x}$.
- Another important point is that the Lorentz group $SO(D-1,1)$ acts on the celestial sphere S^{D-2} as conformal transformations.

- Strominger has shown that these asymptotic symmetries are also symmetries of quantum-gravity S matrix.
- From a holographic perspective - assuming that such a thing exists in flat space - then S-matrix is the natural observable and the asymptotic symmetries are infinite dimensional global symmetries of the dual theory. The (infinite) global symmetry acts on the set of S-matrix elements and, hopefully, constrain their form to some extent.
- Remember that this infinite global symmetry is an extra ingredient on top of the usual unitarity, crossing, analyticity...........But, how to use it ??

Ward-identities

- Now, like Poincare $\to$ BMS, there is a parallel story for electromagnetic gauge transformation in the bulk. In this case, the asymptotic symmetries are large $U(1)$ gauge transformations of the form $e^{ief(\vec{x})}$, where the rotation angle $f(\vec{x})$ is now an arbitrary function on the celestial sphere. Let me try to motivative this from a QFT perspective.

- Let us consider a theory of free massless charged scalar fields.

- This theory has a $U(1)$ global symmetry and the conserved charge can be written as,

$$Q_0 = e \int d\mu(\omega, x)(a^\dagger(\omega, x)a(\omega, x) - b^\dagger(\omega, x)b(\omega, x)) \quad (9)$$

- We can generalize this by defining $Q_0(f)$ as ,

$$Q_0(f) = e \int d\mu(\omega, x) f(x) (a^\dagger(\omega, x) a(\omega, x) - b^\dagger(\omega, x) b(\omega, x))$$

where $f(x)$ is an arbitrary function. In free theory $Q_0(f)$ is also conserved and we have an infinite number of them, corresponding to each function $f(x)$. (Banerjee)

- It acts on the states as,

$$\begin{aligned} e^{iQ_0(f)} a^\dagger(\omega, x) e^{-iQ_0(f)} &= e^{ief(x)} a^\dagger(\omega, x) \\ e^{iQ_0(f)} b^\dagger(\omega, x) e^{-iQ_0(f)} &= e^{-ief(x)} b^\dagger(\omega, x) \end{aligned} \tag{10}$$

We can see that $Q_0(f)$ generates a $U(1)$ rotation at every point x. So the free theory has an infinite-dimensional global $U(1)$ symmetry.

- Using the Lorentz transformation property of the creation-annihilation operator it is easy to check that,

$$U(\Lambda)Q_0(f)U(\Lambda)^{-1} = Q_0(f'), \quad f'(x) = f(\Lambda^{-1}x) \tag{11}$$

- Now, in the interacting charged scalar theory, with non-trivial S-matrix, the analog of $Q_0(f)$ is not conserved, unless, you have photon in the theory. In other words, the interacting theory must be a gauge theory.
- In this case the conserved charge can be written as, $Q(f) = Q_H(f) + Q_S(f)$. $Q_H(f)$ is called the hard-charge and is required to generate the $U(1)$ transformation on the charged particles, i.e,

$$\begin{aligned} Q_H^{in}(f)\,|\alpha, in\rangle &= \left(\sum_{k\in\alpha} q_k f(x_k)\right)|\alpha, in\rangle \\ \langle\beta, out|\,Q_H^{out}(f) &= \langle\beta, out|\left(\sum_{k\in\beta} q_k f(x_k)\right) \end{aligned} \tag{12}$$

(Strominger; He et al. ; Kapec et al. ; Kapec et al. ; Campiglia et al. ; Kapec et al.)

- But, unlike in the case of free theory, in the interacting theory,

$$Q_H^{in}(f) \neq Q_H^{out}(f) \tag{13}$$

- The statement of conservation is,

$$Q(f) = Q_H^{in}(f) + Q_S^{in}(f) = Q_H^{out}(f) + Q_S^{out}(f) \tag{14}$$

- This can be written in the form of a Ward-identity as,

$$\begin{aligned} &\langle \beta, out|\, Q_S^{out}(f)\, |\alpha, in\rangle - \langle \beta, out|\, Q_S^{in}(f)\, |\alpha, in\rangle \\ &= \left(\sum_{i\in\alpha} q_i f(x_i) - \sum_{i\in\beta} q_i f(x_i) \right) \langle \beta, out|\alpha, in\rangle \end{aligned} \tag{15}$$

  This is the Ward-identity for the infinite-dimensional (asymptotic) global $U(1)$ symmetry.

- For the purpose of this talk the Ward-identity is an assumption. In other words, the theories we study are partly defined by the ward-identity.

Lorentz transformation of soft-charge $Q_S(f)$

- Now using Lorentz invariance of the S-matrix, one can show, using the Ward-identity that,

$$\begin{aligned} U(\Lambda)Q_S^{out}(f)U(\Lambda)^\dagger &= Q_S^{out}(f') \\ U(\Lambda)Q_S^{in}(f)U(\Lambda)^\dagger &= Q_S^{in}(f') \end{aligned} \tag{16}$$

where

$$f'(x) = f(\Lambda^{-1}x) \tag{17}$$

(Banerjee, Pandey, Paul)

- Let us now write,

$$Q_S(f) = \int d^n x f(x) O(x) \tag{18}$$

where $O(x)$ is another (local) operator and $n = D - 2$.

- We can write this because it follows from the Ward-identity that $Q_S(f)$ is a linear (operator-valued) functional of f, i.e, $Q_S(\alpha f + g) = \alpha Q_S(f) + Q_S(g)$ for any $\alpha \in \mathbb{C}$. Now if we think of f as a vector $|f\rangle$ in an infinite dimensional Hilbert space then we can write, $Q_S(|f\rangle) = Q_S(\int d^n x f(x) |x\rangle) = \int d^n x f(x) Q_S(|x\rangle) = \int d^n x f(x) O(x)$. Here we have defined $O(x_0) = Q_S(|x_0\rangle) \equiv Q_S(f(x) = \delta^n(x - x_0))$.

- Now if we use the Lorentz transformation property of the soft-charge $Q_S(f)$ then one check that,

$$U(\Lambda)O(x)U(\Lambda)^{-1} = O'(x) = \left|\frac{\partial x'}{\partial x}\right| O(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta}{n}} O(x')$$

where $x' = \Lambda x$ and $\Delta = n$.

- Since Lorentz transformation acts on the $x \in R^n$ coordinates as conformal transformations, this shows that $\underline{O(x) \text{ is a scalar conformal primary of weight}}$ $\Delta = n$.

Two assumptions

- There are an infinite number of operators denoted by $S^{\Delta}_{a_1 a_2 \ldots a_l}(x)$, not all of which are trivial and which transform under (Lorentz) conformal transformation as a primary operator of weight Δ and spin l. We also add to this list all the conformal descendants of all the primaries. So each $S^{\Delta}_{a_1 a_2 \ldots a_l}(x)$ together with its descendants form a complete representation of the (Lorentz) conformal group $SO(n+1,1)$. We further assume that the primary operators $S^{\Delta}_{a_1 a_2 \ldots a_l}(x)$ and their descendants carry zero energy-momentum.
- We can call these operators "soft-operators", but, they are not necessarily the same as the ones appearing in the soft-theorems. This identification is part of the problem. Showing this is equivalent to deriving soft-theorem from the Ward-identity.

- The second assumption is :
  The operator $O(x)$ is either a primary by itself or a (primary) descendent of another primary $S^{\Delta}_{a_1 a_2 \cdots a_l}$ or a sum of (primary) descendants of more than one $S^{\Delta}_{a_1 a_2 \ldots\ldots a_l}$. This is a useful assumption. We will show that the operator $O(x)$, so constructed, is
  almost uniquely determined by conformal invariance.
- This assumption is just for the sake of simplicity.
- In this approach, one should also prove that hard-operators do not contribute to the soft-charge. But we do not know how to do that systematically.
- One way may be to go to the Mellin-space. But we will not discuss this possibility in this talk.

Various possibilities for $O(x)$ when $D = 6$ or $n = 4$

- A potential candidate for $O(x)$ is a $(\Delta = n = 4, l = 0)$ (primary) descendant constructed from $S^{\Delta}_{a_1 a_2 \dots . a_l}$.

- There are an infinite number of possibilities.

- Below we have listed the $(\Delta \geq 0, l)$ primary operators which have $(\Delta = 4, l = 0)$ descendent :

$$\begin{aligned} O(x) &= (\partial^2)^2 S^0(x), \quad \partial^2 \partial_a \partial_b S^0_{ab}(x), \quad \partial_a \partial_b \partial_c \partial_d S^0_{abcd}(x) \\ O(x) &= \partial^2 \partial_a S^1_a(x), \quad \partial_a \partial_b \partial_c S^1_{abc}(x) \\ O(x) &= \partial^2 S^2(x), \quad \partial_a \partial_b S^2_{ab}(x) \\ O(x) &= \partial_a S^3_a(x) \end{aligned} \tag{19}$$

Now which of them are primary ? Let us start with the operator $O(x) = \partial^2 \partial_a S^1_a(x)$.

- Under infinitesimal SCT the operator $O(x) = \partial^2 \partial_a S^1_a(x)$ transforms as :

$$O'(x') = (1 + 8\,\epsilon \cdot x)\, O(x) + \boxed{4\,\epsilon_a\, \partial_b F_{ab}(x)} \qquad (20)$$

where we have defined,

$$F_{ab}(x) = \partial_a S_b(x) - \partial_b S_a(x) \qquad (21)$$

The first term in (20) gives the standard transformation of a $(\Delta = 4, l = 0)$ primary. So if we want $O(x) = \partial^2 \partial_a S_a(x)$ to be primary then we have to set the additional piece $\partial_a F_{ab}(x)$ to zero. Now *the equation $\partial_a F_{ab}(x) = 0$ is consistent or conformally invariant only if $\partial_a F_{ab}(x)$ itself is a primary operator.* One can easily check that this is indeed the case. So we can set,

$$\boxed{\partial_a F_{ab} = \partial_a(\partial_a S_b - \partial_b S_a) = 0} \qquad (22)$$

Therefore $\partial_a F_{ab}$ is a primary descendant or null-state of S_a which decouples from the S-matrix.

Differential equation for S-matrix element

- Let us now define

$$A_a(x, \{\omega_\alpha, x_\alpha, q_\alpha\}) = \langle \{\omega_i, x_i, q_i, out\}|\, S_a(x)\, |\{\omega_j, x_j, q_j, in\}\rangle \tag{23}$$

- Now using the constraint equation we get,

$$\boxed{\langle \{\omega_i, x_i, q_i, out\}|\, \partial_a(\partial_a S_b - \partial_b S_a)(x)\, |\{\omega_j, x_j, q_j, in\}\rangle} = 0 \tag{24}$$

  Since there is no ordering between the $(x, \{x_i, out\}, \{x_j, in\})$ coordinates we can pull the derivates outside the S-matrix without producing contact-terms. It is also true that x coordinates are space-like.

- So we can write,

$$\partial_a(\partial_a A_b - \partial_b A_a) = 0 \tag{25}$$

Solving the equation

- This is Euclidean maxwell's equation. In order to solve this we need boundary condition which can be obtained using inversion.
- $$A_a(x, \{\omega_\alpha, x_\alpha, q_\alpha\}) = \langle\{\omega_i, x_i, q_i, out\}|\, S_a(x)\, |\{\omega_j, x_j, q_j, in\}\rangle$$
$$\xrightarrow{x\to\infty} \frac{1}{x^2} I_{ab}(x) M_b(\{\omega_\alpha, x_\alpha, q_\alpha\}) + O(\frac{1}{x^3}) \tag{26}$$

- Since we are in Euclidean space, instead of the wave equation, the Electric fields $E_i (= F_{i4})$, and the magnetic fields $B_i (= \frac{1}{2}\epsilon_{ijk} F_{jk})$ now satisfy the four dimensional Laplace's equation,

$$\partial_a \partial_a E_i = \partial_a \partial_a B_i = 0 \tag{27}$$

  This, together with the falloff condition $F_{ab} \sim O(\frac{1}{x^3})$ as $x \to \infty$ – derived from the fall-off condition of A_a – implies that $E_i = B_i = 0$. Here we have used the fact that a function which is harmonic everywhere and vanishes at infinity is identically zero.

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$$\boxed{\partial_a A_b - \partial_b A_a = 0 \Longleftrightarrow A_a(x, \{\omega_\alpha, x_\alpha, q_\alpha\}) = \partial_a \Lambda(x, \{\omega_\alpha, x_\alpha, q_\alpha\})}$$

- Therefore the four soft S-matrices A_a corresponding to four helicity states of the photon are determined in terms of a single scalar function $\Lambda(x, \{\omega_\alpha, x_\alpha, q_\alpha\})$.

- To be more precise, we can write,

  $$\langle\{\omega_i, x_i, q_i, out\}|\, S_a^{out}(x)\, |\{\omega_j, x_j, q_j, in\}\rangle = \partial_a \Lambda_{out}(x, \{\omega_\alpha, x_\alpha, q_\alpha\})$$

  and

  $$\langle\{\omega_i, x_i, q_i, out\}|\, S_a^{in}(x)\, |\{\omega_j, x_j, q_j, in\}\rangle = \partial_a \Lambda_{in}(x, \{\omega_\alpha, x_\alpha, q_\alpha\})$$

Other operators in the list

- We have verified that there are no other operators in the list except $(\partial^2)^2 S^0$ and $\partial_a S_a^3$, which can be primary with or without constraint. These two operators are primary without any constraint.
- We have also verified using the results of (*Penedones, Trevisani and Yamazaki*) that there are no potential candidate for $O(x)$ which is a descendant of $S^{\Delta}_{a_1,\ldots,a_l}$ with $\Delta < 0$.
- Therefore the potential candidate for $O(x)$ in the case of $U(1)$ is,

$$\boxed{O(x) = \partial^2 \partial_a S_a^1(x) + \alpha(\partial^2)^2 S^0(x) + \beta \partial_a S_a^3(x)} \tag{28}$$

where α and β are numbers.

Gravity and Supetranslation

- It works in the same way as the $U(1)$.
- The supertranslation Ward-identity can be written as,

$$\begin{aligned} &\langle \beta, out|\, Q_S^{out}(f)\, |\alpha, in\rangle - \langle \beta, out|\, Q_S^{in}(f)\, |\alpha, in\rangle \\ &= \left(\sum_{i\in\alpha} \omega_i f(x_i) - \sum_{i\in\beta} \omega_i f(x_i) \right) \langle \beta, out|\alpha, in\rangle \end{aligned} \tag{29}$$

- The Lorentz transformation of the soft-charge is given by,

$$U(\Lambda) Q_S(f) U^{-1}(\Lambda) = Q_S(f'), \quad f'(x) = \left| \frac{\partial \Lambda^{-1} x}{\partial x} \right|^{-\frac{1}{n}} f(\Lambda^{-1} x) \tag{30}$$

- The corresponding transformation of $O(x)$ is given by,

$$U(\Lambda) O(x) U(\Lambda)^{-1} = \left| \frac{\partial x'}{\partial x} \right|^{\frac{\Delta}{n}} O(x'), \quad x' = \Lambda x, \quad \underline{\Delta = n+1} \tag{31}$$

So, $O(x)$ is a scalar conformal primary of weight $\Delta = n + 1$.

Potential candidates for $O(x)$ in $D = 6$ or $n = 4$

- The potential candidates for $O(x)$ are given by :

$$O(x) = (\partial^2)^2 \partial_a S_a^0(x), \quad \partial^2 \partial_a \partial_b \partial_c S_{abc}^0(x), \quad \partial_a \partial_b \partial_c \partial_d \partial_e S_{abcde}^0(x)$$
$$O(x) = (\partial^2)^2 S^1(x), \quad \partial^2 \partial_a \partial_b S_{ab}^1(x), \quad \partial_a \partial_b \partial_c \partial_d S_{abcd}^1(x)$$
$$O(x) = \partial^2 \partial_a S_a^2(x), \quad \partial_a \partial_b \partial_c S_{abc}^2(x)$$
$$O(x) = \partial^2 S^3(x), \quad \partial_a \partial_b S_{ab}^3(x)$$
$$O(x) = \partial_a S_a^4(x)$$

  Again the requirement that $O(x)$ must be a $(\Delta = 5, l = 0)$ primary rules out most of the above operators except the two, $\partial^2 \partial_a \partial_b S_{ab}^1(x)$ and $\partial_a \partial_b S_{ab}^3(x)$.

- If we consider operators with $\Delta < 0$, then one can show that there is one more operator given by $(\partial^2)^3 S^{-1}$.

Constraint

- The operators $\partial_a\partial_b S^3_{ab}(x)$ and $(\partial^2)^3 S^{-1}$ are primary <u>without any constraint</u>.
- Now in case of $\partial^2\partial_a\partial_b S^1_{ab}(x)$, one can check by applying SCTs, that $S^1_{ab}(= h_{ab})$ has to satisfy the constraint,

$$\partial^2 h_{ab} - \frac{2}{3}\bigg(\partial_a\partial_c h_{cb} + \partial_b\partial_c h_{ca}\bigg) + \frac{1}{3}\delta_{ab}\partial_c\partial_d h_{cd} = 0 \quad (32)$$

- Incidentally, like in the case of $U(1)$, this is also an equation of a gauge theory with gauge transformation law,

$$h_{ab}(x) \to \tilde{h}_{ab}(x) = h_{ab}(x) + \bigg(\partial_a\partial_b - \frac{1}{4}\delta_{ab}\partial^2\bigg)\phi(x) \quad (33)$$

  (Erdmenger, Osborn ; Dolan, Nappi, Witten ; Beccaria, Tseytlin)
- Remember, that in <u>large</u> $U(1)$ gauge-transformation and supertranslation, the transformation parameter is a <u>scalar field</u>.

Change of operator basis

- In the case of $U(1)$ the operator $O(x)$ was finally written as,

$$\boxed{O(x) = \partial^2 \partial_a S_a^1(x) + \alpha(\partial^2)^2 S^0(x) + \beta \partial_a S_a^3(x)} \tag{34}$$

- Now the constrint satisfied by $S_a^1(x)$ is

$$\partial_a(\partial_a S_b^1 - \partial_b S_a^1) = 0 \tag{35}$$

- So we can redefine our $S_a^1(x)$ as,

$$S_a^1(x) \to \tilde{S}_a^1(x) = S_a^1(x) + \alpha \partial_a S^0(x) \tag{36}$$

  With this redefinition we can write,

$$O(x) = \partial^2 \partial_a \tilde{S}_a^1(x) + \beta \partial_a S_a^3(x) \tag{37}$$

  This is a pure spin-1 contribution.

- This is a valid redefinition because $\tilde{S}_a^1(x)$ is a $(\Delta = 1, l = 1)$ primary which also satisfies Maxwell's equation.

- We can do the same thing in case of gravity. In this case, the final form of $O(x)$ is given by,

$$O(x) = \partial^2 \partial_a \partial_b S^1_{ab}(x) + \alpha (\partial^2)^3 S^{-1}(x) + \beta \partial_a \partial_b S^3_{ab}(x) \quad (38)$$

- So we make the redefinition,

$$S^1_{ab}(x) \to \tilde{S}^1_{ab}(x) = S^1_{ab}(x) + \alpha \frac{4}{3} \left(\partial_a \partial_b - \frac{1}{4} \delta_{ab} \partial^2 \right) S^{-1}(x) \quad (39)$$

- Remember that this is a symmetry of the constraint equation,

$$\partial^2 S^1_{ab} - \frac{2}{3} \left(\partial_a \partial_c S^1_{cb} + \partial_b \partial_c S^1_{ca} \right) + \frac{1}{3} \delta_{ab} \partial_c \partial_d S^1_{cd} = 0 \quad (40)$$

- With redefinition $O(x)$ becomes,

$$O(x) = \partial^2 \partial_a \partial_b \tilde{S}^1_{ab}(x) + \beta \partial_a \partial_b S^3_{ab}(x) \quad (41)$$

which is a pure spin-2 contribution.

Can the Ward-identity be solved ?

- For concreteness let us focus on the $U(1)$ symmetry.
- Now let us write the $U(1)$ Ward-identity in the unintegrated form,

$$\begin{aligned}&\langle\beta, out|\, O^{out}(x)\,|\alpha, in\rangle - \langle\beta, out|\, O^{in}(x)\,|\alpha, in\rangle \\ &= \left(\sum_{i\in\alpha} q_i\delta^4(x-x_i) - \sum_{j\in\beta} q_j\delta^4(x-x_j)\right)\langle\beta, out|\alpha, in\rangle\end{aligned} \tag{42}$$

- Now this is a differential equation for the S-matrix elements with the insertion of soft-operators. This may or may not be solvable depending on the structure of $O(x)$. For example, if we take $O(x) = \partial_a S_a^3(x)$ then there is no way to solve this equation because there is <u>one</u> differential equation and <u>four</u> (or eight) unknown functions corresponding to four helicities.

- The *simplest* theory corresponds to the choice $O(x) = \partial^2 \partial_a S_a(x)$. In this case we know, from the decoupling of null-states, that the following relations hold,

$$\begin{aligned} \langle \beta, out| \, S_{a,out}(x) \, |\alpha, in\rangle = \partial_a \Lambda_{out}(x) \\ \langle \beta, out| \, S_{a,in}(x) \, |\alpha, in\rangle = \partial_a \Lambda_{in}(x) \end{aligned} \tag{43}$$

- Now Substituting these in the Ward-identity we get,

$$(\partial^2)^2 \Lambda(x) = \left(\sum_{i \in \alpha} q_i \delta^4(x - x_i) - \sum_{j \in \beta} q_j \delta^4(x - x_j) \right) \langle \beta, out|\alpha, in\rangle \tag{44}$$

where

$$\Lambda = \delta \Lambda_{out} - \delta' \Lambda_{in} \tag{45}$$

- This equation can be easily solved subject to the boundary condition,

$$\partial_a \Lambda(x) \xrightarrow{x \to \infty} \frac{1}{x^2} I_{ab}(x) M_b + O(\frac{1}{x^3}) \tag{46}$$

where M_a is some constant vector dependent only on the coordinates of the hard particles.

- The solution for $\partial_a \Lambda$ is given by Weinberg's soft-photon theorem, upto undetermined normalisation.

- Now Λ_{out} and $\Lambda_i n$ are related by <u>crossing</u>, although we are not able to completely determine the relation.

- But in any case the <u>additional</u> equations coming from the <u>decoupling of primary descendants</u> allow us to solve the Ward-identity and the solution must be Weinberg's soft-photon theorem.

Some similarities to string quantization

- Think of $S_a(x)$ as the "vertex operator" for a (soft) photon. One can do the same with graviton.
- $$S_a(x) \sim i e_\mu(p) \partial_z X^\mu e^{ip.X}$$
$$\partial_a(\partial_a S_b - \partial_b S_a) = 0 \sim p^2 = 0, \quad p^\mu e_\mu(p) = 0$$
$$\partial_a S^0(x) \sim L_{-1} e^{ip.X}$$
$$S_a \to \tilde{S}_a = S_a + \partial_a S^0 \sim e_\mu(p) \to \tilde{e}_\mu(p) = e_\mu(p) + \alpha p_\mu$$
$$\partial_a(\partial_a \tilde{S}_b - \partial_b \tilde{S}_a) = 0 \sim p^\mu \tilde{e}_\mu(p) = 0$$
- In string theory there is one-one correspondence between null-states in the world-sheet CFT and space-time gauge transformation.
- We now understand that a similar thing may be at play here, i.e, : Null-states in the soft-sector $\sim$ large gauge transformations at null-infinity.
- At this stage this is a rule of thumb. But this seems to work and may be a better starting point than Ward-identity.