Towards a full solution of the large N double-scaled SYK model

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Based on work in with P. Narayan and J. Simon, arXiv:1806.04380 ; and M. Isachenkov, V. Narovlansky and G. Torrents, arXiv:1811.02584.

Outline

- Double scaled SYK and Chords
- Analytic evaluation, auxiliary Hilbert space (and "bulk"?)
- Observables, Correlation functions and quantum groups
- Conclusions

Double scaled SYK model and Chords

The Sachdev-Ye-Kitaev (SYK) model

Conventions: N Majorana fermions ψ_i, $i = 1, \cdots, N$. satisfying

$$\{\psi_i, \psi_j\} = 2\delta_{ij}$$

with random (disordered) all-to-all p-local interaction

$$H = i^{p/2} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

J are random gaussian couplings [1]. The usual large N limit is $N \to \infty,\ p\ fixed$.

After ensemble average[2], it was shown to be almost conformal have the maximal chaos exponent (Sachdev, Ye; Georges, Pacollet, Sachdev; Kitaev; Polchinksi, Rosenhaus; Maldacena, Stanford), and it is dual to AdS_2 and 2D JT gravity (Sachdev; Jensen; Maldacena, Stanford; Almheiri, Polchinksi;...).

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[1] $(\langle J^2_{i_1 \cdots i_p} \rangle_J = \binom{N}{p}^{-1} \mathcal{J}^2)$

[2] Although self averaging

* After taking ($N \to \infty,\ p\ fixed$) one may also take $p \to \infty$, and consider the theory in an expansion in $1/p$.

* We are interested in a different "double scaled" limit (Erdos and Schroder; Cotler et al; Garcia-Garcia, Jia, Verbaarschot)

$$N \to \infty,\ \lambda = 2\frac{p^2}{N}\ fixed \qquad (alternate\ parameter\ q = e^{-2\lambda})$$

and obtain results exact in λ. Following Cotler et al we will refer this as the "double scaled SYK".

* If we take then $\lambda \to 0$ and $E - E_0 \to 0$ at the same time, we recover the Schwarzian theory ("triple scaled SYK", Cotler et al).

We can perhaps think of λ and N^{-1} as the analogues of the α' and g_s expansion. Both λ and α' describe a non-linear theory which is fixed when the number of microscopic quantum degrees of freedom is taken to infinity.

$q \to 1$ has a AdS_2 IR with JT gravity.

$q \to 0$ is probably a strongly curved space (note however that the latter limit gives Gaussian RMT on the nose).

Next we would like to introduce the Chord diagrams.... (*for now following Erdos and Schroder, and Cotler et al*)

Chord diagrams for the partition function

Evaluate the averaged partition function by expanding and computing moments of H

$$\langle \operatorname{tr} e^{-\beta H} \rangle_J = \Sigma \frac{(-\beta)^k}{k!} \langle \operatorname{tr} H^k \rangle_J$$

The trace is normalized $\operatorname{tr} 1 = 1$, $\langle\rangle_J$ stands for ensemble average.

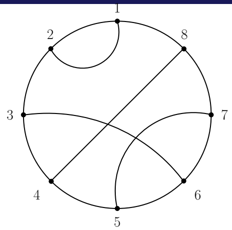
Denote sets of p distinct sites $\{i_1, \cdots, i_p\}$ by capital I, so $H = i^{p/2} \sum_I J_I \cdot \psi_I$ where $\psi_I = \psi_{i_1} \cdots \psi_{i_p}$.

$$\langle \operatorname{tr} H^k \rangle_J = i^{kp/2} \sum_{I_1, \cdots, I_k} \underbrace{\langle J_{I_1} \cdots J_{I_k} \rangle_J}_{\sim \binom{N}{p}^{-k/2}} \operatorname{tr} \psi_{I_1} \cdots \psi_{I_k}.$$

By Wick's theorem, the I_j come in pairs[3].

[3] assume Gaussian - can be relaxed, in large N

The division into pairs can be naturally represented by a chord diagram (circular since trace). Each node $\leftrightarrow$ H insertion.

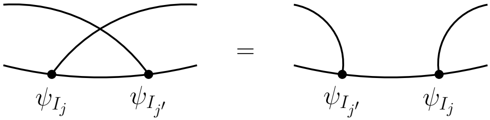


Means $I_1 = I_2, \cdots$.

For each chord diagram left with

$$\binom{N}{p}^{-k/2} i^{kp/2} \sum_{I_1 \neq \cdots \neq I_{k/2}} \operatorname{tr} \psi_{I_1} \cdots \psi_{I_1} \cdots$$

Now commute nodes to bring all pairs to be neighboring:



$\times(-1)^{|I_j \cap I_{j'}|}$

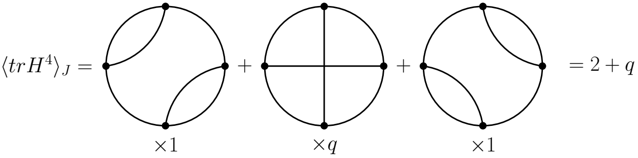
For $p \ll N$

$$|I_j \cap I_{j'}| \sim Pois\left(\frac{p^2}{N}\right)$$

Since $p^2/N \sim O(1)$, there are no triple intersections. Each intersection then gives $\sum_{n=0}^{\infty}\left(\frac{(p^2/N)^n}{n!}e^{-p^2/N}\right)(-1)^n = e^{-2p^2/N} = e^{-\lambda} = q$ and we get

$$\langle \mathrm{tr}\, H^k \rangle_J = \sum_{\text{Chord diagrams}} q^{\#\text{ intersections}}$$

For example

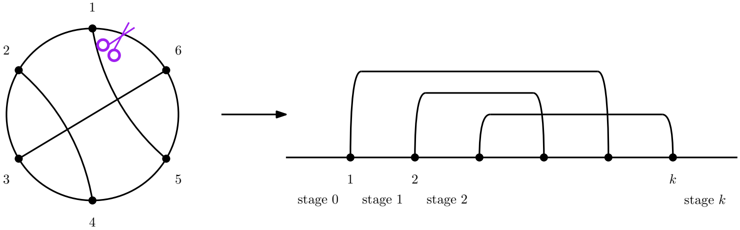


Next, find a cute way to evaluate and get the analogue of an all scale boundary dynamics for free.....

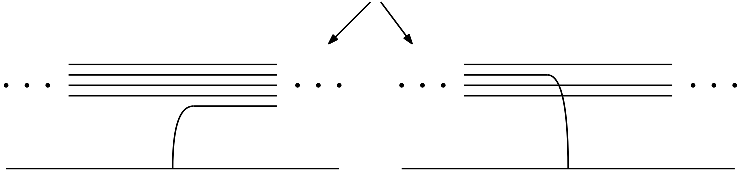
Analytic evaluation, Auxiliary Hilbert space (and "bulk"?)

Partition function

To evaluate the sum over chord diagrams efficient we will proceed in different way (then Erdos-Schroder), by cutting open the chord diagrams



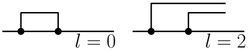
and writing a "transfer matrix" that implements this as process going from left to right. At each stage we can either "open" a new chord (at the bottom of the stack) or "close" one of the chords from the stack.



At each stage have a particular number of chords $l = 0, 1, 2, \cdots$. Introduce an auxiliary Hilbert space $\mathcal{H}$ with basis $|l\rangle$.

Denote partial sums over open chord diagrams: $v_l^{(i)}$ = sum over chord diagrams until stage i having l open chords, each weighted by $q^{\#\text{ intersections}}$.

For example, $v^{(2)} = (1, 0, 1, ...)$ corresponds to



The recursion relation then becomes

$$v_l^{(i+1)} = v_{l-1}^{(i)} + (1 + q + \cdots + q^l) v_{l+1}^{(i)} =$$
$$= v_{l-1}^{(i)} + \frac{1 - q^{l+1}}{1 - q} v_{l+1}^{(i)}$$

$$v^{(i+1)} = T v^{(i)}$$

$$T = \begin{pmatrix} 0 & \frac{1-q}{1-q} & 0 & 0 & \cdots \\ 1 & 0 & \frac{1-q^2}{1-q} & 0 & \cdots \\ 0 & 1 & 0 & \frac{1-q^3}{1-q} & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

T can be conjugated and made diagonal

$$T \to \hat{T} = A_q + A_q^\dagger$$
$$[N, A_q] = -A_q,\ [N, A_q^\dagger] = A_q^\dagger,$$
$$A_q A_q^\dagger - q A_q^\dagger A_q = 1$$

Arik-Coon q-deformed oscillator algebra,

$$A_q = \begin{pmatrix} 0 & \sqrt{\frac{1-q}{1-q}} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{1-q^2}{1-q}} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{1-q^3}{1-q}} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

At stages 0 and k we have no chords , so

$$\langle \operatorname{tr} H^k \rangle_J = \langle 0 | T^k | 0 \rangle$$

Any insertion $e^{itH} \to e^{it\hat{T}}$. So $\hat{T}$ is the Hamiltonian in the auxiliary Hilbert space $\mathcal{H}$ and the spectrum of the system is the spectrum of $\hat{T}$ (T).

To complete the solution, we diagonalize T. Changing variables $v_l = \frac{(1-q)^{l/2}}{(q;q)_l} u_l$, the recursion relation becomes $2xu_l = u_{l+1} + (1-q^l)u_{l-1}$. So u_l are the q-Hermite polynomials

$$H_0(x|q) = 1, H_1(x|q) = 2x, H_2(x|q) = 4x^2 - (1-q), ...\ x = \cos(\theta),\ \theta \in [0, \pi].$$

The energies (eigenvalues of T) are thus

$$E(\theta) = \frac{2\cos(\theta)}{\sqrt{1-q}}, \qquad \theta \in [0,\pi]$$

and eigenvectors $\langle l|\psi_\theta\rangle$ known.

$$\langle \mathrm{tr}\, e^{-\beta H}\rangle_J = \int_0^\pi d\theta \rho(\theta) |\langle 0|\psi_\theta\rangle|^2 e^{-\beta E(\theta)} =$$
$$= \int_0^\pi d\mu(\theta)\, e^{-\beta E(\theta)}, \quad d\mu(\theta) \equiv \frac{d\theta}{2\pi}(q, e^{\pm 2i\theta}; q)_\infty$$

where $(e^{\pm 2i\theta}; q)_n = (e^{2i\theta}; q)_n (e^{-2i\theta}; q)_n$ and
$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ are q-pochhammer symbols.

More on the chords and the auxiliary Hilbert space

1. As a sanity check one can see that in the limit $q \to 1$,if we also scale $E - E_0 \to 0$, then we obtain the Schwarzian theory. Basically T can be approximated at low energies by a differential operator which turns out to be the Liouville operator (Altland, Bargets and Kamenev; (Lam), Mertens, Turiaci, Verlinde). But our formula is exact for the entire model.

2. In the limit $q \to 0$, the theory goes over to a RMT with Wigner semi-circle distribution. Diagrammatically, chord intersections are suppressed, leading to planar correlators. One can argue that there is no weakly coupled GR description in this case (no factorization of correlators).

3. Relation to complexity of the operator (?).

1. The chord diagrams are an efficient way of keeping a minimal set of correlations, while "coarse graining" all other microscopic data. T can be thought of as a book keeping device to propagating these correlations in time.

2. They are an effective description obtained only after averaging over J.

3. Microscopic ($2^{N/2}$ worth of) data cannot be obtained from manipulating chord diagrams.

4. Given that T reduces to the Schwarzian, and captures the full energy spectrum - is this the same as the SYK/AdS_2 duality? Do the same statements above apply for gravity?

Observables, Correlation functions and more quantum groups

Operators

Similarly to the Hamiltonian, consider random operators with different $p' \sim \sqrt{N}$

$$M = i^{p_A/2} \sum_{1 \le i_1 < \cdots < i_{p_A} \le N} {J'}_{i_1 \cdots i_{p_A}} \psi_{i_1} \cdots \psi_{i_{p_A}}$$

The J''s are again random, independent, with zero mean, with similar normalization (and independent of the Hamiltonian couplings).

1. It ordinary RMT models (say GUE, $P = exp(-NV(H))$) correlators do not factorize (planarity). The ensemble above, for ${p'}^2/N << 1$ does.

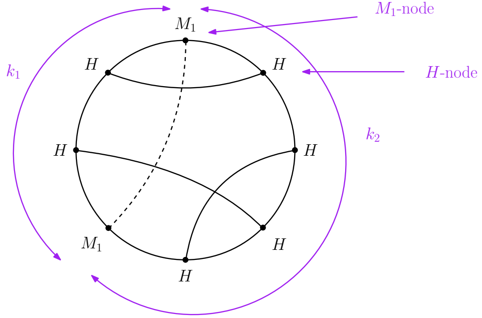
2. If AdS_2 is embedded in a higher dim space, then the allowed probes are determined by the boundary theory of the latter. What is the universal data in AdS_2?

3. We think of the ψ as spanning the d.o.f. of the black hole - not single trace operators. H acts as a random operator on the Fock space of ψ's - why should other single trace operators not be in a similar statistical class?

Correlation function moments

$$\langle \operatorname{tr} H^{k_1} M_1 H^{k_2} M_1 \cdots \rangle_J$$

$\langle \cdot \rangle_J \Rightarrow$ the index sets of the H insertions come in pairs, those of M_1 come in pairs, and so on.

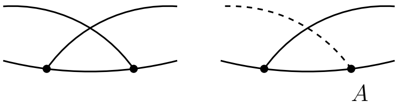


All "M_i chords" are denoted by dashed lines, distinguished by the nodes. The solid chords are "H-chords".

The only difference is the probability distribution of the number of sites in the intersection. For sets of size p, p' the intersection is distributed $Pois\left(\frac{pp'}{N}\right)$ and

$$\sum_{n=0}^{\infty} \frac{(pp'/N)^n}{n!} e^{-pp'/N} (-1)^n = e^{-2pp'/N}$$

So

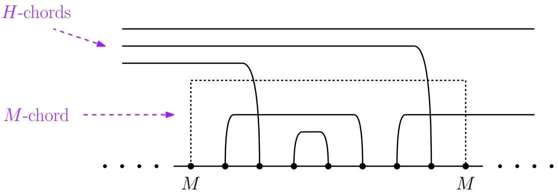


$$\Downarrow \qquad\qquad \Downarrow$$

$$q = e^{-2p^2/N} \qquad\qquad \tilde{q}_A = e^{-2pp_A/N}$$

$$\langle \operatorname{tr} H^{k_1} M_1 H^{k_2} M_1 \cdots \rangle_J \propto \quad = \sum_{\text{Chord diagrams}} q^{\#\ H\text{-}H \text{ intersections}} * \tilde{q}^{\#\ H\text{-}M \text{ intersections}}$$

Consider a region enclosed by a contracted pair of M-nodes.



But suppose we have a two point function. We can then cut the chord diagram just before the first insertion of the first M. In this case we get

$$\langle \mathrm{tr}\, M H^{k_1} M H^{k_2} M_1 \rangle_J = \langle 0 | T^{k_1} S T^{k_2} | 0 \rangle, \qquad S = diag(1, \tilde{q}, \tilde{q}^2, \tilde{q}^3, ...)$$

Since we know the eigenvectors of T in the $|l\rangle$ basis, we can evaluate this.

$$\langle \mathrm{tr}\, e^{-\beta_1 H} M e^{-\beta_2 H} M \rangle_J = \int_0^\pi d\mu(\theta_1) d\mu(\theta_2) e^{-\beta_1 E(\theta_1) - \beta_2 E(\theta_2)} \frac{(\tilde{q}^2; q)_\infty}{(\tilde{q} e^{i(\pm\theta_1 \pm \theta_2)}; q)_\infty}$$

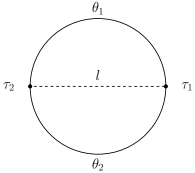
and obtain an effective insertion vertex in the chord Hilbert space.

* In IR, when $q \to 1$, dimension of operator $= p'/p$..

Diagrammatic rules

The 4-pt function and higher is much more complicated and one needs to use some additional iterative procedures to compute correlators. But doable.

It is convenient to organize the results for correlation functions using (non-perturbative) diagrammatic rules (similar to Mertens, Turiaci and Verlinde; and Lam, Mertnes, Turiaci and Verlinde) . The diagrams arise naturally here; these are just chord diagrams.

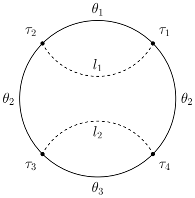


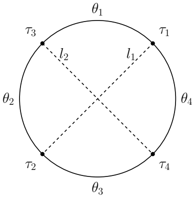
- At a segment along the circle, evolution with T matrix gives a ropagator:

$$= e^{-\Delta\tau\cdot E(\theta)}$$

- Sum over energy eigenstates that propagate, or equivalently over θ, with measure $d\mu(\theta) = \frac{d\theta}{2\pi}(q, e^{\pm 2i\theta}; q)_\infty$.
- Vertex – matrix element of operator between energy eigenstrates:

$$= \gamma_l(\theta_1, \theta_2) = \sqrt{\frac{(\tilde{q}_A^2; q)_\infty}{(\tilde{q}_A e^{i(\pm\theta_1 \pm\theta_2)}; q)_\infty}}, \qquad \tilde{q}_A = q^{l_A}$$





$$\langle M_1 M_2 M_1 M_2 \rangle = \int \prod_{j=1}^{4} d\mu(\theta_j) e^{-\sum \beta_j E(\theta_j)} \gamma_{l_1}(\theta_1, \theta_4) \gamma_{l_1}(\theta_2, \theta_3) \gamma_{l_2}(\theta_1, \theta_2) \gamma_{l_2}(\theta_3, \theta_4) \cdot R$$

with R is associated to the crossing of chords.

For the Schwarzian, the R-matrix is the $6j$ symbol of $SU(1,1)$.

In double-scaled SYK, the R-matrix seems to be related to the $6j$ symbol of $SL_{q^{1/2}}(2)$ (up to some factors) (Groenevelt). This is the R-matrix for the entire model.

$$R = \frac{\left(\tilde{q}_1 e^{-i(\theta_2+\theta_3)}, \tilde{q}_1\tilde{q}_2 e^{i(\theta_3\pm\theta_1)}, \tilde{q}_1\tilde{q}_2 e^{i(\theta_2\pm\theta_4)}; q\right)_\infty}{\left(\tilde{q}_1\tilde{q}_2^2 e^{i(\theta_2+\theta_3)}; q\right)_\infty} \cdot$$
$$\cdot \frac{(\tilde{q}_2^2; q)_\infty}{\left[\left(\tilde{q}_1 e^{i(\pm\theta_2\pm\theta_3)}, \tilde{q}_1 e^{i(\pm\theta_1\pm\theta_4)}, \tilde{q}_2 e^{i(\pm\theta_1\pm\theta_2)}, \tilde{q}_2 e^{i(\pm\theta_3\pm\theta_4)}; q\right)_\infty\right]^{1/2}} \cdot$$
$$\cdot \, {}_8W_7\left(\frac{\tilde{q}_1\tilde{q}_2^2 e^{i(\theta_2+\theta_3)}}{q}; \tilde{q}_1 e^{i(\theta_2+\theta_3)}, \tilde{q}_2 e^{i(\theta_2\pm\theta_1)}, \tilde{q}_2 e^{i(\theta_3\pm\theta_4)}; q, \tilde{q}_1 e^{-i(\theta_2+\theta_3)}\right).$$

$${}_8W_7\left(a; b, c, d, e, f; q, z\right) = {}_8\phi_7\begin{bmatrix} a, & qa^{1/2} & -qa^{1/2}, & b, & c, & d, & e, & f \\ & a^{1/2}, & -a^{1/2}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & \frac{aq}{f}; q, z \end{bmatrix}$$

The basic hypergeometric series is the q-deformation of the generalized hypergeometric function ${}_pF_q$

$${}_{k+1}\phi_k\begin{bmatrix} a_1 & \cdots & a_k & a_{k+1}; q, z \\ b_1 & \cdots & b_k & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_k, a_{k+1}; q)_n}{(b_1, \cdots, b_k, q; q)_n} z^n$$

Summary

- Chord diagrams are quite tractable objects,
- They capture the minimal set of correlations between operators needed to describe a universal set of QM correlators, coarse graining over everything else,
- The auxiliary Hilbert space directly related to the boundary gravity dynamics,
- Reducing a dynamical problem into an algebraic one is always useful. Enable exact computation in the full double scaled limit - at least parts of the model are governed by a rigid algebraic quantum group structure (maybe all of it? solvable?).

A question

- Usually the Schwarzian is obtained by going through JT gravity. Here the Hilbert space of chords, and the Hamiltonian on it, gave it in a simple way (for the full theory). Is this a sleight of hand way of bulk reconstruction - just follow the chords?