CONSTRUCTING THE BULK OF SYK MODEL

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- The SYK model may provide us a unique opportunity to understand how internal degrees of freedom get metamorphosed into space-time dimensions in a tractable toy model of holography.
- This talk will discuss one approach to this issue and the extent to which we are able to do this.
- Based on

S.R.D., A. Jevicki and K. Suzuki, JHEP 1709 (2017) 17

S.R.D., A. Ghosh, A. Jevicki and K. Suzuki, JHEP 1802 (2018) 162

S.R.D., A. Ghosh, A. Jevicki and K. Suzuki, JHEP 1807 (2018) 184

S.R.D., A. Ghosh, A. Jevicki and K. Suzuki – 1902.xxxx

The Model

- The SYK model is a model with N real fermions with a hamiltonian

$$H = (i)^{\frac{q}{2}} \sum_{1\le i_1<i_2<\cdots<i_q\le N} j_{i_1 i_2\cdots i_q}\, \chi_{i_1}\, \chi_{i_2} \cdots \chi_{i_q}\,, \qquad \{\chi_i, \chi_j\} = \delta_{ij}$$

- The couplings are random which are drawn from a Gaussian ensemble

$$< j^2_{i_1 i_2\cdots i_q} >= \frac{J^2 (q-1)!}{N^{q-1}}$$

- The averaging is a quenched averaging. However for large N one can replace this by an annealed averaging (*Sachdev, Kitaev*)
- We will consider the annealed version of the theory. The model will be defined to be in Euclidean space via a path integral.

- Averaging over the couplings leads to the Euclidean action

$$S = \frac{1}{2}\int dt \sum_{i=1}^{N} \chi_i \partial_t \chi_i - \frac{J^2 N^{q-1}}{2q}\int dt_1 \int dt_2 (\sum_{i=1}^{N} \chi_i(t_1)\chi_i(t_2))^q$$

- The model now acquires an O(N) global symmetry and becomes a vector-like model.
- All O(N) invariant correlation functions can be expressed in terms of bi-local fields

$$\Psi(t_1, t_2) \equiv \frac{1}{N}\sum_{i=1}^{N} \chi_i(t_1)\chi_i(t_2)$$

- By a standard change of variables in the path integral one gets a large N action for the bi-locals (*Jevicki, Suzuki & Yoon, 2016*)

$$S_{\rm col}[\Psi] = \frac{N}{2}\int dt \left[\partial_t \Psi(t,t')\right]_{t'=t} + \frac{N}{2}\,\mathrm{Tr}\log\Psi - \frac{J^2 N}{2q}\int dt_1 dt_2\, \Psi^q(t_1, t_2)$$

- In the context of vector model / Vasilev theory duality, it was argued that bi-local field packages higher spin fields in AdS (*S.R.D. & A. Jevicki, 2003*)

Emergent Conformal Symmetry

- At energy scales small compared to the coupling J one can ignore the kinetic term. Then the theory has reparametrization invariance $t \to f(t)$

$$\Psi(t_1, t_2) \to [f(t_1) f(t_2)]^\Delta \Psi(f(t_1), f(t_2)) \qquad \Delta = \frac{1}{q}$$

  This includes SL(2,R) conformal transformations.

- At finite temperature $T = 1/\beta$ the conformal limit is $\beta J \gg 1$
- In this limit the saddle point solution of the bi-local equations of motion are simple. At zero temperature this breaks the symmetry to SL(2,R)

$$\Psi_0(t_1, t_2) = \frac{b}{|t_{12}|^{\frac{2}{q}}} \mathrm{sgn}(t_{12}) \qquad b^q = \frac{\tan(\frac{\pi}{q})}{J^2 \pi} \left(\frac{1}{2} - \frac{1}{q} \right)$$

- The finite temperature solution can be obtained by performing a reparametrization

$$f(t) = \tan(\frac{\pi t}{\beta})$$

The Strong Coupling Spectrum

- The spectrum is obtained by expanding the collective action around the large-N saddle point

$$\Psi(t_1, t_2) = \Psi_0(t_1, t_2) + \sqrt{\frac{2}{N}}\, \eta(t_1, t_2)$$

- This leads to a quadratic action for the fluctuations

$$S^{(2)} = \int dt_1 \cdots dt_4\ \eta(t_1, t_2)\mathcal{K}(t_1, t_2; t_3, t_4)\eta(t_1, t_2)$$

- The Kernel is given by

$$\mathcal{K}(t_1, t_2; t_3, t_4) = \Psi_0^{-1}(t_1, t_3)\Psi_0^{-1}(t_2, t_4) + (q-1)J^2\,\delta(t_{13})\delta(t_{24})\,\Psi_0^{q-2}(t_1, t_2)$$

- The spectrum is then obtained by diagonalizing the kernel

$$\int dt_3 dt_4 \mathcal{K}(t_1, t_2; t_3, t_4) u_{\nu,\omega}(t_3, t_4) = \tilde{\kappa}(\nu) u_{\nu\omega}(t_1, t_2)$$

- The complete orthonormal set of eigenfunctions are

$$u_{\nu w}(t_1,t_2) = e^{iw\frac{t_1+t_2}{2}}\,\mathrm{sgn}\,(t_1-t_2) Z_\nu\left(|w(t_1-t_2)/2|\right)$$

- While the eigenvalues are

$$\tilde{\kappa}(\nu) = -\frac{1}{(q-1)}\frac{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})}{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})}\frac{\Gamma(\frac{5}{4}-\frac{1}{q}+\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{1}{q}-\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{1}{q}+\frac{\nu}{2})\Gamma(\frac{1}{4}+\frac{1}{q}-\frac{\nu}{2})}$$

- Here

$$Z_\nu(x) \;=\; J_\nu(x) \,+\, \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2)+1}{\tan(\pi\nu/2)-1}$$

- The combination of Bessel functions appear since the kernel is diagonalized simultaneously with the SL(2,R) Casimir (*Kitaev;Polchinski & Rosenhaus*)

$$\mathcal{C}_{1+2} = -(t_1-t_2)^2\partial_{t_1}\partial_{t_2}$$

- They are fourier transforms of the conformal 3 point functions

$$\boxed{\int dt_0\; e^{i\omega t_0} < \mathcal{O}_\Delta(t_1)\mathcal{O}_\Delta(t_2)\mathcal{O}_h(t_0) > \sim |\omega|^\nu \frac{sgn(z)}{|z|^{2\Delta-1/2}} e^{i\omega t}\; Z_\nu(|\omega z|)} \qquad h = \frac{1}{2}+\nu$$

- This leads to the quadratic action

$$S^{(2)} \sim \int d\nu \int d\omega \tilde{\Phi}^{\star}_{\nu,\omega}[\tilde{\kappa}(\nu) - 1]\tilde{\Phi}_{\nu,\omega}$$

The spectrum is therefore given by the solutions of the equation

$$\tilde{\kappa}(\nu) = 1 \qquad \nu = p_m$$

- There are always an infinite number of solutions.
- For any q $p_m = 3/2$ is always a solution – the action for this vanishes.
- This is a zero mode at strong coupling coming from reparametrization symmetry broken by the saddle point solution.

$$\int dt\ e^{i\omega t}\delta\Psi_0(t,z) \sim \epsilon(\omega) \quad (\omega z)^{1/2} J_{3/2}(\omega z)$$

- The space of bilocals is in fact Lorentzian AdS_2 or dS_2. In terms of center of mass and relative coordinates

$$z = \frac{1}{2}(t_1 - t_2) \qquad t = \frac{1}{2}(t_1 + t_2)$$

- The Casimir is the Laplacian in a metric

$$ds^2 = \frac{1}{z^2}[-dt^2 + dz^2]$$

- It is tempting to regard this space itself as the dual space of the theory – this would be a special case of what we suggested earlier. In higher dimensions the angular part of the relative coordinate provide the spin degree of freedom while the magnitude provides the additional dimension.
- However the latter *cannot* be directly identified with the Poincare radial coordinate. The usual higher spin fields are related to the bilocal by an integral transform (*de Mello Koch, Jevicki, Jin & Rodrigues, 2011)*
- In 2 dimensions the symmetries work out – however this is a *Lorentzian* AdS_2. In standard AdS/CFT the dual of an Euclidean theory should be Euclidean – so this cannot be quite right.

The Bilocal 2 Point Function

- The bilocal two point function – which is the fermion 4 point function

$$< \chi(t_1)\chi(t_2)\chi(t_1')\chi(t_2') >_c$$

- Is given by

$$\boxed{\mathcal{G}(t,z;t',z') = \sum_{p_m} \frac{R_{p_m}}{N_{p_m}} \mathcal{G}_m(t,z;t',z')}$$

$$t = \frac{1}{2}(t_1+t_2) \qquad z = \frac{1}{2}(t_1-t_2)$$

$$t' = \frac{1}{2}(t_1'+t_2') \qquad z' = \frac{1}{2}(t_1'-t_2')$$

- Where

$$\boxed{\mathcal{G}_m(t,z;t',z') = \begin{cases} \sin(\pi p_m)\frac{1-\tan\frac{\pi p_m}{2}}{1+\tan\frac{\pi p_m}{2}} \boxed{Q_{p_m-\frac{1}{2}}(\xi)} & |\xi|>1 \\ \pi \boxed{P_{p_m-\frac{1}{2}}(\xi)} + 2\frac{1-\tan\frac{\pi p_m}{2}}{1+\tan\frac{\pi p_m}{2}} \boxed{Q_{p_m-\frac{1}{2}}(\xi)} & |\xi|<1 \end{cases}}$$

- ξ is a AdS_2 invariant distance $\quad \xi = \frac{1}{2zz'}\left[z^2 + z'^2 - (t-t')^2\right]$

- The usual way to write this is in terms of conformal blocks. For $t_1 > t_2 > t_1' > t_2'$ the relationship between ξ and the cross-ratio is

$$\xi = 1 - \frac{2}{x} \qquad x = \frac{(t_1 - t_2)(t_1' - t_2')}{(t_1 - t_1')(t_2 - t_2')}$$

- Then we have the more commonly used expressions in terms of hypergeometric functions (e.g. *Maldacena and Stanford*)
- However writing these in terms of ξ highlights the fact that this is sum over non-standard propagators in the AdS_2 "kinematic space" on which the bilocals live.
- The standard Feynman propagator is proportional to the Legendre function Q and does not involve the Legendre P function.
- It will be useful to give a spectral representation of the bilocal 2 point function, (*Polchinski and Rosenhaus*)

$$\boxed{\mathcal{G}(t, z; t', z') \sim |zz'|^{1/2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{R_{p_m}}{N_{p_m}} Z_{-p_m}(|\omega z^>|) J_{p_m}(|\omega z^<|)}$$

Normalization

Residues at poles

- The behavior near the "boundary" z=0 is the same as that of the Feynman propagator. If either z or z' approaches 0, $|\xi| \to \infty$ - in this regime we only have the Legendre Q function.
- Approaching the "boundary" is in fact the OPE limit in terms of the fermions. When

$$\mathcal{G}(t,z;t',z') \sim \sum_{p_m} C_{p_m}^2 \left(\frac{zz'}{(t-t')^2}\right)^{p_m+\frac{1}{2}}$$

- This looks pretty much like standard AdS/CFT : one has a sum of 2 point functions of CFT operators with conformal weights

$$h_m = p_m + \frac{1}{2}$$

- In bilocal space, the quadratic action involves all powers of the AdS_2 Laplacian – this is why a single function represents an infinite tower of fields

$$S^{(2)} = \int dtdz \{(z^{1/2}\eta(t,z)\left[\tilde{\kappa}(\sqrt{\mathcal{D}_B}) - 1\right](z^{1/2}\eta(t,z)\}$$

$$\mathcal{D}_B \equiv z^2(-\partial_t^2 + \partial_z^2) + \frac{1}{4}$$

- The contribution from the $p_m = 3/2$ mode diverges. This is because this is the zero mode of broken reparametrization invariance.
- However at any finite J the reparametrization is explicitly broken. This shifts the pole which can be calculated perturbatively
- This leads to a corrected contribution from this mode $\boxed{p_0 = 3/2 - \alpha \frac{\omega}{J}}$

$$\mathcal{G}_{zero-mode}(t, z; t', z') \sim J \int_{-\infty}^{\infty} \frac{d\omega}{2\pi|\omega|} \; e^{-i\omega(t-t')} \; J_{3/2}(|\omega z|) J_{3/2}(|\omega z'|)$$

- This is proportional to J – hence called the enhanced propagator.
- The dynamics is captured by the Schwarzian action – which is also the action of a whole class of 2d gravity theories – e.g. Jackiw-Teitelboim theory (*Maldacena, Stanford & Yang*; *Engelsoy, Martens & Verlinde*) or Liouville theory *(Mandal, Nayak and Wadia),* or throat limits of higher dimensional black holes *Almheiri & Kang*; *Nayak, Shukla, Soni, Trivedi & Vishal*; *Larsen*; *Gaikwad, Joshi, Mandal and Wadia)*

Large q limit

- In the large q limit $p_m = 3/2$ remains an exact solution. The other poles are (*Gross and Rosenhaus; Maldacena and Stanford*)

$$p_m = 2m + \frac{1}{2} + \frac{2}{q}\frac{2m^2+m+1}{2m^2+m-1} + \cdots \qquad m = 1, 2, \cdots$$

- The residue at $p_m = 3/2$ remains finite

$$R(3/2) = \frac{2}{3} - \frac{1}{q}\left(\frac{5}{2} + \frac{\pi^2}{3}\right) + O(1/q^2)$$

- The residues at the other poles vanish for large q - these do not contribute to the propagator.

$$R(p_m) \to \frac{1}{q}\frac{4(2m^2+m)}{(2m^2+m-1)^2} + O(1/q^2)$$

- Unlike in string theory, the infinite tower modes have dimensions of the same order even at large q. They decouple in the propagator due to vanishing residues.

Interaction Vertices

- The expansion around the saddle point leads to an infinite set of terms containing higher powers of the fluctuation fields. The theory thus contains interaction vertices of arbitrary order.
- For example the 3 point vertex of bi-locals contain terms which are local in the bi-local space

$$\int dt \int_0^\infty \frac{dz}{z^2} \, [z^{\frac{1}{2}} \Phi(t,z)]^3$$

- As well as terms which are non-local

$$\sum_{\nu_1,\nu_2,\nu_3} \int \left[\prod_{i=1}^{3} dw_i \right] c^{w_1 w_2 w_3}_{\nu_1 \nu_2 \nu_3} \widetilde{\Phi}_{\nu_1,w_1} \widetilde{\Phi}_{\nu_2,w_2} \widetilde{\Phi}_{\nu_3,w_3}$$

- Explicit expressions in *Gross and Rosenhaus*

A Three dimensional view

(S.R. Das, A. Jevicki and K. Suzuki : JHEP 1709 (2017) 17
(S.R. Das, A. Ghosh, A. Jevicki and K. Suzuki : JHEP 1802 (2018) 162)

- It turns out that the infinite tower of fields can be interpreted as the KK tower of a Horava-Witten compactification of a 3 dimensional theory in a fixed background.
- For q=4 the background is $$\boxed{ds^2 = \frac{1}{z^2}[-dt^2 + dz^2] + (1 + \frac{a}{z})^2 dy^2}$$ $a \sim 1/J$
- The third direction is an interval S^1/Z_2 with Dirichlet boundary conditions
- There is a single scalar field which is subject to a delta function potential

$$\boxed{S = \int dzdt \left[(\partial_t \phi)^2 - (\partial_z \phi)^2 + \frac{1}{4z^2} - \frac{1}{z^2}\{(\partial_y \phi)^2 + V(y)\phi^2\} \right]}$$ $$V(y) = V\delta(y)$$

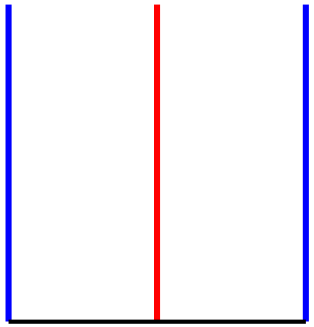
Schrodinger problem in an infinite well with a delta function in the middle

- The background for a general q is conformal to $AdS_2 \times S^1/Z_2$

$$\boxed{ds^2 = |x|^{\frac{4}{q}-1}\left[\frac{-dt^2+dz^2}{z^2}+\frac{dx^2}{4|x|(1-|x|)}\right]}$$

- And the potential is given by

$$V(x) = \frac{1}{|x|^{\frac{4}{q}-1}}\left[4\left(\frac{1}{q}-\frac{1}{4}\right)^2 + m_0^2 + \frac{2V}{J(x)}\left(1-\frac{2}{q}\right)\delta(x)\right] \qquad J(x) = \frac{|x|^{\frac{2}{q}-1}}{2\sqrt{1-|x|}}$$



- Solving the eigenvalue problem in the 3rd direction leads to eigenfunctions which are hypergeometric functions and eigenvalues ν^2 which satisfy the equation

$$\tilde{\kappa}(\nu) = -\frac{1}{(q-1)}\frac{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})}{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})}\frac{\Gamma(\frac{5}{4}-\frac{1}{q}+\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{1}{q}-\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{1}{q}+\frac{\nu}{2})\Gamma(\frac{1}{4}+\frac{1}{q}-\frac{\nu}{2})}$$

- Furthermore, for a=0 a non-standard 3d propagator with the two ends lying on the center of the interval precisely reproduces the bi-local 2 point function

$$\mathcal{G}(t,z,x=0;t',z',x=0) = \sum_{p_m} \frac{(f_m(0))^2}{N_{p_m}} \mathcal{G}_m(t,z;t',z')$$

- Here $f_{p_m}(y)$ is an eigenfunction of the part of the Laplacian in the 3rd direction.
- The residue at the poles – or the OPE coefficients of the composite operators – are now given by the 3rd direction wavefunctions at these end points.

$$f_m(0) \propto R_m$$

- This suggests the correspondence

$$\phi(t,z,0) \sim \eta(t,z)$$

- One can view this as an unpacking of a single field $\eta(t,z)$ with a complicated kinetic term in terms of a sum over fields each of which have a conventional kinetic term.

- We can now use this correspondence to rewrite the cubic vertex in terms of the three dimensional field.
- This vertex is not local in the third direction – in fact there was no reason to expect why it would be local.
- We have now expressed the vertex in the 3d language – but we have not yet obtained much insight.

The Enhanced Propagator

- The 3d non-standard propagator is of course divergent, once again due to the contribution of the $p_m = 3/2$ mode.
- So far we have ignored the warp factor in front of the 3rd direction – this is the term which breaks the conformal isometry. The quadratic action is

$$\boxed{S = \frac{1}{2}\int dz dy \int \frac{d\omega}{2\pi}\, \chi_{-\omega}\left(\mathcal{D}_0 + \mathcal{D}_1\right)\chi_\omega}$$

$$\mathcal{D}_0 = \partial_z^2 + \omega^2 - \frac{m_0^2}{z^2} + \frac{1}{z^2}\Big(\partial_y^2 - V(y)\Big)\,,$$

$$\mathcal{D}_1 = \frac{a}{z}\left[\partial_z^2 - \frac{1}{z}\partial_z + \omega^2 - \frac{m_0^2}{z^2} - \frac{1}{z^2}\Big(\partial_y^2 + V(y)\Big)\right]$$

- The effect can be now computed perturbatively. This changes the eigenvalues. In particular the eigenvalue $p_m = 3/2$ gets modified as

$$p_0 = 3/2 + \beta a \omega$$

- Substituting this in the spectral representation of the propagator we exactly reproduce the SYK enhanced propagator (Note $a \sim 1/J$) upto a numerical factor.

Towards an Euclidean Dual

- According to usual AdS/CFT rules we should have obtained a dual description in Euclidean space - the bi-local theory however naturally lives in a Lorentzian space.
- This is of course consistent with the symmetries, since the isometries of both Euclidean and Lorentzian AdS_2 are identical – SO(2,1) or SO(1,2).
- We now describe a way to obtain a Euclidean signature theory from the collective theory. (*S.R.D., A. Ghosh, A. Jevicki & K. Suzuki*)

- Let us write the metric on the bilocal space as

$$ds^2 = \frac{1}{\eta^2}[-dt^2 + d\eta^2]$$

- Whose isometries are

$$\boxed{\hat{D}_{1+2} = t_1\, p_1 + t_2\, p_2\,, \qquad \hat{P}_{1+2} = -p_1 - p_2\,, \qquad \hat{K}_{1+2} = -\,t_1^2\, p_1 - t_2^2\, p_2}$$

$$t = 1/2(t_1 + t_2) \qquad \eta = 1/2(t_1 - t_2)$$

- Let us denote the metric on EAdS_2 as

$$ds^2 = \frac{d\tau^2 + dz^2}{z^2}$$

- Whose isometries are

$$\boxed{\hat{D}_{\mathrm{EAdS}} = \tau\, p_\tau + z\, p_z\,, \qquad \hat{P}_{\mathrm{EAdS}} = -p_\tau\,, \qquad \hat{K}_{\mathrm{EAdS}} = (z^2 - \tau^2)\, p_\tau - 2\tau z\, p_z}$$

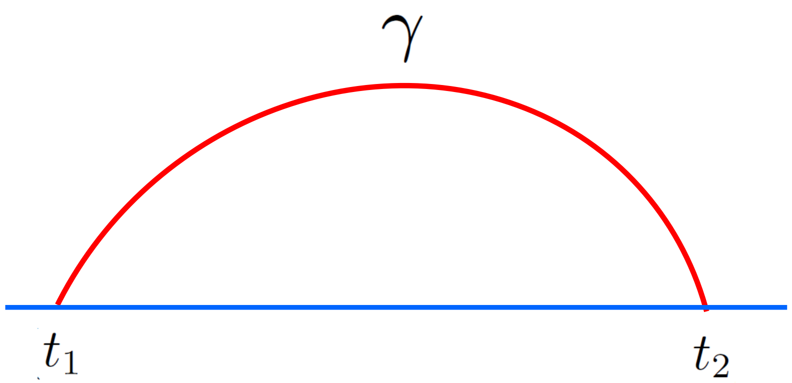
- Is there a canonical transformation which takes one to the other ?
- A priori, this is an *overdetermined* problem. However the answer is YES.

$$\tau \;=\; \frac{t_1\,p_1 - t_2\,p_2}{p_1 - p_2}\,, \qquad p_\tau \;=\; p_1 + p_2\,, \qquad z^2 \;=\; -\left(\frac{t_1 - t_2}{p_1 - p_2}\right)^2 p_1 p_2\,, \qquad p_z^2 \;=\; -4p_1p_2$$

- This can be implemented as transformations on momentum space fields.
- It turns out that – written in position space – is the X-ray (or radon) transformation acting on a function on EAdS_2

$$\left[\mathcal{R}f\right](\eta, t) \;=\; 2\eta \int_{t-\eta}^{t+\eta} d\tau \int_0^\infty \frac{dz}{z}\,\delta\!\left(\eta^2 - (\tau - t)^2 - z^2\right) f\,(\tau, z)$$

- The X-ray transform of a function is the integral of the function evaluated on a geodesic with both end-points on the boundary.



$$\boxed{\mathcal{R}f(\tau, z) = \int_\gamma ds\, f(\gamma)}$$

$$t \equiv \frac{t_1 + t_2}{2} \qquad \eta. \equiv \frac{t_1 - t_2}{2}$$

- The transform intertwines generators of EAdS_2 isometries and those of AdS_2 which implies

$$\boxed{\nabla^2_{LAdS_2}\mathcal{R} = \mathcal{R}\nabla^2_{EAdS_2}}$$

- In the context of AdS/CFT, Radon transforms have been used to relate OPE blocks of 2d CFT to bulk fields in (*Czech et.al*). This reproduces the bulk reconstruction of *Hamilton, Kabat, Lifschytz & Lowe*. For a different perspective see *Bhaumick et.al.*
- This has also been mentioned in the SYK context *(Maldacena and Stanford)*

- An evidence that this is on the right track is that this map transforms the usual normalizable eigenfunctions of the Euclidean Laplacian

$$\phi_{EAdS}(\tau, z) = z^{\frac{1}{2}} e^{-i\omega\tau} K_\nu(\omega z) \qquad \nu = ir$$

- To the precise combinations of Bessel functions which diagonalize the SYK kernel

$$[\mathcal{R}\phi_{EAdS}](t,\eta) = -\frac{\pi^{3/2}}{\sin(\pi\nu)} \frac{\Gamma(\frac{1}{4}+\frac{\nu}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2})} \eta^{\frac{1}{2}} e^{-i\omega t} \left[J_\nu(\omega\eta) + \frac{\tan(\frac{\pi\nu}{2})+1}{\tan(\frac{\pi\nu}{2})-1} J_{-\nu}(\omega\eta) \right] \qquad \nu = ir$$

$$Z_\nu(x) \;=\; J_\nu(x) \;+\; \xi_\nu \, J_{-\nu}(x)$$

- In fact this continues to work at finite temperature
- In the conformal limit of the SYK model the finite temperature background is obtained from the zero temperature background by a reparametrization

$$t = \tan(\frac{\pi\theta}{\beta})$$

- This takes the bilocal space metric to

$$-\frac{4dt_1dt_2}{|t_1 - t_2|^2} \to \frac{-dt^2 + d\rho^2}{\sin^2\rho} \qquad \text{FTSYK}$$

$$t = \frac{\pi}{\beta}(\theta_1 + \theta_2) \qquad \rho = \frac{\pi}{\beta}(\theta_1 - \theta_2)$$

- One might have thought that this would be a Lorentzian black hole. It is not. The Lorentzian black hole has a metric

$$\boxed{\frac{-dt^2 + d\rho^2}{\sinh^2\rho}}$$

- The Laplacian in this metric intertwines with the Laplacian in the Euclidean Black Hole background under a Radon transform

$$ds_B^2 = \frac{d\tau_B^2 + d\xi^2}{\sinh^2 \xi} \qquad\qquad \nabla^2_{FTSYK}\mathcal{R} = \mathcal{R}\nabla^2_{EBH_2}$$

- The eigenfunctions of the Laplacian in this background with the appropriate periodicity conditions have been obtained by *Maldacena and Stanford*

$$\boxed{\psi_{n,\nu}(t,\rho) = d_{n,\nu}e^{-int}(\sin\rho)^{\nu+\frac{1}{2}}\ {}_2F_1[\frac{1}{2}(\frac{1}{2}+\nu-n), \frac{1}{2}(\frac{1}{2}+\nu+n); \frac{1}{2}; \cos^2\rho]}$$

- The normalizable eigenfunctions of $\nabla^2_{EBH_2}$ are

$$\boxed{\phi_{n,\nu} = c_{n,\nu}e^{-in\tau_B}(\sinh\xi)^{1/2}Q^{\nu}_{|n|-\frac{1}{2}}(\cosh\xi) \qquad \nu = ir}$$

- We found numerically that these are related by a radon transform.

- The radon transform relates fields non-locally. However the near boundary region of the Lorentzian AdS_2 maps to near boundary regions of Euclidean AdS_2
- This is why using the extrapolate dictionary leads to the same two point functions of the CFT operators.
- However points in the interior see this non-locality.
- The radon transform of the Euclidean propagator, however, does not lead to the bilocal propagator in all regimes of the invariant distance

$$\boxed{[\mathcal{R}\mathcal{G}_{E,p_m}](t,\eta;t'\eta') = \begin{cases} F(p_m)\mathcal{G}_m^{SYK}(\xi) & |\xi|>1 \\ F(p_m)\mathcal{G}_m^{SYK}(\xi) - \sum_{n=0}^{\infty} H_n(p_m)\ P_{2n}(-\xi) & |\xi|<1 \end{cases}}$$

$$F(p_m) = \frac{1}{\sin \pi p_m} \frac{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})}{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}$$

$$H_n(p_m) = 4\frac{4n+1}{(2n+1/2)^2-p_m^2}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^2$$

- The gamma function ratios are like "leg pole factors".
- Once again – for the region close to the boundary the relevant regime is $|\xi| > 1$ for which there is a direct connection to the SYK propagator.
- We do not have a clear understanding of the additional terms which appear for $|\xi| < 1$ - this is the region near the horizon.
- It is possible to invent an additional transformation which removes these terms – but we do not yet have a nice geometric meaning of that.
- Finally this map has nothing to do with the three dimensional perspective – this acts individually on each KK mode. The 3rd dimension goes along for a ride.

Epilouge

- We are beginning to have a better idea of the relationship between the space on which bi-locals live and the space-time in the dual theory.
- In our low dimensional case this relationship involves the radon transform – which is known to relate the OPE blocks of a CFT to bulk operators.
- In higher dimensions, a “momentum space” version of this map has been proposed in a covariant gauge recently (*de Mello Koch et.al.)* – a real space geometrical meaning is not yet available.
- For the SYK model there is this intriguing three dimensional packaging – the agreements are too good to think this is a coincidence. However its usefulness is yet to reveal itself.

THANK YOU