Symmetries of Feynman Integrals

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- Introduction
- Main Results
- Mechanism
- Applications
- More results
- Conclusions

based on work w. Philipp Burda, Subhajit Mazumdar, Amit Schiller and Ruth Shir in 8 papers since 2015.

Introduction

Motivation

At risk of stating the obvious:

Feynman diagrams are at the computational core of Quantum Field Theory.

Necessary for experimental design and data analysis including at the LHC.

Background

- Feynman diagrams 1948.
- Schwinger parameters.
- Landau singularities 1959.
- Cutkosky discontinuities 1960.
- Dimensional regularization, Giambiagi & Bollini, 't Hooft & Veltman 1972.

Background - well known, but less

- Integration By Parts (IBP), Chetyrkin & Tkachov 1981
- Differential Equations (DE), Kotikov 1991, Remiddi 1997.
- Dimensional Recurrence, Tarasov 1996.
- Canonical Basis for DE, Henn 2013.

Modern perturbative field theory

- On shell unitarity, Bern, Dixon & Kosower 1994.
- Scattering amplitudes: Witten 2003; Britto, Cachazo, Feng & Witten (BCFW) on-shell recursion relations 2005; Cachazo, He & Yuan amplitudes 2013; Amplituhedron, Arkani-Hamed & Trnka 2014.

A general theory
is still lacking

Personal trajectory

- Got interested through binary problem in Einstein’s gravity; Effective Field Theory for Post-Newtonian approx; Higher loop; Integration By Parts.
- Is it possible to contribute to a longstanding issue?

Main results

Definitions

A Feynman integral

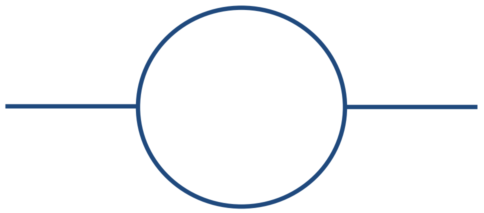
$$I(\mu_1, \ldots, \mu_P, p_1^\mu, \ldots, p_X^\mu) = \int \frac{d^d l_1 \ldots d^d l_L}{\Pi_{i=1}^P (k_i^2 - \mu_i + i0)}$$

$$\mu := m^2$$

Parameter space: masses and kinematical invariants

$$X = \{\mu_i,\, p_r \cdot p_s\}$$

Example:
bubble diagram



$$I(\mu_1, \mu_2, p^2) = \int \frac{d^d l}{(l^2 - \mu_1)((l+p)^2 - \mu_2)}$$

Definitions

A Feynman integral

$$I(\mu_1, \ldots, \mu_P, p_1^\mu, \ldots, p_X^\mu) = \int \frac{d^d l_1 \ldots d^d l_L}{\Pi_{i=1}^P (k_i^2 - \mu_i + i0)}$$

$$\mu := m^2$$

Comment: Numerators, spacetime dimension

Main Result: reduction

Reduction $$\hat{I}(x) = \hat{I}(x_0) + \int_{x_0}^{x} J^{\alpha}(\xi) d\xi_{\alpha}$$

Leading singularity normalization $\hat{I} := I/I_0$

I_0 - a homogeneous soln., incl. maximal cut.

x_0 - conveniently chosen base point
within G orbit (more below).

Integrand depends on simpler diagrams,
namely, with an edge contracted.

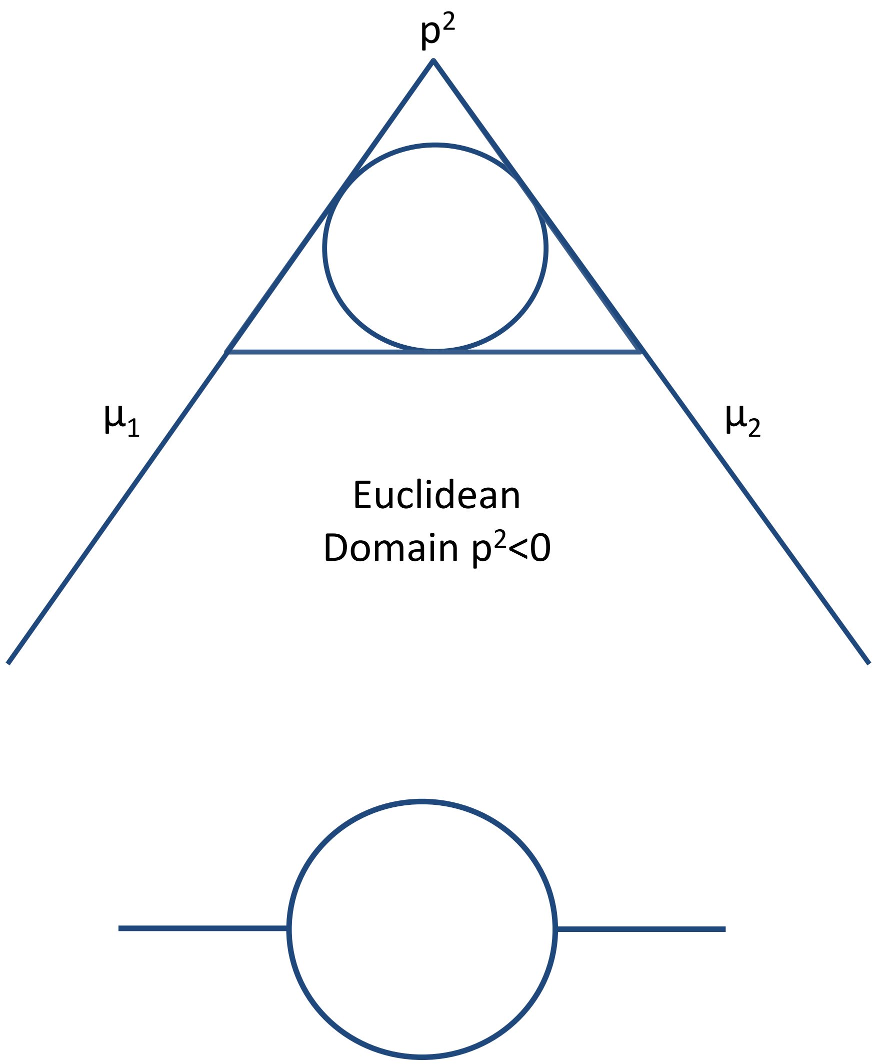
SFI equation system

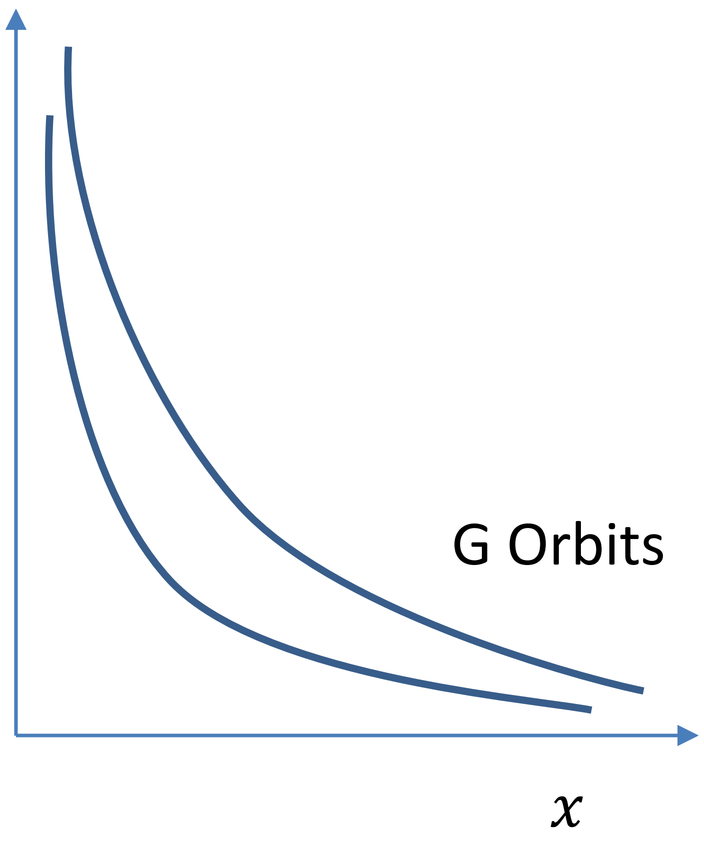
$$c^a\, I + (T^a)^i_j\, x_i\, \frac{\partial}{\partial x_j} I = J^a$$

$$a = 1, \ldots, \dim G$$

- c^a - x-indep. const's
- T^a - group generators, acting on X
- J^a - simpler diagrams
- Symmetries of Feynman Integrals - SFI
- Closely related to Integration By Parts (IBP) and Differential Equations (DE) methods.

Foliation of X into G orbits





Singularity locus

$$I(x) = \sum_{\alpha} c_{\alpha}(x) J^{\alpha}(x)$$

A linear combination of simpler diagrams.

General theory - SFI
Summary of main results

$$\hat{I}(x) = \hat{I}(x_0) + \int_{x_0}^{x} J^{\alpha}(\xi) d\xi_{\alpha}$$

$$c^a \, I + (T^a)^i_j \, x_i \, \frac{\partial}{\partial x_j} I = J^a$$

$$I(x) = \sum_{\alpha} c_{\alpha}(x) J^{\alpha}(x)$$

Mechanism

Current freedom

$$G \subset T_{L,n-1}$$

upper block triangular,

$L = \#$ loops, $\mathrm{n} = \#$ external legs

$$\delta l = l + p$$

$$\delta p = p$$

l – loop currents

p – external currents

Geometric characterization: G is defined by preserving the subspace of squares inside quadratics

$$Q = Sp\{l_a \cdot l_b,\ l_a \cdot p_r,\ p_r \cdot p_s\}$$

$$a, b = 1, \ldots, L,\ r, s = 1, \ldots n - 1$$

$$S \subset Q$$

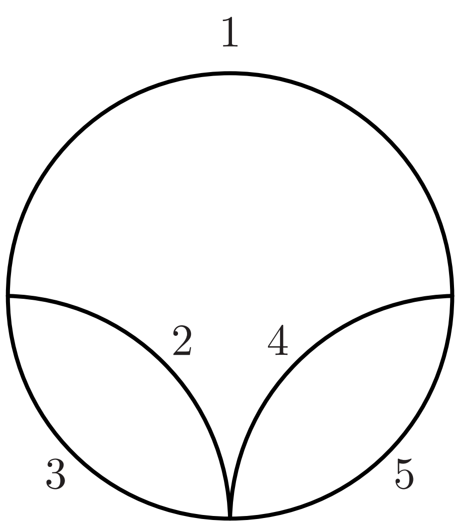
$$S = Sp\{k_i^2,\ p_r \cdot p_s\}$$

$$i = 1, \ldots, P$$

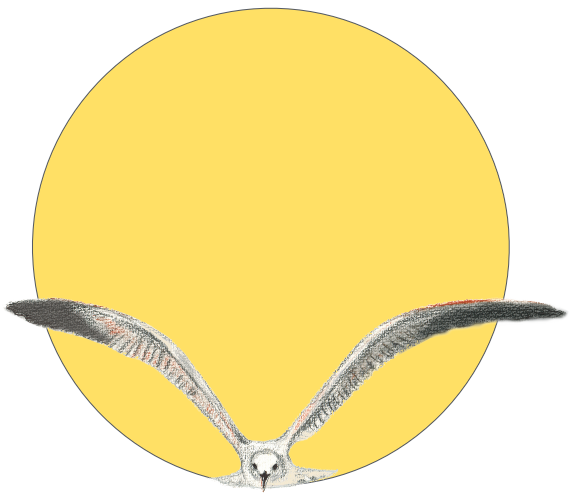
$$Num = Q/S$$

Applications

Vacuum seagull



(a)



(b)

evaluated a sector w. 3 mass scales (novel).

$$I(m^2,m^2x,0,m^2y,0)=(m^2)^{\frac{3d}{2}-5}I_0(x,y,d)$$
$$(I_1(x,y,d)+I_1(y,x,d)+I_2(x,d)+I_2(y,d)$$
$$+I_3(x,y,d)+c_4(d)) \tag{53a}$$

where

$$I_0(x,y,d)=i\pi^{\frac{3d}{2}}((1-x)(1-y))^{d-3}, \tag{53b}$$

$$I_1(x,y,d)=\Bigg[c_{1a}(d)xy^{\frac{3d}{2}-5}{}_2F_1\left(5-\frac{3d}{2},4-d,3-\frac{d}{2}\middle|\frac{x}{y}\right)$$
$$+c_{1b}(d)x^{\frac{d}{2}-1}y^{d-3}{}_2F_1\left(3-d,2-\frac{d}{2},\frac{d}{2}-1\middle|\frac{x}{y}\right)\Bigg]$$
$$\times F_1(3d/2-4,d-2,d-3,3d/2-3|x,y), \tag{53c}$$

$$I_2(x,d)=c_2(d)x^{\frac{d}{2}-1}{}_2F_1(d/2-1,d-2,d/2|x), \tag{53d}$$

$$I_3(x,y,d)=c_3(d)(xy)^{\frac{d}{2}-1}{}_2F_1(d/2-1,d-2,d/2|x)$$
$$\times{}_2F_1(d/2-1,d-2,d/2|y), \tag{53e}$$

$$c_{1a}(d)=-3\Gamma\left(3-\frac{3d}{2}\right)\Gamma(4-d)\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}-1\right), \tag{53f}$$

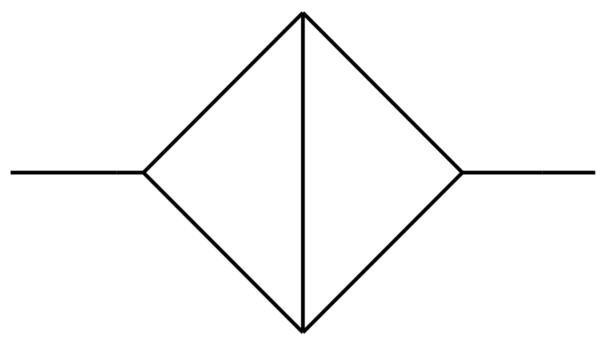
$$c_{1b}(d)=-\frac{4\pi\mathrm{Csc}(\frac{\pi d}{2})\Gamma(2-d)\Gamma(2-\frac{d}{2})}{3d-8}, \tag{53g}$$

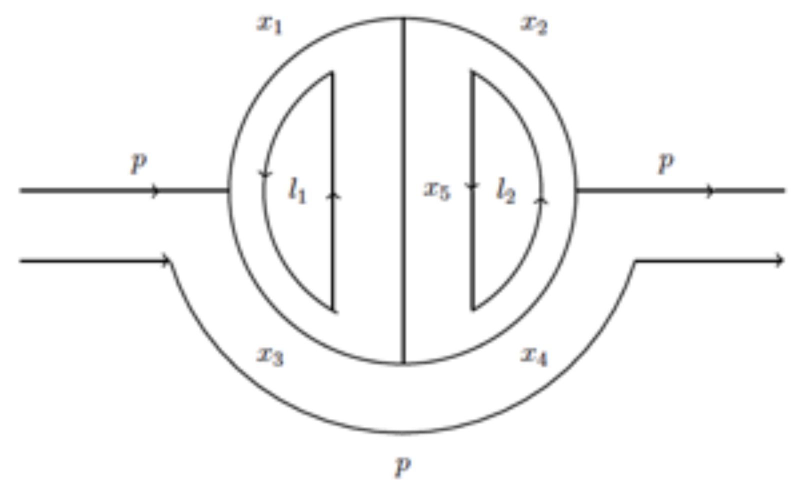
$$c_2(d)=\pi d\mathrm{Csc}\left(\frac{\pi d}{2}\right)\Gamma(2-d)\Gamma\left(-\frac{d}{2}\right), \tag{53h}$$

$$c_3(d)=\Gamma\left(1-\frac{d}{2}\right)^3. \tag{53i}$$

$c_4(d)$ was defined in (48e).

Kite - singularity locus





$$B_3(x) = 0$$

$$\begin{aligned} B_3 = {} & x_1\, x_4(x_1 + x_4) + x_2\, x_3(x_2 + x_3) + x_5\, x_6(x_5 + x_6) + \\ & + x_1\, x_2\, x_5 + x_1\, x_3\, x_6 + x_2\, x_4\, x_6 + x_3\, x_4\, x_5 + \\ & - (x_1\, x_4(x_2 + x_3 + x_5 + x_6) + x_2\, x_3(x_1 + x_4 + x_5 + x_6) + x_5\, x_6(x_1 + x_2 + x_3 + x_4)) \end{aligned} \tag{3.2}$$

$$I|_{B_3} = -\frac{1}{d-4} \frac{\vec{u} \cdot \vec{J}}{\partial^5 B_3} = \ldots$$

Generalizes the massless case [ChetyrkinTkachov1981], and the more general “diamond rule” [Ruijl, Ueda and Vermaseren 2015].

Singularity locus

More results

Symmetry adapted basis

Equation basis chosen to be compatible with the standard discrete symmetry of the diagram

Maximal minors

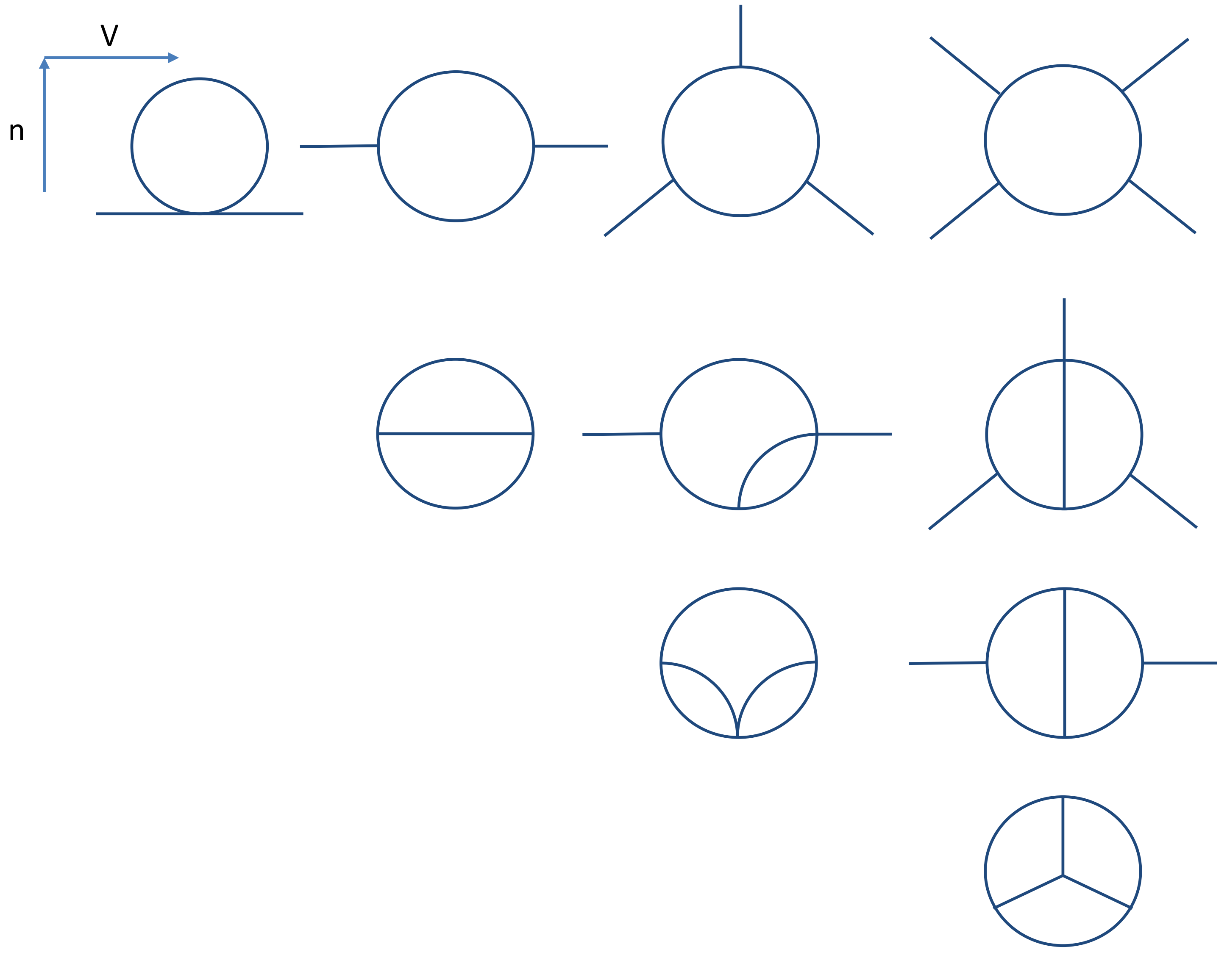
- Factorisation of maximal minors

$$M^I{}_A(x) = \epsilon_{Aa_1\ldots a_r}\,\epsilon^{Ii_1\ldots i_r}\,Tx^{a_1}_{i_1}\ldots Tx^{a_r}_{i_r}$$

$$M^I{}_A(x) = S(x)\,Orb^I(x)\,Stb_A(x)$$

Partial group invariants, stabilizers

Hierarchy of diagrams



Conclusions

Conclusions

- Every diagram defines an SFI group G.
- The group acts on parameters and allows to change parameters at the price of a line integral over simpler diagrams.
- Achieved a novel and general theory.
- Continuing to apply and refine the method.

Conclusions

Even 70 years later innovation is possible.

Thank you for your attention!