Quantum quench of fermions in 1 and 2 dimensions: Lessons from c=1 for cold atoms

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GM-Morita 2013, Kulkarni-GM-Morita 2018-19
based on
Dhar-GM-Wadia 1992-94, GM 2005, Dhar-GM-Suryanarayana 2006-07

Long time dynamics of non-equilibrium states

Questions of long time dynamics of systems:

1. Does the system equilibrate? (In terms of what variables, which states…)

2. Nature of the equilibrium: thermal, GGE, memory (MBL), "texture" density (Kibble-Zurek).

3. Rate of approach to equilibrium, exponential, power law: relaxation rates.

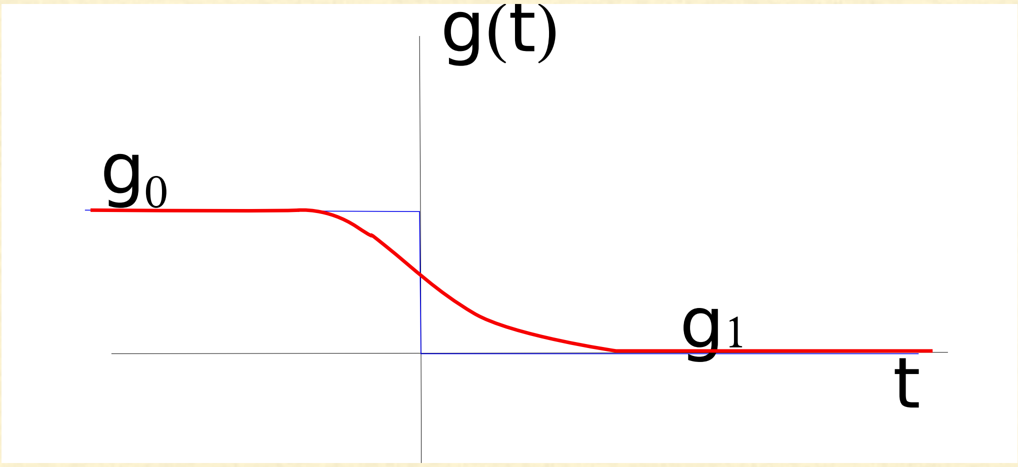
4. Universality of equilibrium state, of relaxation rates.

Quantum quench

A theoretically as well as experimentally popular method of creating non-equilibrium states is to do quantum quench

$$H = J_x \sum_i \sigma_x(i)\sigma_x(i+1) + J_y \sum_i \sigma_y(i)\sigma_y(i+1) + h(t) \sum_i \sigma_z(i)$$

$$S = -\int d^d x \left(\partial_\mu \phi \partial^\mu \phi - m^2(t)\phi^2 \right)$$



Methods

Primary method: numerical

Analytical methods:

1. Post-quench system= CFT (esp 1+1), free or large N in d+1 (d=2,3,…). The thermal or GGE state, as well as relaxation rate, dictated by universal CFT properties

Calabrese-Cardy 2005-09, Rigol et al 2007, Calabrese, Srednicki, ….
GM-Morita 2013, GM-Sinha-Sorokhaibam 2015, GM-Paranjape-Sorokhaibam 2017, Banerjee-Gaikwad-Kaushal-GM 2019, …

$$\langle O(t)\rangle \to \langle O\rangle_{\rm GGE} + e^{-\gamma t}, \text{ if d=odd}$$

$$\langle O(t)\rangle \to \langle O\rangle_{\rm GGE} + \frac{1}{t^p}, \quad \text{if d=even}$$

BGKM 2019

“New” method

2. In this talk we will discuss the application of phase space hydrodynamics (droplet method) to quantum quench in fermi gas (also hard-core Bose gas) in 1+1 as well as (in a limited way) in 2+1 dimensions.

Kulkarni-GM-Morita 2018, 2019 (in progress)

These problems have experimental significance in cold atom systems. The method arose in early applications, e.g. c=1, giant gravitons, SYM on 2D torus, …

Polchinski 1992, Dhar-GM-Wadia 1992-95, GM 2005, Dhar-GM-Suryanaranyana 2006-2008, GM-Morita 2009-2013

Plan for the rest of the talk

- Statement of the problem
- Review of phase space hydrodynamics for large N fermions
- Result 1: Curing deficiencies of conventional hydrodynamics: Treatment of shock fronts
- Result 2: Derivation of equilibration to Generalized Gibbs Ensemble
- Result 3: Universal relaxation exponents from shape of the confining potential
- 2+1 dimensions: Fermions in a rotating trap (restricted to "lowest Landau level")

1. Statement of the Problem

Experiments in a number of ultra cold atom systems involve the following question:

Suppose we have a system of non-interacting fermions or infinitely repelling bosons (Tonks gas) which are suddenly released from a confining potential to free space (typically modelled by a periodic box) or to a different confining potential (typically with a different value of the shape parameter, e.g. $kx^2 \to k'x^2$).

The problem is to find the subsequent quantum evolution of the state of the system if we started, e.g. from the ground state, and ask questions mentioned in the introduction, e.g. thermalisation (or lack of it), relaxation, chaos, etc.

The exact treatment requires finding single-particle eigenstates before and after the quench, computing the Bogoliubov coefficients, and expressing the old Fermi sea in terms of the many-body eigenstates of the new Hamiltonian. This is complicated and except in certain cases, can only be done numerically.

Conventional hydrodynamics

When the number of particles is large, it is natural to try a continuum description in terms of the density and velocity of particles: $\rho(x,t)$ and $v(x,t)$

The evolution of the particle density is assumed to satisfy the continuity and Euler equations

$$\partial_t \rho + \partial_x(\rho v) = 0 \quad \text{Continuity}$$

$$\partial_t v + \partial_x(v^2/2 + \pi^2 \rho^2/2 + V(x)) = 0 \quad \text{Euler equation}$$

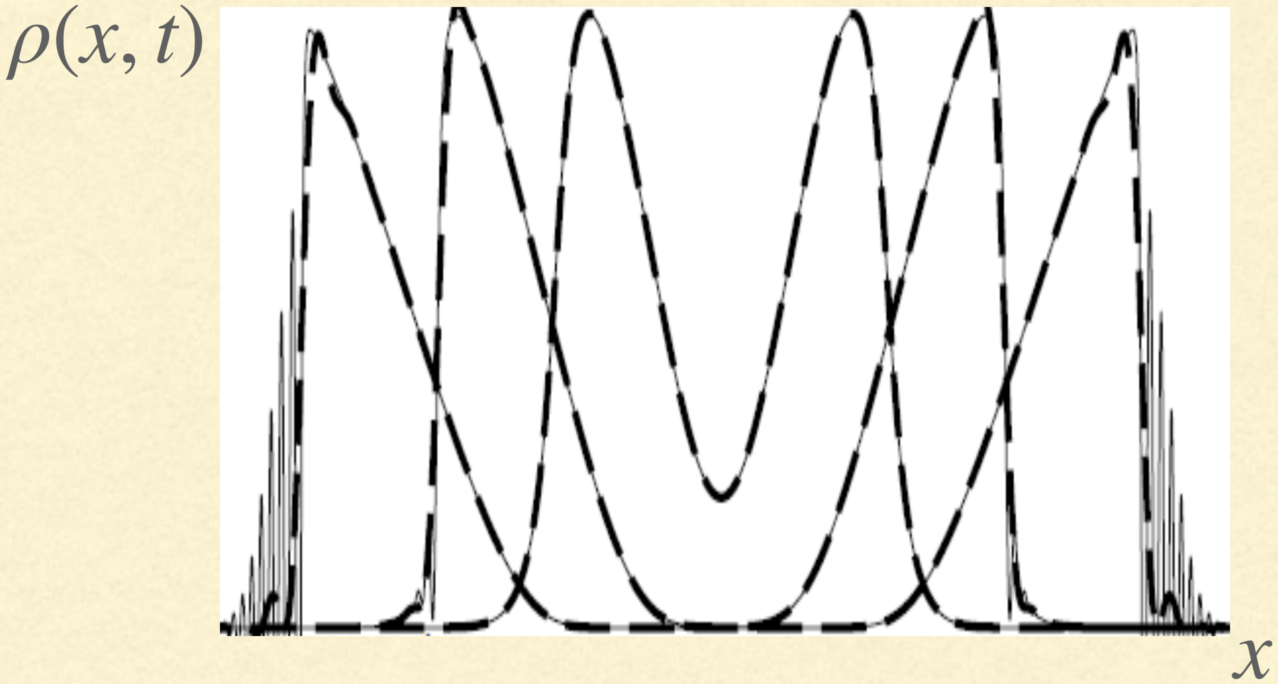
There are several drawbacks of this equation. One is that it has spurious singularities when shock fronts appear (because of high density regions moving faster), where $\partial\rho/\partial x = \infty$

Attempts have been made to add *ad hoc* terms to the Euler Equation to tame the singularity; e.g.

$$\partial_t v + \partial_x (v^2/2 + \pi^2 \rho^2/2 + V(x) - \frac{1}{2}\frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}}) = 0$$

However, these do not solve the problem. The time-development beyond the shock cannot be computed reliably from conventional hydrodynamics.

Damski, Kulkarni et al 2013



Density at three instants of time (before, at and after the shock). Dashed lines are N-body numerical simulations, which show regular behaviour even after the shock. Solid lines are analytic solutions with conventional hydro + ad hoc terms. These show irregular behaviour beyond the shock.

Damski 2006

We will solve this problem by using an exact large N description of the many-fermion systems, which leads to phase space hydrodynamics (coming up next).

Conventional hydro appears as an approximation from this, which breaks down at the time of shock formation.

Phase space hydrodynamics remains valid at all times, and can address long time evolution and equilibration.

2. Hydrodynamics in fermion phase space

Consider N non-relativistic free Fermions in one dimension, with single particle Hamiltonian

$$\hat{h} = -\frac{1}{2}\hbar^2\partial_x^2 + V(x)$$

Introduce the second quantised fermion field

$$\psi(x,t) = \sum_n \hat{c}_n \chi_n(x)\exp[-iE_n t/\hbar], \quad \hat{h}\varphi_n(x) = E_n\varphi_n(x)$$

This satisfies EOM and constraint:

$$i\hbar\ \partial_t\psi(x,t) = -\frac{1}{2}\hbar^2\ \partial_x^2\psi(x,t) + V(x)\psi(x,t)$$

$$\int dx\ \psi^\dagger(x,t)\psi(x,t) = N$$

Define the Wigner phase space distribution operator

$$U(x,p,t) = \int d\eta \; \psi^\dagger(x+\eta/2,t)\psi(x-\eta/2,t) \;\; \exp[i\eta p/\hbar]$$

The fermion path integral can be rewritten in terms of a phase density variable u(x,p,t) which is an expectation value of U(x,p,t) in a W-infinity coherent state. This satisfies the EOM

$$\frac{\partial}{\partial t}u(x,p,t) + \{h(x,p), u(x,p,t)\}_{MB} = 0,$$

and constraints that reflect Pauli exclusion principle and the conserved total fermion number:

$$(u \star u)(x,p,t) = u(x,p,t), \qquad \int \frac{dxdp}{2\pi\hbar} u(x,p,t) = N$$

Here the star product and Moyal bracket are defined by

$$(f \star g)(x,p) \equiv \left[\cos\frac{\hbar}{2}(\partial_p\partial_{x'} - \partial_{p'}\partial_x)(f(x,p,t)g(p',x',t))\right]_{p'=p,x'=x}$$

$$\{f,g\}_{MB} = f \star g - g \star f$$

Dhar-GM-Wadia 1992-1994

Large N limit

Let us now assume that we have a large number of fermions. We further assume that highest occupied states (at low energies, this means states near the Fermi level) are describable by WKB. This leads us to the following large N limit

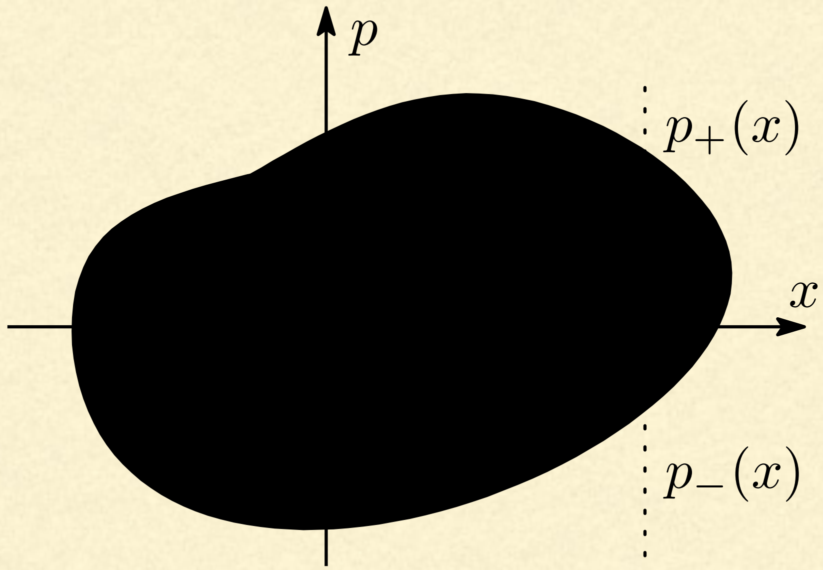
$$N \to \infty, \quad \hbar \to 0, \quad N\hbar = 1$$

In this limit the EOM and constraints become

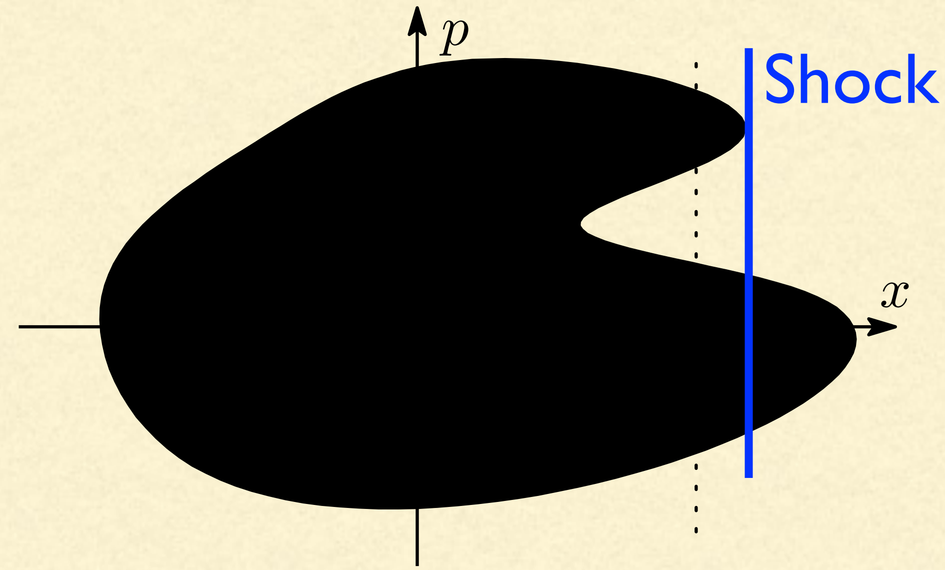
$$\frac{\partial}{\partial t} u(x,p,t) + \{h(x,p), u(x,p,t)\}_{PB} = 0,$$
$$u^2 = u, \quad \int \frac{dp dx}{2\pi}\, u(x,p,t) = N\hbar = 1.$$

The first constraint has solutions u(x,p)=0 or 1. Thus the entire configuration space of fermions is described by droplets in phase space:

Phase space droplets



A quadratic droplet



A droplet with a fold

The cartesian (x,p) parameterisation of fluid profiles works for “quadratic droplets ”: The quantities $p_+(x), p_-(x)$ get simply related to to $\rho(x), v(x)$, and the EOM of u(x,p) lead to the Euler equation of conventional hydro.

However, this description turns singular when the droplet develops a fold. This happens often, since the higher fluid elements are faster. This is the place where the real space density develops a shock. However, such singularities are a coordinate artefact.

Result 1: Shock treatment

The u(x,p,t) EOM is equivalent to the statement that motion of the droplet can be tracked by following fluid particles at each phase space point (x,p). In other words, $u(x,p,t) = u_0(x',p')$, where (x',p') denotes the location of a fluid particle at (x,p) in time t.

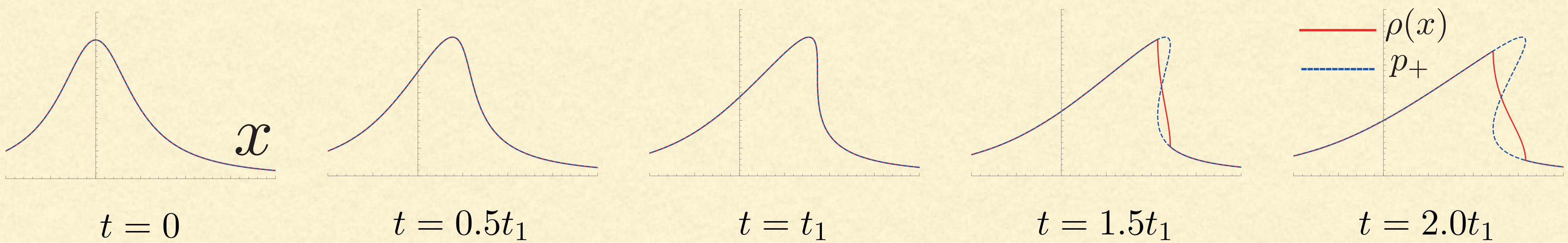


Figure 9: Development of a shock front (a single fold) in the $V = 0$ case. The five panels represent snapshots at times $t = 0, 0.5t_1, t_1, 1.5t_1, 2.0t_1$, respectively, where t_1 is the instant of time the overhang (fold) develops. The blue dashed curve represents the fluid boundary p_+ in the phase space, while the red curve represents the Fermion density $2\pi\hbar\rho(x,t)$. We have taken $p_- = 0$. The horizontal axis represents x. The x-turning points lead to $\partial\rho/\partial x = \infty$ which characterize shock fronts.

Kulkarni, GM, Morita 2018

Result 2: Equilibration to GGE

The basic idea: filamentation and equilibration through long time average:

GM-Morita 2013, Kulkarni-GM-Morita 2018

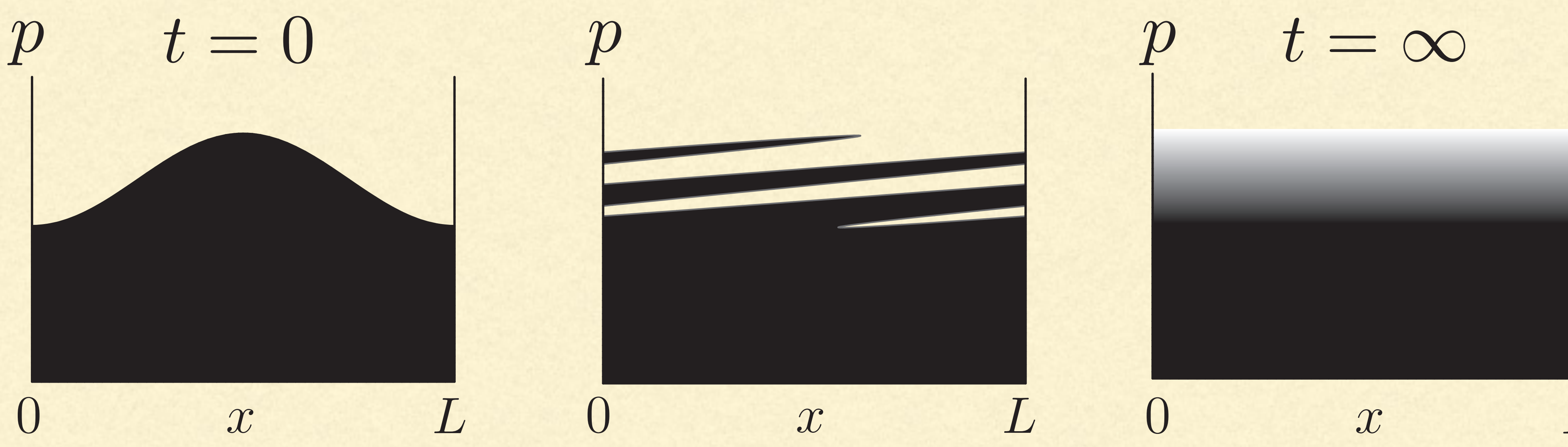
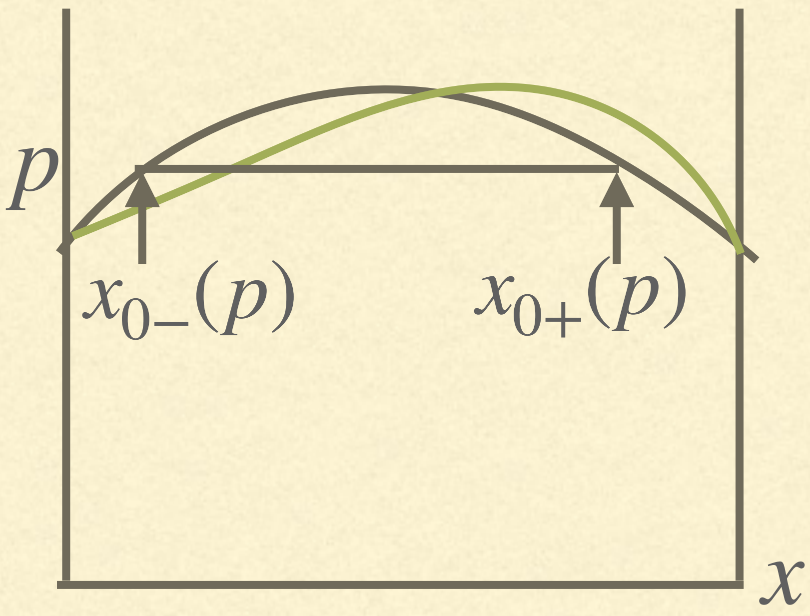


Figure 2: Cartoon of the time evolution of a droplet on a circle in the $V = 0$ case. The speed of the particle increases as p increases and the droplet will be tilted as time evolves. Finally it will be smeared uniformly on the circle and reach a steady state.



Some details in a simple case

$$u_0(x,p) = \theta(x_{0+}(p) - x)\ \theta(x - x_{0-}(p))$$

$$u(x,p,t) = \theta(x_{0+}(p) - x - pt)\ \theta(x + pt - x_{0-}(p))$$

Circle (length L) $\downarrow$

$$u(x,p,t) = \sum_m \theta(x_{0+}(p) - x - mL - pt)\ \theta(x + mL + pt - x_{0-}(p))$$

$$= \sum_k \frac{1}{L} \int dz\ e^{2\pi i k z/L}\ \ (,\,,)_{m \to z/L}$$ Poisson summation

At long times, only the k=0 term survives:

$$u(x,p,t) \to \frac{1}{L}\left(x_{0+}(p) - x_{0-}(p)\right)$$

which is the initial number of fermions with momentum p.

Relation to GGE

The Fermi gas (or infinitely repelling Bose gas) has an infinite number of conserved charges, e.g. the occupation numbers. These are given by the initial values. In the present example

$$\langle N_k \rangle_{GGE} = \frac{1}{L}\left(x_{0+}(k) - x_{0-}(p)\right)$$

$$\langle u(x,p) \rangle_{GGE} = \sum_k \langle N_k \rangle_{GGE}\, u_k(x,p) - - - (\,*\,)$$

$u_k(x,p)$ is the single-state Wigner phase space distribution:

$$u_k(x,p) = \int d\eta\ \psi_k^*(x+\eta/2)\psi_k(x-\eta/2)e^{i\eta p}, \quad \psi_k(x) = e^{ikx}$$
$$= \delta(k-p)$$

Putting in eq (*), we recover the asymptotic u(x,p) from last page

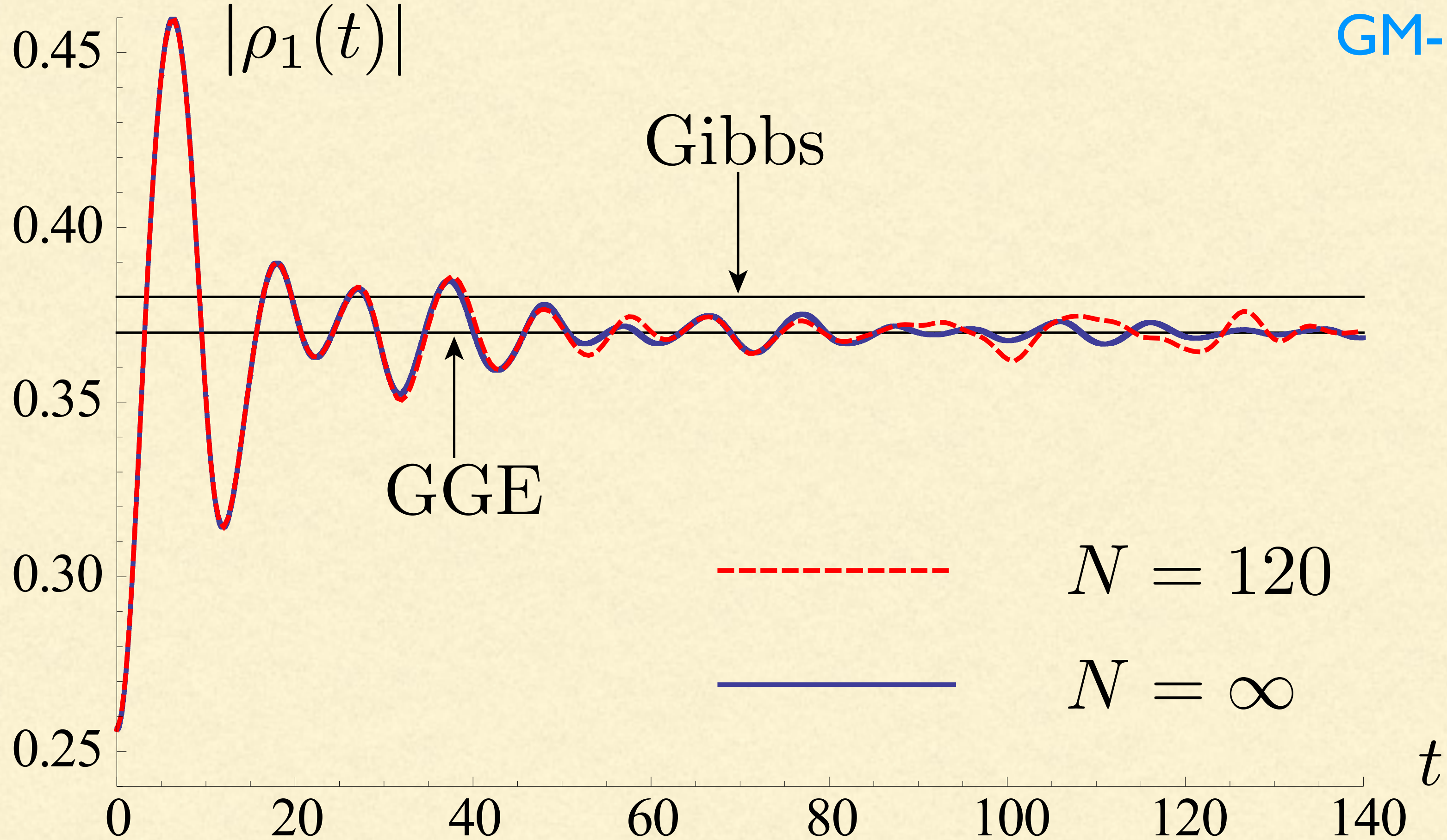
$$u(x,p,t) \to \langle u(x,p) \rangle_{GGE},\ t \to \infty$$

Example

$$V(\theta) = a\cos\theta \to b\cos\theta$$

$$\rho_1(t) = \int_0^{2\pi} d\theta\ \rho(\theta, t)\cos\theta$$

GM-Morita 2013



(a) $\rho_1(t)$ and GGE vs. Gibbs ensemble

Result 3: Universal relaxation rates

Quench Protocol	*Power Law Exponent*
released from a potential x^{2m} to a periodic circle	$t^{-\frac{2m+1}{2m}}$
released from a box potential to a periodic circle	t^{-1}
introduction of a cosine potential from $V=0$	$t^{-3/2}$

Table 1: Examples of power law relaxation of one point functions, e.g. particle densities, at late times. The results may work for arbitrary local observables defined by (32).

Kulkarni,GM,Morita 2018

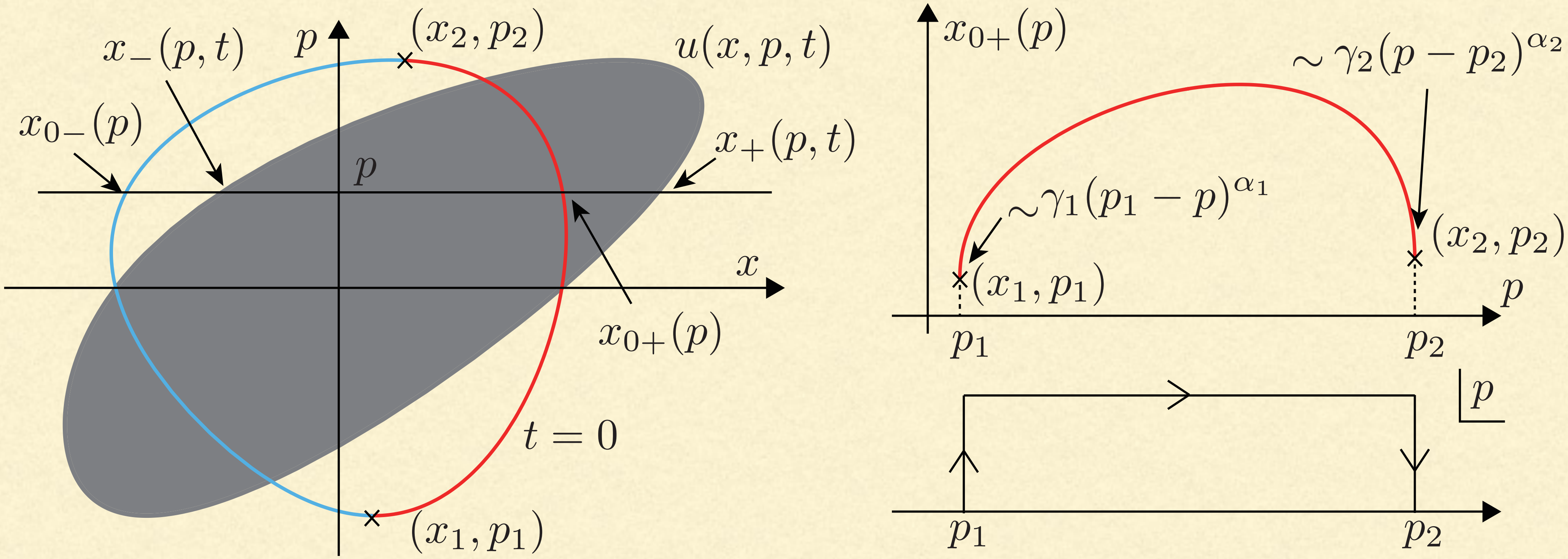


Figure 1: (Left) Fermion droplet in the phase space. The red-blue curve describes the boundary of the droplet at $t=0$. $x_+(p,t)$ and $x_-(p,t)$ denote the boundaries of the droplet for a given p at time t, and $x_{0+}(p)$ (red) and $x_{0-}(p)$ (blue) are those at time $t=0$. At (x_1,p_1) and (x_2,p_2), the curves $x_{0+}(p)$ and $x_{0-}(p)$ meet each other. (Right Top) Plot of $x_{0+}(p)$. (Right Bottom) Integral contour (24) on the complex p-plane.

Suppose the initial droplet (blue-red) has two extrema. The final shape is given by the solid blob. The density at time t is given by

$$\rho(x,t) = \int_{blob} dp\ u(x,p,t) = N/L + O\left(\frac{1}{t^{1+\alpha}}\right)$$

Example

Consider a quantum quench where fermions are released from a potential $V(x) = x^{2m}$ to free motion on a circle.

In this case, the initial droplet is given by the Fermi sea

$$p^2/2 + V(x) = E_F, \quad x_{0,\pm}(p) = 2(E_F - p^2)^{1/2m}, \quad \alpha = 1/(2m)$$

Hence, the asymptotic density is given by

$$\rho(x,t) = N/L + O\left(\frac{1}{t^{1+\alpha}}\right)$$

More generally, the expectation value of an operator O, corresponding to a phase space quantity O(x,p), has the asymptotic form

$$\langle O(t)\rangle = \int_{blob} dp\ O(x,p)u(x,p,t) = \langle O\rangle_{GGE} + O\left(\frac{1}{t^{1+\alpha}}\right)$$

Thus, various local quantities like density, velocity, even two-point functions have the <u>*same*</u> relaxation exponent.

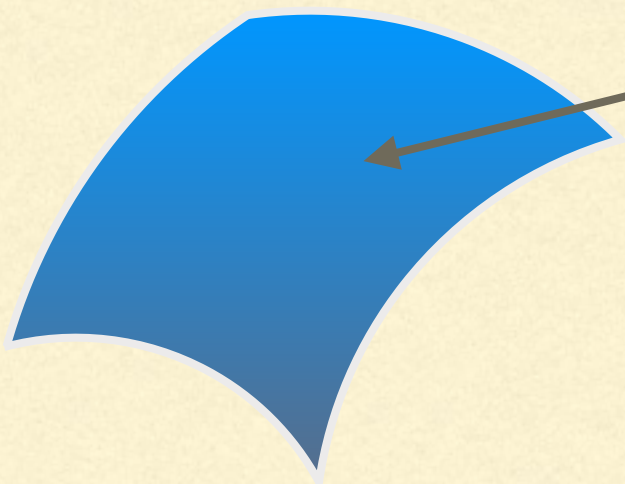
2+1 dimensional fermions

Rotating trap: $H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2+y^2\right) + \Omega\left(xp_y - yp_x\right)$

Lacroix-Majumdar-Schehr 2018

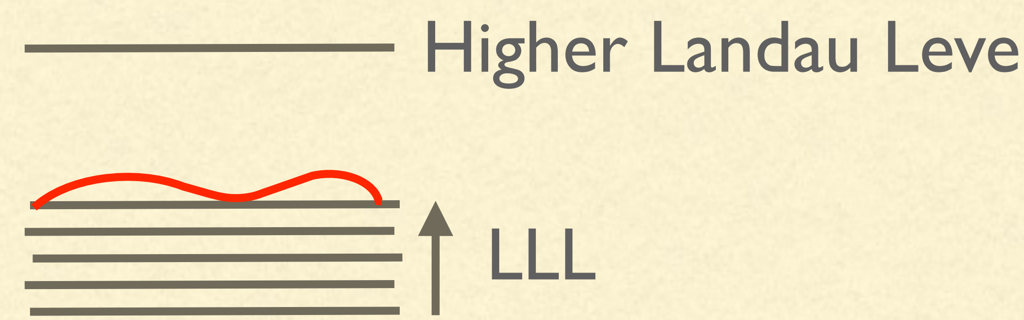
$$H = \frac{1}{2m}\left(\left(p_x - \frac{eB}{2}y\right)^2 + \left(p_y + \frac{eB}{2}x\right)^2\right) - \epsilon(xp_y - yp_x),\ eB = 2m\omega,\ \frac{eB}{m} - \epsilon = \Omega$$

Lowest Landau Level assumption: $p_x = eBy,\ p_y = -eBx$



$$\{x, y\}_{DB} = \{x, y\}_{PB} - \{x, f_1\}C_{12}\{f_2, y\} = eB$$

$$H_{LLL} = \tilde{\epsilon}(x^2 + y^2) = \tilde{\epsilon}(x^2 + p_x^2)$$



Kulkarni,GM,Morita 2019

Effective one dimensional fermi fluid in the LLL sector. We can introduce a droplet description as before and discuss long time behaviour of the Fermi fluid.

Conclusions

- Quantum quench dynamics studied in 1 and 2 dimensional fermi gas.
- Large N limit leads to phase space hydrodynamics (“droplets”), which is exact and does not suffer from spurious singularities of real space hydrodynamics at shock fronts.
- This allows study of long time dynamics and approach to equilibrium.
- We described how universal relaxation rates can be obtained using this method.
- Outlook: 1/N corrections, double scaling, criticality.