

Aspects of black hole formation in SYK-KM model

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(Arinash Dhar, Adwait Gaikwad,
Lata Joshi, Gautam Mandal & SRW)

The SYK Model

QM of N real fermions: $\Psi_a, a=1, \dots, N$

$$\{\Psi_a, \Psi_b\} = \delta_{ab}$$

$$H = \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \Psi_a \Psi_b \Psi_c \Psi_d$$

$$\langle J_{abcd} \rangle = 0, \quad \langle J_{abcd}^2 \rangle = 3! \frac{J^2}{N^3}$$

Relevance / Interest for black hole physics

- (i) Simple microscopic degrees of freedom
- (ii) Soluble in large N and long wavelengths $\omega \ll J$; can compute $\lambda_L = \frac{2\pi}{\beta} + \text{corrections}$
- (iii) Easy to compute at finite N numerically
- (iv) There is a simple semi-classical gravity dual
- (v) SYK at finite temp. is dual to a geometry in 2-dim AdS_2 spacetime

Motivation:

One way to gain insight towards the resolution of the information paradox is to study (track) black hole formation from a microscopic theory in quantum gravity (??).

AdS/CFT enables some progress in this direction provided we can solve the QFT on the boundary.

SYK model:

(Kitaev, Maldacena, Stanford)

- Soluble at large N and strong coupling
- Explicit construction of thermal state in terms of microstates (Kouckova - Maldacena)

Results:

- 1) We compute the evolution of a 'pure state' (that evaluates thermal averages of many observables), using a Hamiltonian that is tuned to the 'pure state'. By varying the strength of the Hamiltonian (piecewise in time) we can extract or pump in energy into the system, via 2 simple quenches.
- 2) The second quench, pumps in energy into the system and enables an explicit time dependent evolution of a state which asymptotically can be identified with a BH (geom with a horizon) with
$$T_{bh} \propto (\epsilon_{critical} - \epsilon_2)^{1/2}$$

Large N mean field theory

Bilocal variables:

$$G(z_1, z_2) = \frac{1}{N} \sum_{a=1}^N \langle \Psi_a(z_1) \Psi_a(z_2) \rangle$$

$$\Sigma(z_1, z_2) = \frac{1}{N} \sum_{a=1}^N \langle \chi_a(z_1) \chi_a(z_2) \rangle$$

$$\chi_a = \sum_{b,c,d} J_{abcd} \Psi_b \Psi_c \Psi_d$$

$$\text{Hamiltonian} \sim \sum_a \Psi_a \chi_a$$

Schwinger-Dyson eqns:

$$\Sigma(z_1, z_2) = J^2 G(z_1, z_2)^3, \quad G(\omega) = (i\omega - \Sigma(\omega))^{-1}$$

are derived from

$$S = -\frac{1}{2} \text{Pf}(\partial_z - \Sigma) + \int dz_1 dz_2 \left[\Sigma G - \frac{J^2}{4} G^3 \right]$$

$$Z = \int \mathcal{D}G \mathcal{D}\Sigma e^{-NS}$$

Solution in infra-red

$$\underline{W \ll J}$$

SD eqns + action has an infinite dim symmetry:

$$z \rightarrow f(z) \quad (\text{Diff 1})$$

$$G^{[f]} = [f'(z_1) f'(z_2)]^{-4} G(z_1, z_2)$$

$$\Sigma^{[f]} = [f'(z_1) f'(z_2)]^{-3\Delta} \Sigma(z_1, z_2)$$

$$\Delta = \frac{1}{4}$$

$$\text{UV dim}[\Psi] = 0, \quad \text{IR dim}[\Psi] = \frac{1}{2}$$

Solution:

$$G_{\text{IR}}(z) = \frac{b}{|z|^{2\Delta}} \text{sgn}(z), \quad b = \left(\frac{1}{4\pi J^2}\right)^{\frac{1}{4}}$$

Diff 1 \rightarrow $SL(2, \mathbb{R})$

Finite temperature $T = \beta^{-1}$: $f(z) = \tan \frac{\pi z}{\beta}$

$$G_{\text{IR}}(z, \beta) = c \left[\frac{\pi}{J\beta \sin \frac{\pi z}{\beta}} \right]^{\frac{1}{2}} \text{sgn}(z)$$

\Rightarrow

$$G_{\text{IR}}(t, \beta) = c \left[\frac{\pi}{J\beta \sinh \frac{\pi t}{\beta}} \right]^{\frac{1}{2}} \quad \text{in real time}$$

Low energy dynamics

Degrees of freedom $\tau \rightarrow f(z)$, mod $SL(2, \mathbb{R})$

Effective action:

$$NS = -\frac{Nd_s}{J} \int d\tau \{f(z), \tau\}$$

Finite temp: $f(z) = \tan \frac{\pi}{\beta} g(z)$

$$NS = \frac{Nd_s}{2J} \int_0^\beta dz \left[\left(\frac{g''}{g'} \right)^2 - \left(\frac{2\pi}{\beta} \right)^2 (g')^2 \right]$$

e.g. $g(z) = z$ is a solution of EOM and

it corresponds to a BH spacetime in the bulk.

SYK micro-states

$$\{\Psi_a, \Psi_b\} = 2\delta_{ab} \Rightarrow \Psi_a = \frac{\gamma_a}{\sqrt{2}} \quad (\gamma\text{-matrices})$$

$$\hat{S}_k = 2i \Psi_{2k-1} \Psi_{2k}, \quad k=1, 2, \dots, \frac{N}{2}$$

$$\hat{S}_k^2 = 1 \Rightarrow \text{eigenvalues } s_k = \pm 1$$

Hence Hilbert space \mathcal{H} is spanned by

the complete set $\{ |s_1, s_2, \dots, s_{N/2}\rangle \}$

$$\text{and } \dim \mathcal{H} = 2^{N/2}$$

Define $|B_S\rangle = |s_1, s_2, \dots, s_{N/2}\rangle$

These are high energy states:

$$\langle B_S | H_{\text{SYK}} | B_S \rangle \sim 0 \quad \text{and}$$

$$\langle \text{GND} | H_{\text{SYK}} | \text{GND} \rangle \sim -N \quad (\text{constant})$$

Low energy projection:

$$|B_S(l)\rangle = e^{-l H_{\text{SYK}}} |B_S\rangle \quad \text{for large } l = \frac{\beta}{2}.$$

Properties of $|B_s(l)\rangle$

$$1) \sum_{\{s_k\}} \langle B_s(l) | B_s(l) \rangle = \text{Tr} e^{-2lH}$$

$$2) \sum_{\{s_k\}} \langle B_s(l) | \prod_i \mathcal{O}_i | B_s(l) \rangle = \text{Tr} (e^{-2lH} \prod_i \mathcal{O}_i)$$

3) Flip symmetry: $\Psi_{2k} \rightarrow -\Psi_{2k}, \Psi_{2k-1} \rightarrow \Psi_{2k-1}$
(subgroup of $O(N)$)

4) For flip invariant operators

$$\sum_{\{s_k\}} \langle B_s(l) | \mathcal{O}_1 \dots \mathcal{O}_n | B_s(l) \rangle = 2^{N/2} \langle B_{s_0}(l) | \mathcal{O}_1 \dots \mathcal{O}_n | B_{s_0}(l) \rangle$$

For flip invariant operators

$$\sum_s \langle B_s(l) | \mathcal{O}_1 \dots \mathcal{O}_n | B_s(l) \rangle = 2^{N/2} \langle B_{s_0}(l) | \mathcal{O}_1 \dots \mathcal{O}_n | B_{s_0}(l) \rangle$$

$$\Rightarrow \text{Tr} (e^{-2lH} \mathcal{O}_1 \dots \mathcal{O}_n) = 2^{N/2} \langle B_{s_0}(l) | \mathcal{O}_1 \dots \mathcal{O}_n | B_{s_0}(l) \rangle$$

$\Rightarrow |B_{s_0}(l)\rangle$ acts a thermal state for
any choice of $\{s_k\}$

$$5) \langle B_s(l) | \Psi_a(t) \Psi_a(t') | B_s(l) \rangle = \frac{c_a}{\left[\frac{\beta J}{T_c} \sin \frac{\pi}{\beta} (t-t') \right]^{1/2}}$$

State dependent Operator added to H_{SYK}

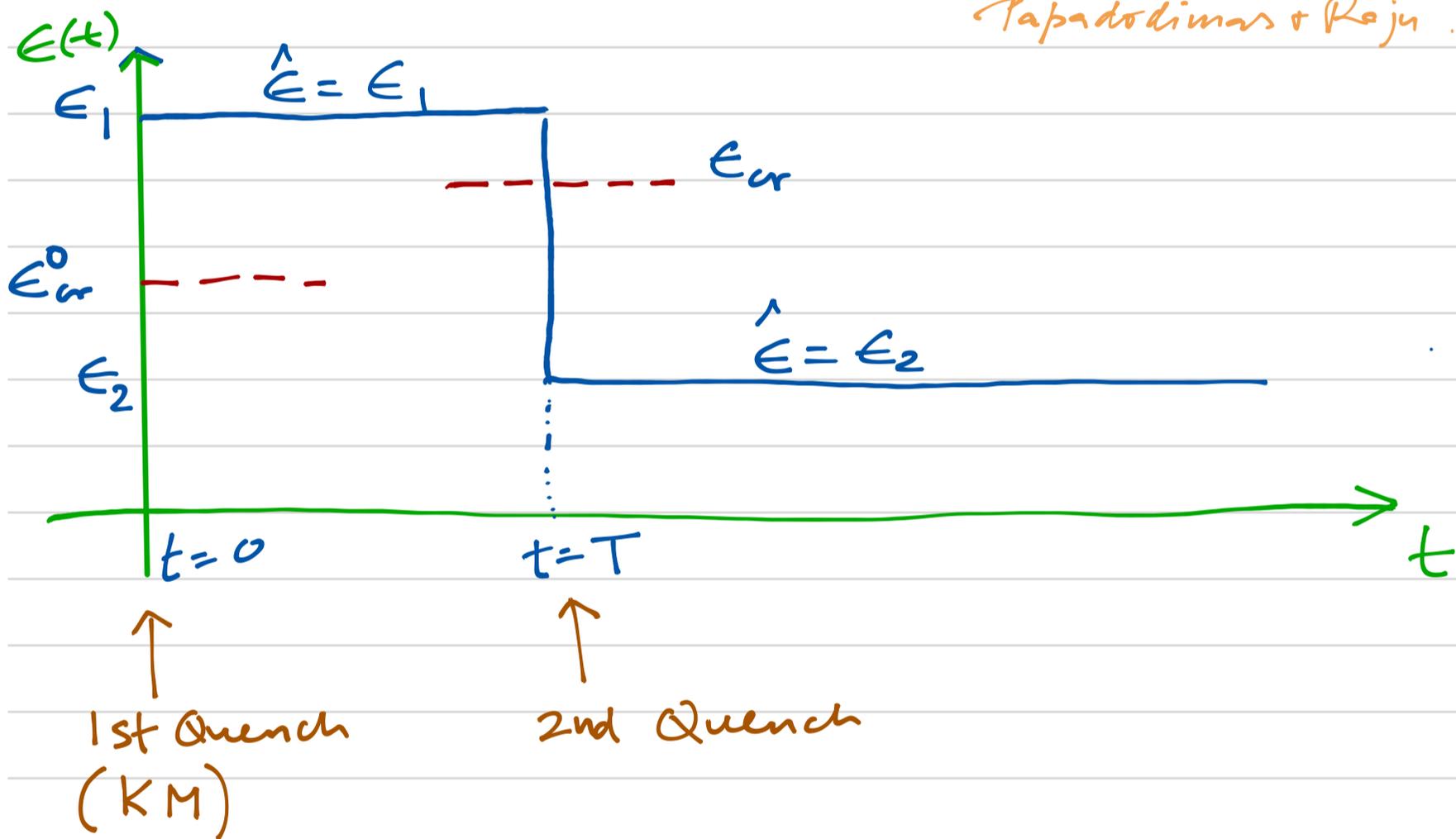
$$H = H_0 + \epsilon(t) H_M, \quad H_0 = H_{SYK}$$

$$H_M = -iJ \sum_{k=1}^{N/2} \Delta_k \Psi_{2k-1} \Psi_{2k}$$

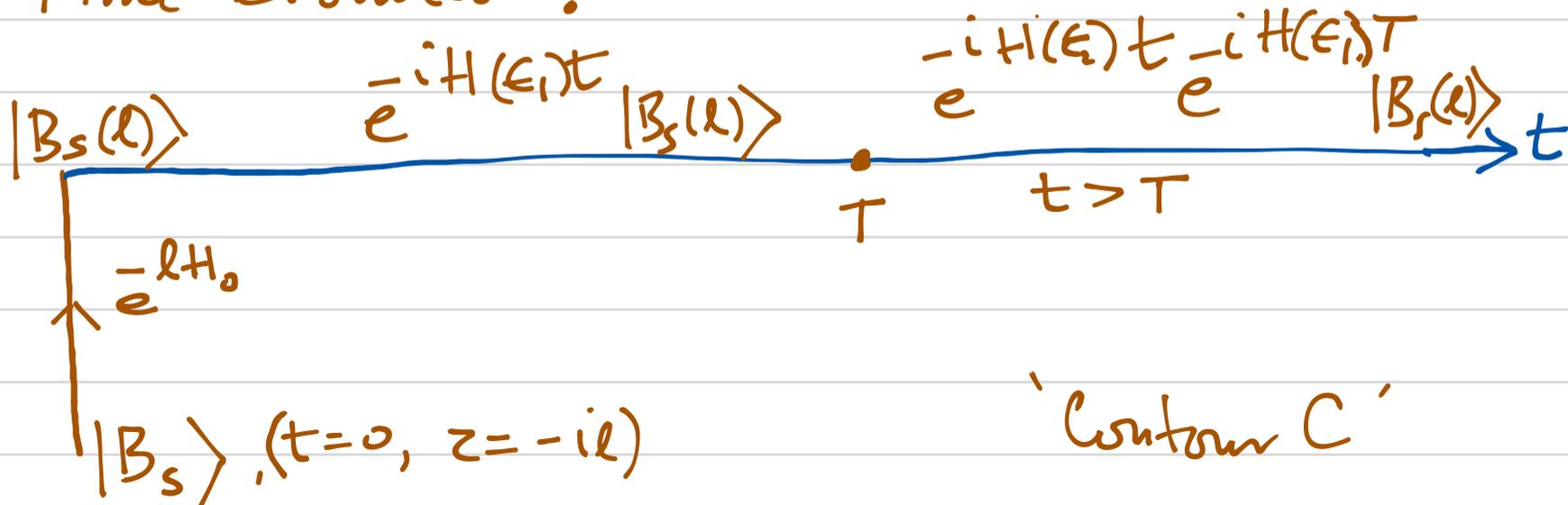
where $\{\Delta_k\}$ are the same as those in $|B_S(\ell)\rangle$

(state dependent operator)

Papadodimas + Raju



Time evolution:



Compute :

$$A_s(t, l) = \langle B_s(l) | e^{-iHt} | B_s(l) \rangle$$

$$H = H_{\text{SYK}} + \hat{\epsilon} H_M$$

$$e^{-iHt} = e^{-iH_0 t} T \left(e^{-i \int_0^t H_I(t') dt'} \right)$$

$$\Rightarrow A_s(t, l) = \int \mathcal{D}\theta \mathcal{D}\Sigma e^{i S_{\text{SYK}}(\Sigma, \theta) + \Delta S}$$

$$\Delta S = (-i \epsilon N J) \int_C dt G^2(t, -il)$$

"sees the microstates"

(we have used the large N limit)

$$G(t, -il) = \left\langle \sum_{a=1}^N \frac{1}{N} \psi^a(t) \psi^a(-il) \right\rangle$$

In the SYK model

$$\overline{G}(t, -il) = \overline{G}(t+il, 0) = \frac{c_\Delta}{\left[\frac{J\beta}{\pi} \cosh \frac{\pi}{\beta} t \right]^{\frac{1}{2}}}$$

Low energy effective action: $t \rightarrow \varphi(t)$

$$\Delta S \propto \epsilon J N \int dt \frac{|\varphi'(t)|^{2\Delta}}{\left[\frac{\beta J}{\pi} \cosh \frac{\pi}{\beta} \varphi(t) \right]^{4\Delta}} \propto \epsilon J N \int dt (f')^{2\Delta}$$

$$f(t) = \frac{\pi}{J^2 \beta} \tanh \frac{\pi}{\beta} \varphi$$

Compute :

$$A_s(t, l) = \langle B_s(l) | e^{-iHt} | B_s(l) \rangle$$

$$H = H_{\text{SYK}} + \hat{\epsilon} H_M$$

$$e^{-iHt} = e^{-iH_0 t} T \left(e^{-i \int_0^t H_I(t') dt'} \right)$$

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$$f(t) = \frac{\pi}{J^2 \beta} \tanh \frac{\pi}{\beta} \varphi$$

$$1) \quad 0 \leq t \leq T$$

$$\epsilon_1 > \frac{\pi}{\beta J}, \quad e^{\phi(0)} = \left(\frac{\pi}{\beta J}\right)^2, \quad \dot{\phi}(0) = 0$$

$$\bar{\epsilon}_0 = \frac{\pi}{\beta J}, \quad \bar{\epsilon}_1 = \epsilon_1 - \frac{\pi}{\beta J}$$

$$\dot{f}(t) = e^{\phi(t)} = \frac{\bar{\epsilon}_0^2}{\left[\left(1 - \epsilon_1 \bar{\epsilon}_1^{-1}\right) \cos \frac{Jt}{2} \sqrt{\bar{\epsilon}_0 \bar{\epsilon}_1} + \frac{\epsilon_1 \bar{\epsilon}_1^{-1}}{2} \right]^2}$$

The boundary conditions come from the matching of the Euclidean solution at $t=0$.

$\phi(t)$ is bounded & periodic

$$\therefore \dot{f}(t) \neq 0 \quad \forall t \in [0, T]$$

(KM \rightarrow corresponds to a geometry without a horizon, because near the spacetime

boundary: $z \approx f'(t)\delta$

$$2) T \leq t < \infty, \quad \epsilon(t) = \epsilon_2$$

$$\dot{f}(t) = e^{\phi(t)} = \bar{\epsilon}_1^2 \left[\left(1 + \frac{\epsilon_2 \bar{\epsilon}_2^{-1}}{2} \right) \cosh \frac{J}{2} (t-T) \sqrt{\bar{\epsilon}_1 \bar{\epsilon}_2} - \frac{\epsilon_2 \bar{\epsilon}_2^{-1}}{2} \right]^{-2}$$

$$\bar{\epsilon}_2 \equiv \bar{\epsilon}_1 - \epsilon_2$$

$$\text{at } t = T, \quad \phi(T-\epsilon) = \phi(T+\epsilon) \quad \text{as } \epsilon \rightarrow 0$$

$$\dot{\phi}(T-\epsilon) = \dot{\phi}(T+\epsilon) \quad \text{as } \epsilon \rightarrow 0$$

$$\text{but } \ddot{\phi}(T-\epsilon) \neq \ddot{\phi}(T+\epsilon)$$

$$\Delta \ddot{\phi}(T) = -\Delta V'(\phi(T)) = (\epsilon_1 - \epsilon_2) e^{\phi(T)/2}$$

$$t \rightarrow \infty$$

$$\dot{f} = e^{\phi} \rightarrow e^{-Jt \sqrt{\bar{\epsilon}_1 (\bar{\epsilon}_1 - \epsilon_2)}}$$

Comparing with the large time behavior of

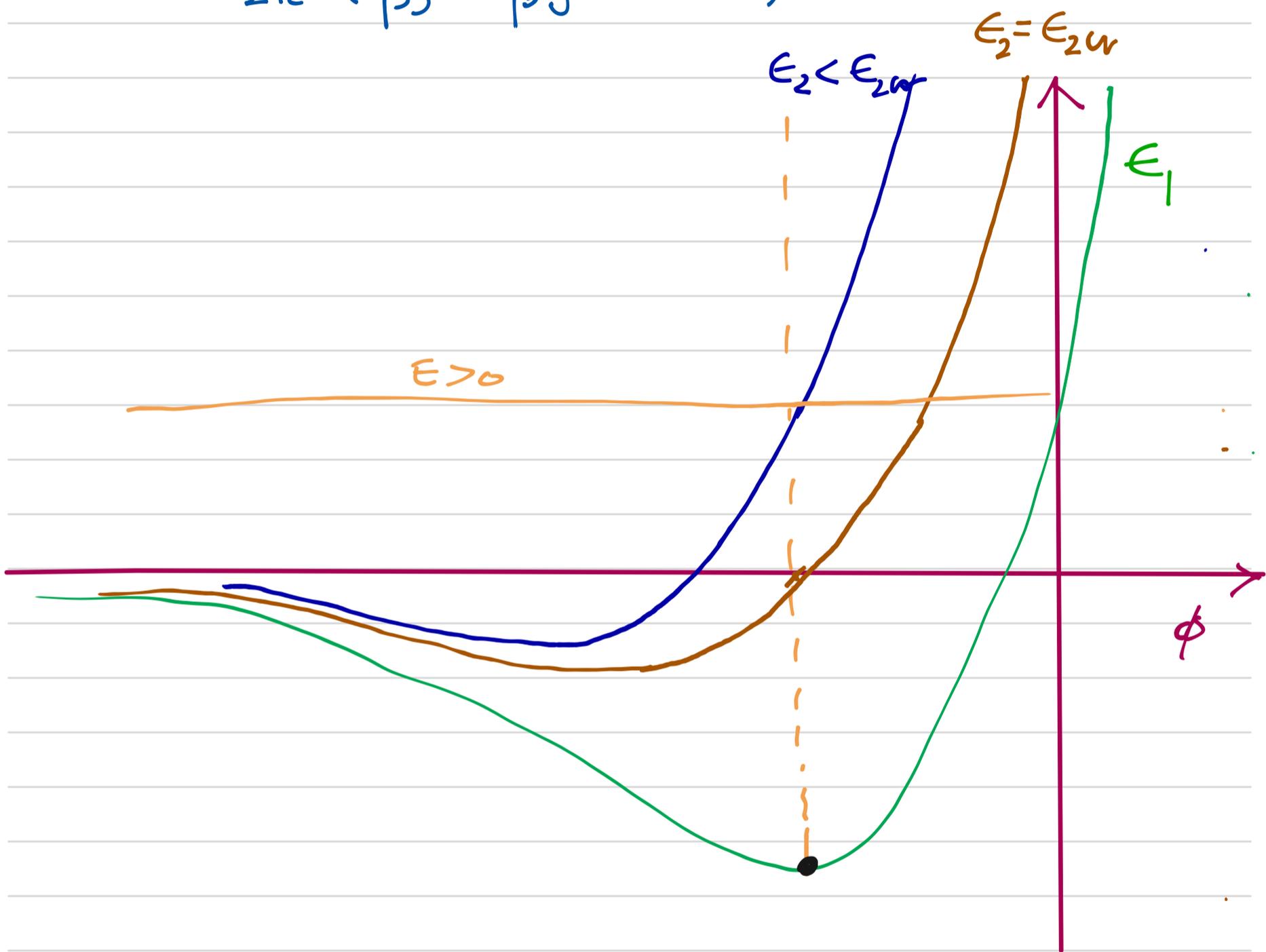
$$f_{bh} = \frac{\pi}{\beta J^2} \tanh \frac{\pi}{\beta} t \Rightarrow \dot{f}_{bh} = \left(\frac{\pi}{\beta J} \right)^2 \text{sech}^2 \left(\frac{\pi}{\beta} t \right)$$

$$\dot{f}_{bh} \sim \left(\frac{\pi}{\beta J} \right)^2 e^{-\frac{2\pi t}{\beta}}$$

$$T_{bh} = \frac{J}{2\pi} \sqrt{(\epsilon_1 - \frac{\pi}{\beta J})(\epsilon_1 - \epsilon_2 - \frac{\pi}{\beta J})}$$

In the case when $\epsilon_1 = \frac{2\pi}{\beta J}$

$$T_{bh} = \frac{J}{2\pi} \left(\frac{\pi}{\beta J} \left[\frac{\pi}{\beta J} - \epsilon_2 \right] \right)^{1/2}$$



Bulk Interpretation :

Bulk dual description has 2 'equivalent' candidates :

1) Polyakov gravity, that arises from quantisation of the co-adjoint orbit of $\text{Diff}(1)/\text{SL}(2, \mathbb{R})$ (Nayak, Mandal, SRW)

2) Jakin-Teitelboim dilaton gravity.

S-wave reduction of a near extremal charged black hole in 4-dim. $\rightarrow \text{AdS}_2 \times S_2$

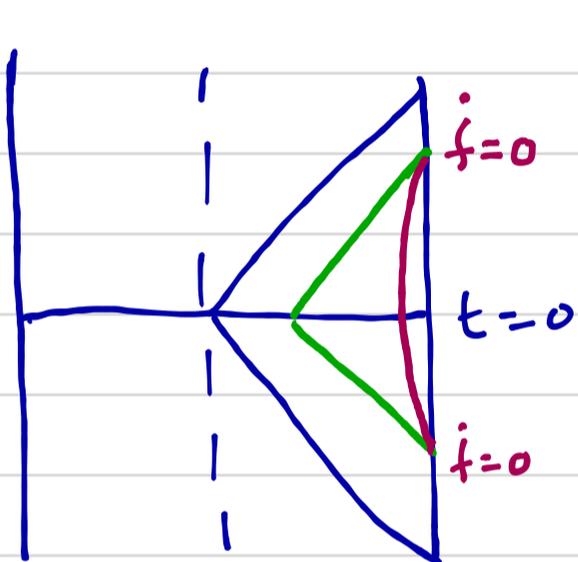
In both versions the only propagating degrees of freedom are large diffs. and the gravity action is the Schwarzian.

$$f(z) \rightarrow ds^2 = -\frac{dt^2}{z^2} \left(1 + \frac{z^2}{2} \{f, t\} \right)^2 + \frac{dz^2}{z^2}$$

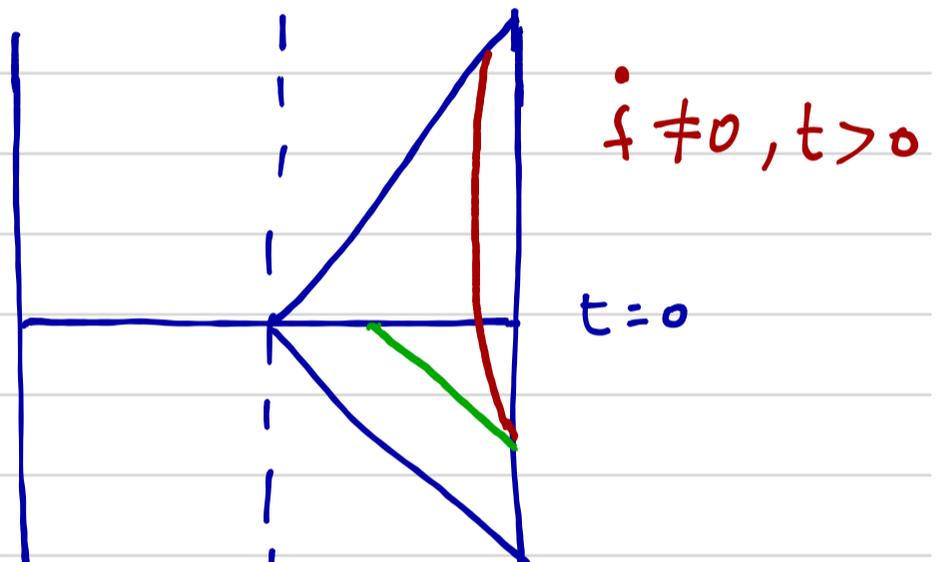
$$f(t) = \tanh \frac{\pi t}{\beta} \text{ is a bh. } \{f, t\} = -\frac{\pi^2}{\beta^2}$$

In fact for each of the 3 functions we found, there is an associated geometry.

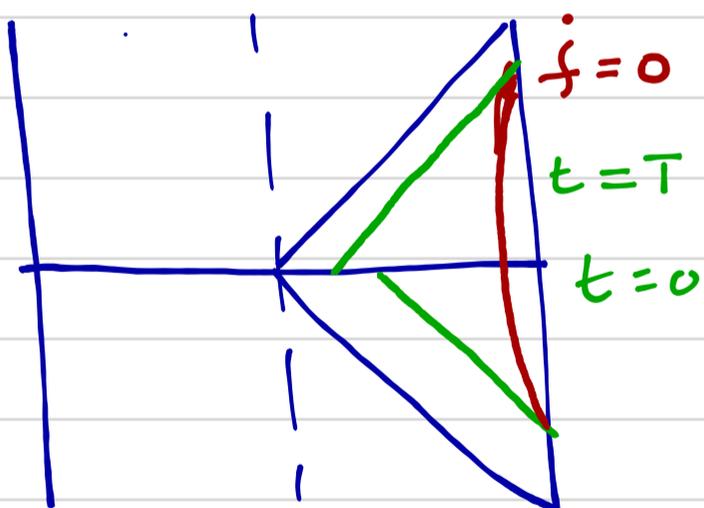
$$ds^2 = -\frac{dt^2}{z^2} \left(1 - \frac{z^2}{2} \left[\frac{J^2}{4} e^{\phi/2} - E \right] \right)^2 + \frac{dz^2}{z^2}$$



$$H = H_{\text{SyK}}$$



$$H = H_{\text{SyK}} + \epsilon_1 H_M \quad t \geq 0$$



$$H = H_{\text{SyK}} + \epsilon_1 H_M, \quad 0 \leq t \leq T$$

$$= H_{\text{SyK}} + \epsilon_2 H_M, \quad 0 \leq T < \infty$$

↑ 'end of the world brane'

A massive particle (charged under $SL(2, \mathbb{R})$) that 'reflects' an eigenstate of spins on the left boundary i.e. $|B_s\rangle_L$.

End remarks and questions:

1. Correlation functions can be calculated and studied at the 'cross over' of solutions and reflect the discontinuity of $f'''(t)$ or $\phi''(t)$ at $t=T$.
2. Where are the micro-states in the bulk theory?

Thanks

