Quantum intercept of the string with massive endpoints

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Regge intercept

String theory was originally proposed as a theory of hadrons, motivated by the observation of linear Regge trajectories.

A classical rotating string:

$$J = \alpha' M^2, \qquad \alpha = (2\pi T)^{-1}$$

Quantum spectrum of the open string

$$N = \alpha' M^2 + a$$

Where for the bosonic critical string,

$$a = -\frac{D-2}{2}\sum_{n=1}^{\infty} n = -\frac{D-2}{2} \times \frac{-1}{12} = \frac{D-2}{24} = 1$$

Our goal is to generalize this to the string with massive endpoints (in D dimensions).

The intercept determines the spectrum, in particular the mass of the ground state.

Plays an additional role in scattering amplitudes

$$A(s,t) = \mathrm{B}(-\alpha(s), -\alpha(t))$$

$$\alpha(s) = \alpha' s + 1, \alpha(t) = \alpha' s + 1$$

Regge limit: $A(s,t) \sim s^{\alpha(t)}$

Total cross section: $\sigma \sim s^{\alpha(0)-1}$

Why sum over n?

The bulk EoM:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = \partial\bar{\partial}X^\mu = 0 \Rightarrow X^\mu = X_R^\mu + X_L^\mu$$

Mode expansion

$$X^\mu = X_0^\mu + i\sqrt{N}\sum_n \frac{1}{\omega_n}\left(\alpha_n^\mu e^{-i\frac{\pi}{\ell}\omega_n(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-i\frac{\pi}{\ell}\omega_n(\tau+\sigma)}\right)$$

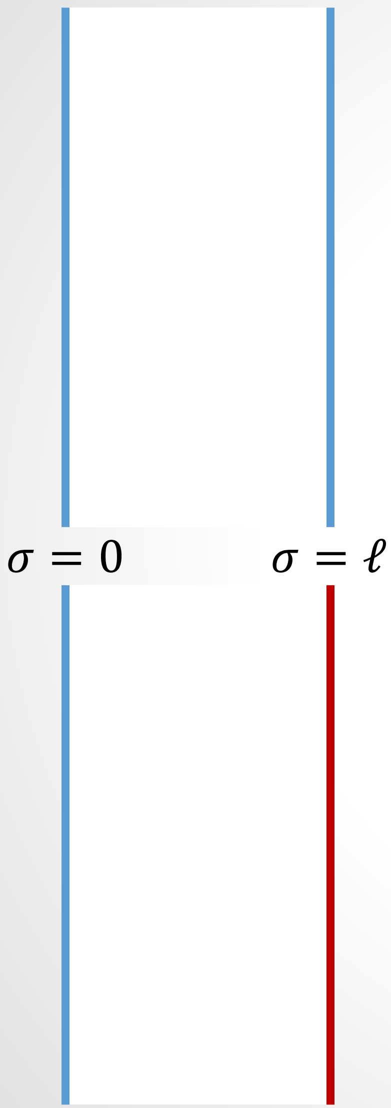
Boundary conditions at $\sigma = 0, \ell$ impose

a) A relation between α_n^μ and $\tilde{\alpha}_n^\mu$ (possibly phase, rotation)
b) The set of eigenfrequencies

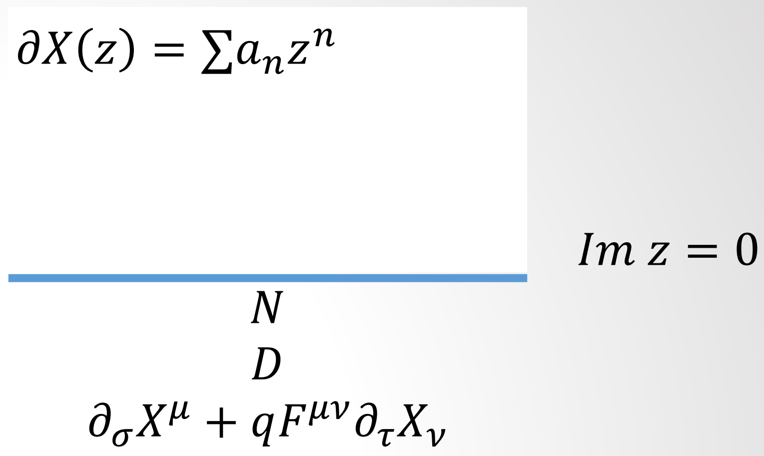
For example:

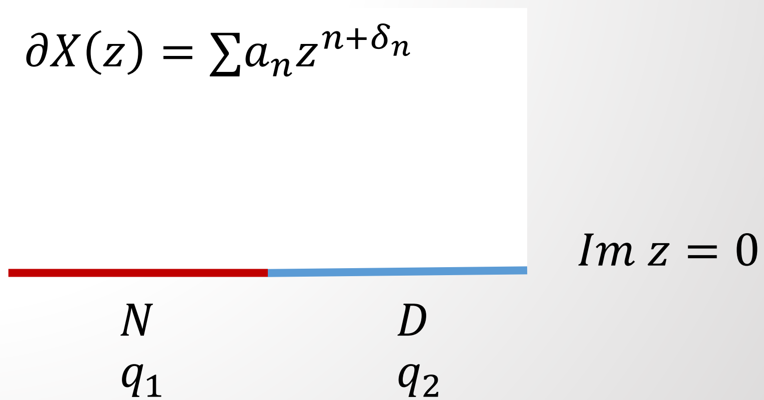
Neumann:	$\tilde{\alpha}_n = \alpha_n$	$\omega_n = n$
Dirichlet:	$\tilde{\alpha}_n = -\alpha_n$	$\omega_n = n$
ND:	$\tilde{\alpha}_n = \alpha_n$	$\omega_n = n - \frac{1}{2}$

Strip



UHP





or masses…

The intercept can be calculated as the normal ordering constant in the Hamiltonian

$$H = \sum \alpha_{-n} \cdot \alpha_n$$

Given $\left[\alpha_m^{\mu}, \alpha_n^{\nu}\right] = \omega_m \eta^{\mu\nu} \delta_{m+n}$

We have to sum the eigenfrequencies for each mode:

$$a \sim \sum_i \sum_n \omega_n^{(i)}$$

The string with massive endpoints

The string with massive endpoints in flat spacetime is described by the Nambu-Goto action, coupled to massive point particles at the string endpoints

$$S = -T\int d\tau d\sigma\sqrt{-h} - m\int d\tau\sqrt{-\dot{X}^2}$$

To the usual eqs. of motion in the bulk is added a boundary condition

$$\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0$$

$$T\sqrt{-h}\partial^\sigma X^\mu \mp m\partial_\tau\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}\right) = 0 \qquad \sigma = \pm\ell$$

Gauge choice can simplify bulk EoM but not simultaneously the boundary condition.

Instead we will expand the action around classical solution.

$$X^0 = \tau, X^1 = R(\sigma)\cos(\omega\tau)\,, X^2 = R(\sigma)\sin(\omega\tau)$$

We can write the energy and angular momentum of the rotating string, in terms of the endpoint velocity β

$$E = 2\gamma m + TL\frac{\arcsin\beta}{\beta}$$

$$J = \gamma\beta m L + \frac{1}{4}TL^2\frac{\arcsin\beta - \beta\sqrt{1-\beta^2}}{\beta^2}$$

Here $\gamma = (1-\beta^2)^{-1/2}$ and the length is determined from the boundary condition

$$\frac{T}{\gamma} = \frac{2\gamma m \beta^2}{L}$$

The three equations define the (classical) relation between J and E, or the classical Regge trajectory.

For small masses/high energies (ultrarelavisitic β)

$$J = \alpha' E^2 (1 - \frac{8\sqrt{\pi}}{3} \left(\frac{m}{E}\right)^{3/2} + \cdots)$$

To compute the quantum corrections, we add fluctuations

$$X^\mu = X^\mu_{rot} + \delta X^\mu$$

For very long strings ($L \gg \frac{1}{m}, \frac{m}{T}, \frac{1}{\sqrt{T}}$) we can truncate the action to quadratic order in δX and get a solvable theory.

One has to distinguish between $D - 3$ transverse modes and single "planar" mode.

We define the intercept as the quantum corrections to the relation between

$$\mathrm{a} \equiv \langle \delta\big(J - J_{cl}(E)\big)\rangle = \langle \delta J - \frac{\partial J}{\partial E}\delta E\rangle = \langle \delta J - \frac{1}{\omega}\delta E\rangle$$

By inserting $X = X_{rot} + \delta X$ into the expressions for J and E, we find that

$$\delta J - \frac{1}{\omega}\delta E = -\frac{1}{\omega}H_{ws}(\delta X)$$

Where H is the Hamiltonian for the fluctuations.

So to get the intercept we solve the quadratic theory for the fluctuations and compute $\langle H \rangle$.

For a rotating string with massive endpoints, we have

$$a = -\frac{D-3}{2}\sum\omega_n^{(t)} - \frac{1}{2}\sum\omega_n^{(p)}$$

$$\delta X = i\sqrt{N}\sum\frac{\alpha_n}{\omega_n}e^{-i\omega_n\tau}f_n(\sigma)$$

The transverse fluctuations, Fourier modes solve

$$f_n'' + \omega_n^2 f_n = 0 \qquad (Tf_n' \pm \gamma m\omega_n^2 f_n)|_{\sigma=\pm\ell} = 0$$

The planar mode

$$f_n'' + \left(\omega_n^2 - \frac{2\omega}{\cos^2(\omega\sigma)}\right)f_n = 0$$

$$[Tf_n' \pm (\gamma m\omega_n^2 + \cdots)f_n]\Big|_{\sigma=\pm\ell} = 0$$

The transverse modes' ω_n^t are given by the solutions to the equation

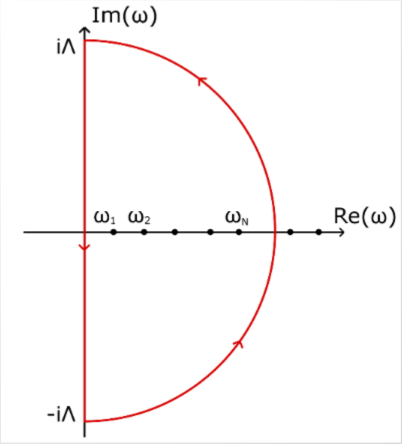
$$f(x) = 2x\beta^2\sqrt{1-\beta^2}\cos\left(\frac{2arcsin\beta}{\beta}x\right) + (\beta^4 - (1-\beta^2)x^2)\sin\left(\frac{2arcsin\beta}{\beta}x\right) = 0$$

where $x = \omega_n \ell$.

The sum can be converted into a contour integral.

$$\sum_{n>0} \omega_n = \frac{1}{2\pi i\ell}\oint dz\, z\frac{f'(z)}{f(z)} = \frac{1}{2\pi i\ell}\oint dz\, z\frac{d}{dz}\log f(z)$$

The contour is a semicircle of radius Λ that we eventually take to infinity.



To understand the renormalization, we can look back at the massless string ($\beta = 1$)

$$f(\omega) = \sin(\pi\omega\ell)$$

We are computing Casimir energy

$$E_C = \frac{1}{2}\sum\omega_n = \frac{1}{4i}\oint \omega\ell \cot(\pi\omega\ell) = \frac{\Lambda^2 L}{8} - \frac{1}{12L}$$

Divergent part proportional to the length of the string $L = 2\ell$, and can be absorbed into string tension.

The finite part is contribution of one mode to the intercept,

$$E_C^{ren} = -\frac{2a}{L} \rightarrow a = \frac{1}{24}$$

We can also think of the subtraction as

$$E_C^{ren} = \lim_{\Lambda \to \infty} \left(E_C(L, \Lambda) - E_C(L \to \infty, \Lambda)\right)$$

This subtraction can be done before computing the full contour integral:

$$E_C = \frac{1}{2\pi} \int_0^{\infty} \log(1 - e^{\pi L\, y})\, dy = -\frac{1}{12L}$$

For the rotating string with masses, the divergent part of the contour integral

$$\frac{\Lambda^2 L \arcsin\beta}{\pi\beta} + \frac{T}{2\gamma m}\log\frac{2\gamma m\Lambda}{T}$$

Compare with $E = 2\gamma m + TL\frac{\arcsin\beta}{\beta}$

Subtracting the divergent part directly as before leaves

$$a_t = -\frac{1}{2\pi\beta}\int_0^\infty \log\left[1 - e^{-4\frac{\arcsin\beta}{\beta}y}\left(\frac{y-\gamma\beta^2}{y+\gamma\beta^2}\right)^2\right]dy$$

The leading order correction at large $\frac{TL}{2m} = \gamma^2 - 1$,

$$a_t = \frac{1}{24} - \frac{11}{360\pi}\left(\frac{2m}{TL}\right)^{\frac{3}{2}} + \cdots$$

The leading order result could have been obtained through Zeta function regularization, using an approximate solution for ω_n

$$\approx n - 1 + \frac{2}{\pi}\arctan(\frac{2\gamma\beta\text{arcsin}\beta + 2}{n\pi})$$

Which we can expand in γ^{-1}

$$a_t \approx \sum(-\frac{1}{2}n + \frac{n - n^3}{3\pi}\gamma^{-3})$$

The coefficient of γ^{-3} is in fact $\frac{1}{3\pi}\left(\zeta(-1) - \zeta(-3)\right) = \frac{-11}{360\pi}$.

A similar, but more complicated calculation yields

$$a_p = \frac{1}{24} + \frac{11}{720\pi}\left(\frac{2m}{TL}\right)^{\frac{3}{2}} + \cdots$$

for the planar intercept.

Non-critical string

For $D < 26$, we can work in effective string theory. In EFT, the next term added to the NG action is the Polchinski-Strominger term (PS, 1991)

$$S_{PS} = \frac{B}{2\pi} \int d\tau d\sigma \frac{(\partial_+^2 X \cdot \partial_- X)(\partial_-^2 X \cdot \partial_+ X)}{(\partial_- X \cdot \partial_+ X)^2}$$

This is related to the Liouville action one finds in the Polyakov formulation when identifying $\gamma_{ab} = e^{\phi}\eta_{ab}$ with the induced metric.

The PS term is inserted in order to maintain conformal symmetry on the world sheet for any D. This fixes $B = \frac{D-26}{12}$.

Without masses, it was found by Hellerman-Swanson (2013) that the PS term cancels the D dependence in

$$a = \frac{D-2}{24} + \frac{26-D}{24} = 1.$$

The leading order correction from the PS term to the intercept is $a_{PS} = -\frac{1}{\omega} E_{PS}, \quad E_{PS} = -\int \mathcal{L}\_PS(X_{rot}) d\sigma$

The massive endpoints in this case act as a regulator to a divergence,

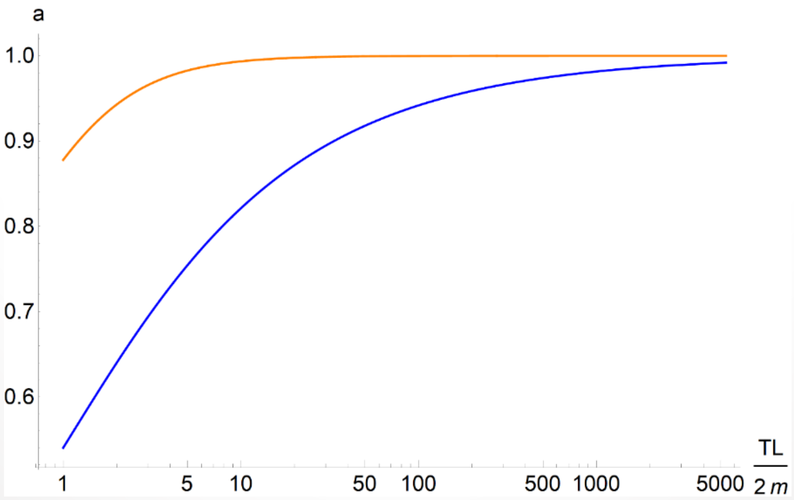
$$E_{PS} = \frac{B}{\pi}\frac{T}{\gamma m} - \omega\frac{26-D}{12\pi}\arcsin\beta$$

If we subtract the part that diverges when the mass is taken to zero (and expand near $\beta \to 1$), we are left with

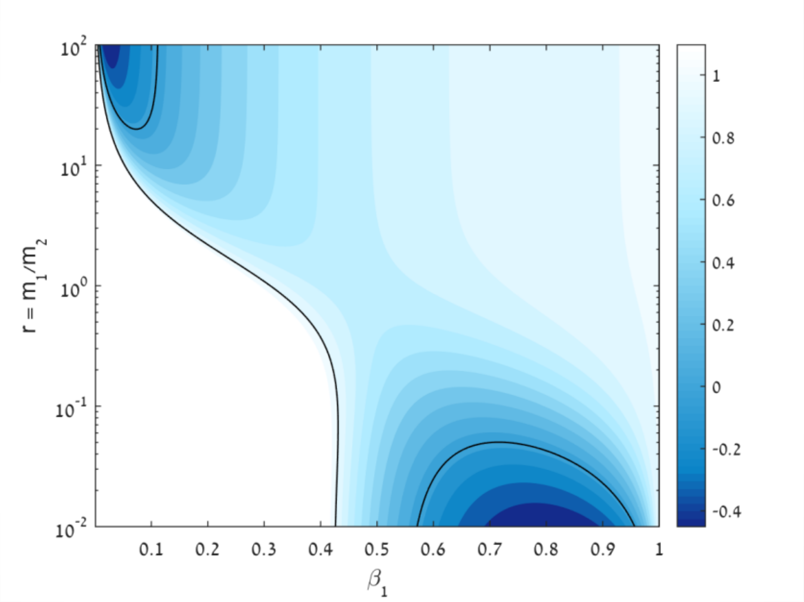
$$a_{PS} = \frac{26-D}{24}(1 - \frac{2}{\pi}\left(\frac{2m}{TL}\right)^{\frac{1}{2}} + \frac{2}{3\pi}\left(\frac{2m}{TL}\right)^{\frac{3}{2}})$$

The total intercept intercept, $a = (D-3)a_t + a_p + a_{PS}, is$

$$a = 1 - \frac{26-D}{12\pi}\left(\frac{2m}{TL}\right)^{\frac{1}{2}} + \frac{199-14D}{240\pi}\left(\frac{2m}{TL}\right)^{\frac{3}{2}}$$



We can generalize to different masses



Summary

We computed the intercept $a = \langle J - J_{cl}(E) \rangle$ for high energy/spin with endpoint masses for any D.

The result is a correction to the $a = 1$ from the bosonic critical string, which is still far from the phenomenological intercept of e.g. $a_\rho = -0.5$.

Future directions: more general boundary conditions, scattering amplitudes.

Thank you!