New approaches to the quantum Heisenberg models: Schwinger boson representations (invited)

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Schwinger boson representations allow us to study ferromagnetic and antiferromagnetic Heisenberg models in a rotationally invariant formulation. The large-N SU(N) Heisenberg models are approximated by the Schwinger boson mean-field theory (SBMFT). In most cases, even for $N=2$ (the physical model), the SBMFT is surprisingly successful: We review recent comparisons with numerical results, spin-wave theory, and renormalization group analysis of the nonlinear sigma model. The mean-field theory, like the nonlinear sigma model, does not include the effects of topological Berry phases, which can appear in the antiferromagnetic spin liquid phases.

I. INTRODUCTION

Recently, experimental and theoretical interest in low-dimensional magnetism has been greatly revived. Much of the activity has been spurred by discovery of the quasi-two-dimensional Heisenberg antiferromagnets (high T_c superconductors[1] La_2CuO_4 and $YBa_2Cu_3O_{7-\delta}$). As a result, we have seen substantial progress in new approaches to the quantum Heisenberg model (QHM).

Since in one and two dimensions, quantum and thermal fluctuations combine to destroy long-range order[2], the use of naïve spin-wave theory[3] is strictly limited to zero temperature and to ground states with broken rotational symmetry. For the antiferromagnetic model in one dimension, it can be that the *ground state* has no long-range order, as shown by Bethe's solution[4] of the $S=1/2$ chain, and Haldane's mapping of the large S QHM onto the nonlinear sigma model (NLSM) in $(1+1)$ dimensions[5] (S is the length of the spin). The field theory includes a topological θ term, which differentiates between the integer and half-odd-integer S ground states,[6] in accordance with the Lieb–Shultz–Mattis theorem.[7] This approach was recently extended to the two-dimensional model, where an ordered ground state exists above a critical value of the spin. In the disordered phase (small S and/or large frustration), interference from topological terms was predicted[8] to depend on the value of $2S$ mod 4.

Here we shall focus on an alternate asymptotic approach[9,10] based on the functional integral steepest descents approximation to the QHM. The resulting Schwinger boson mean-field theory (SBMFT) is the large-N theory of a particular generalization of the QHM to SU(N) generators replacing the usual SU(2) spin operators. Comparisons of our results to exact and numerical calculations for the physical $N=2$ model reveal a surprising success of the low-order approximation.

The generalized SU(N) Hamiltonian is given by

$$H=-\frac{1}{N}\sum_{\substack{\langle ij\rangle\\ \alpha,\beta}}\mathscr{S}^{\alpha}_{\beta}(i)\tilde{\mathscr{S}}^{\beta}_{\alpha}(j)+\tfrac{1}{2}zS^2$$

$$=-\frac{1}{N}\sum_{\langle ij\rangle}:\mathscr{A}^{\dagger}_{ij}\mathscr{A}_{ij}:$$

$$\mathscr{A}_{ij}\equiv\sum_{\alpha}b^{\dagger}_{\alpha i}b_{\alpha j}\quad\text{ferromagnet,}$$

$$\mathscr{A}_{ij}\equiv\sum_{\alpha}b_{\alpha i}b_{\alpha j}\quad\text{antiferromagnet,}\tag{1}$$

where in each bond $\langle ij\rangle$ the site j is taken to be in the second sublattice. Here the generalized spin operators are defined as

$$\mathscr{S}^{\alpha}_{\beta}(i)\equiv b^{\dagger}_{\alpha i}b_{\beta i},\tag{2}$$

which satisfy the algebra

$$[\mathscr{S}^{\alpha}_{\beta}(i),\mathscr{S}^{\rho}_{\sigma}(j)]=\delta^{\rho}_{\beta}\delta^{i}_{j}\mathscr{S}^{\alpha}_{\sigma}(i)-\delta^{\alpha}_{\sigma}\delta^{i}_{j}\mathscr{S}^{\rho}_{\beta}(i)\tag{3}$$

and are subject to the constraint $\Sigma^{N}_{\alpha=1}\,\mathscr{S}^{\alpha}_{\alpha}(i)=NS$. S must be an integer multiple of $1/N$. For the ferromagnet, $\tilde{\mathscr{S}}$ is defined by Eq. (2), while for the antiferromagnet,

$$\tilde{\mathscr{S}}^{\alpha}_{\beta}(i)\equiv-b^{\dagger}_{\beta i}b_{\alpha i}.\tag{3a}$$

We shall review our theory for the case of the ferromagnet (Sec. II) and antiferromagnet (Sec. III) separately. Special emphasis will be placed on the square lattice antiferromagnet.

 0021-8979/90/095734-06$03.00

II. THE FERROMAGNETS

The ferromagnetic partition function is given by[9]

$$Z_F = \int \mathscr{D}(b,\bar{b};Q,\bar{Q};\lambda)\exp[-\mathscr{S}_F(b,\bar{b};Q,\bar{Q};\lambda)]$$

$$\mathscr{S}_F = \int_0^\beta d\tau\Big(\frac{1}{2}\sum_{i,\alpha}(\bar{b}_{\alpha i}\dot{b}_{\alpha i} - \dot{\bar{b}}_{\alpha i}b_{\alpha i}) + N\sum_{\langle ij\rangle}\bar{Q}_{ij}Q_{ij} + \sum_{\substack{\langle ij\rangle \\ \alpha}}(\bar{Q}_{ij}\bar{b}_{\alpha i}b_{\alpha j} + Q_{ij}b_{\alpha i}\bar{b}_{\alpha j}) + \sum_{i,\alpha}\lambda_i(\bar{b}_{\alpha i}b_{\alpha i} - S)\Big). \quad (4)$$

Making the static assumption

$$Q_{ij}^{\mathrm{MF}}(\tau) = Q, \qquad \lambda_i^{\mathrm{MF}}(\tau) = \lambda, \quad (5)$$

the Schwinger bosons can be integrated out explicitly, resulting in a free energy of

$$\frac{1}{N}F^{\mathrm{MF}} = \tfrac{1}{2}zQ^2 - S\lambda + \frac{1}{\beta}\int\frac{d^dk}{(2\pi)^d}\ln(1 - e^{-\beta\omega_{\mathbf{k}}}), \quad (6)$$

where z is the lattice coordination number, d is the number of spatial dimensions, $\mathscr{N}$ is the total number of sites in the lattice, and the integral is performed over the first Brillouin zone. The dispersion $\omega_{\mathbf{k}}$ is defined by

$$\mu \equiv \lambda - zQ, \qquad \epsilon_{\mathbf{k}} \equiv \frac{1}{z}\sum_{\delta}(1 - e^{i\mathbf{k}\cdot\delta}), \qquad \omega_{\mathbf{k}} \equiv \mu + zQ\epsilon_{\mathbf{k}}. \quad (7)$$

In Fig. 1, we plot $\omega_{\mathbf{k}}$ in one dimension. The saddle point equations $\delta F/\delta Q = 0$ and $\delta F/\delta\lambda = 0$ are

$$S = \int\frac{d^dk}{(2\pi)^d}n_{\mathbf{k}}, \quad (8a)$$

$$Q = S - \int\frac{d^dk}{(2\pi)^d}\epsilon_{\mathbf{k}}n_{\mathbf{k}}, \quad (8b)$$

with $n_{\mathbf{k}} = (e^{\beta\omega_{\mathbf{k}}} - 1)^{-1}$. Thus, we obtain a free energy per spin of

$$\frac{1}{N}F^{\mathrm{MF}} = -\tfrac{1}{2}zS^2 + \tfrac{1}{2}z(Q-S)^2 - S\mu - \frac{1}{\beta}\int\frac{d^dk}{(2\pi)^d}\ln(1 + n_{\mathbf{k}}). \quad (9)$$

Upon addition of the reference energy [see Eq. (1)] $+\frac{1}{2}zS^2$, and taking $N = 2$, the first term gives the classical ferromagnetic ground-state energy per spin, $E_0^{\mathrm{cl}} = -\frac{1}{2}zS^2$. We note that the remaining contribution is precisely *twice* Takahashi's result for $(F - E_0^{\mathrm{cl}})$.[11] This factor of 2 is taken to be an artifact of the static constraint and is a generic consequence of approximations of this sort. The SU (N) theory is defined in terms of N bosons and 1 constraint (per site). Uniformizing the field λ amounts to ignoring the nonzero wavelength components of the constraint field, enforcing the local restriction $\Sigma_\alpha b_\alpha^\dagger b_\alpha = NS$ only on average, cf. Eq. (8a). Thus, at the mean-field level, the number of independent degrees of freedom is overcounted by a factor $g = N/(N-1)$. This is

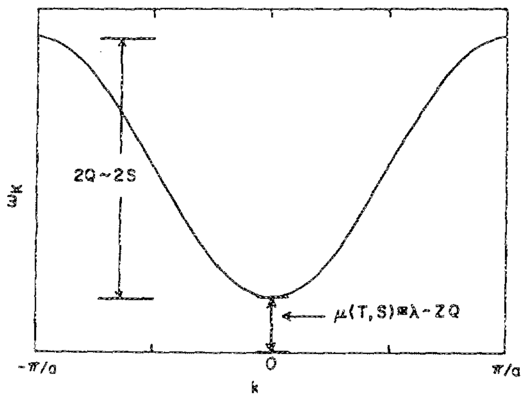

FIG. 1. Schwinger boson excitation spectrum in the mean-field theory for the ferromagnetic chain. See Eq. (7).

partially corrected by the $\mathscr{O}(1/N)$ contribution $F^{(1/N)}$ arising from integration over the Gaussian fluctuations of the constraint field, as was demonstrated in Ref. 9.

The mean-field equations which determine $Q(T,S)$ and $\mu(T,S)$ are identical to those of Takahashi,[12] and we have independently verified his solutions (details may be found in Ref. 11). From Eq. (23), we obtain for the one-dimensional chain

$$(F^{\mathrm{MF}} - E_0^{\mathrm{MF}})_{\mathrm{chain}} = T\Big[-\frac{\zeta(3/2)}{\sqrt{2\pi}}\Big(\frac{T}{2S}\Big)^{1/2} + \frac{1}{2S}\Big(\frac{T}{2S}\Big) + \mathscr{O}(T^{3/2})\Big], \quad (10)$$

where $E_0^{\mathrm{MF}} \equiv F^{\mathrm{MF}}(T=0)$, which is an expansion in the quantity T/S, assumed here to be small. The calculation of $F^{(1/N)}$ was carried out in Appendix B of Ref. 9, where it was found [Eq. (9)]

$$F^{(1/N)} = -\frac{1}{N}\frac{\zeta(3/2)}{2\sqrt{\pi S}}T^{3/2} + \mathscr{O}(T^{5/2}). \quad (11)$$

Combining Eqs. (3.7, 3.8) and setting $N = 2$ yields, to $\mathscr{O}(1/N)$,

$$F - E_0^{\mathrm{MF}} = 2(F^{\mathrm{MF}} - E_0^{\mathrm{MF}} - F^{(1/N)}) = -\sqrt{\frac{2}{\pi}}\zeta\Big(\frac{3}{2}\Big)\Big(\frac{T}{2S}\Big)^{3/2} + \frac{T^2}{2S^2} + \mathscr{O}(T^{5/2}). \quad (12)$$

Comparing our expression with that of Takahashi, we see that the $\mathscr{O}(1/N)$ corrections have brought our $\mathscr{O}(T^{3/2})$ term in line with his, but that our coefficient of the $\mathscr{O}(T^2)$ term remains a factor of 2 too large. The Takahashi result is in remarkable agreement with thermodynamic Bethe änsatz results for $S = \frac{1}{2}$. One unfortunate aspect of Takahashi's variational density matrix is that it is not rotationally invariant, and therefore the longitudinal and transverse susceptibilities in his model will be unequal. Takahashi calculates the static susceptibility

$$\chi = g^2\beta\frac{1}{\mathscr{N}}\sum_{ij}\langle S_i^z S_j^z\rangle \quad (13)$$

by performing a rotational average over $\hat{n}$ in $\langle(\mathbf{S}_i\cdot\hat{\mathbf{n}})(\mathbf{S}_j\cdot\hat{\mathbf{n}})\rangle$ and finds the corresponding result to be in good agreement with known $S=\frac{1}{2}$ results. That this rotational averaging produces the "correct" result in interesting, although we wish to emphasize that Takahashi's underlying theory is not rotationally invariant. Our model preserves rotational invariance, and we find

$$\chi_{\text{chain}} = \tfrac{1}{3}g^2 \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, n_k(1+n_k)$$
$$= g^2S^4T^{-2}\left[1 - \frac{3}{S}\,\frac{\zeta(3/2)}{\sqrt{2\pi}}\left(\frac{T}{2S}\right)^{1/2} + \mathscr{O}(T)\right], \tag{14}$$

which is 3/2 as great as Takahashi's result. For the two-dimensional square lattice, we find

$$(F^{\text{MF}} - E_0^{\text{MF}})_{sq} = -\tfrac{1}{2}T^2\left[\frac{\zeta(2)}{2\pi S} + \frac{\zeta(3)}{8\pi S}\left(\frac{T}{2S}\right) + \mathscr{O}(T^2)\right],$$
$$\chi_{sq} = \frac{g^2}{8\pi S}\exp\left(\frac{4\pi S^2}{T}\right) + \mathscr{O}(Te^{4\pi S^2/T}). \tag{15}$$

The spin-spin correlation function

$$\langle \mathbf{S}_0\cdot\mathbf{S}_R\rangle = \tfrac{3}{2}|f(\mathbf{R})|^2,$$
$$f(R) \equiv \int \frac{d^dk}{(2\pi)^d}\, e^{i\mathbf{k}\cdot\mathbf{R}} n_{\mathbf{k}}. \tag{16}$$

At long distances, one is concerned with the small k behavior of the occupation function $n_{\mathbf{k}}$, and we obtain the following asymptotic expressions:

$$\langle \mathbf{S}_0\cdot\mathbf{S}_{\mathbf{R}}\rangle \simeq \tfrac{3}{2}S^2 e^{-R/\zeta},$$
$$\zeta \simeq S^2/T \quad (d=1),$$
$$\langle \mathbf{S}_0\cdot\mathbf{S}_{\mathbf{R}}\rangle \simeq \frac{3T^2}{8\pi S^2}\,\frac{e^{-R/\zeta}}{(R/\zeta)},$$
$$\zeta \simeq \sqrt{S/T}\,\exp(2\pi S^2/T) \quad (d=2,\text{square}). \tag{17}$$

As discussed in Ref. 11, the correlation function in Eq. (17) differs only in its prefactor from the Ornstein–Zernike correlation function expected for the two-dimensional *classical* Heisenberg model.[13]

III. THE ANTIFERROMAGNETS

The bosonic partition function for the spin-S Heisenberg antiferromagnet is given by

$$Z_A = \int \mathscr{D}[b,\bar{b};Q,\bar{Q};\lambda]\exp(-\mathscr{S}_A[b,\bar{b};Q,\bar{Q};\lambda])$$
$$\mathscr{S}_A = \int_0^\beta d\tau\left(\frac{1}{2}\sum_{i,\alpha}(\bar{b}_{\alpha i}\dot{b}_{\alpha i} - \dot{\bar{b}}_{\alpha i}b_{\alpha i}) + N\sum_{\langle ij\rangle}\bar{Q}_{ij}Q_{ij} + \sum_{\substack{\langle ij\rangle\\ \alpha}}(\bar{Q}_{ij}b_{\alpha i}b_{\alpha j} + Q_{ij}\bar{b}_{\alpha i}\bar{b}_{\alpha j}) + \sum_{i,\alpha}\lambda_i(\bar{b}_{\alpha i}b_{\alpha i} - S)\right). \tag{18}$$

The mean-field theory amounts to a steepest descents approximation, where Q and λ acquire static uniform values, that are determined by extremizing the free energy. The mean-field (MF) Hamiltonian is given by

$$H^{\text{MF}} = \tfrac{1}{2}\mathscr{N}NzQ^2 - \mathscr{N}NS\lambda + \frac{1}{2}\sum_{\mathbf{k},\alpha}[\lambda(b^\dagger_{\mathbf{k}\alpha}b_{\mathbf{k}\alpha} + b^\dagger_{-\mathbf{k}\alpha}b_{-\mathbf{k}\alpha}) + zQ(\bar{\gamma}_{\mathbf{k}}b_{\mathbf{k}\alpha}b_{-\mathbf{k}\alpha} + \gamma_{\mathbf{k}}b^\dagger_{\mathbf{k}\alpha}b^\dagger_{-\mathbf{k}\alpha})], \tag{19}$$

with

$$\gamma_{\mathbf{k}} \equiv \frac{1}{z}\sum_{\delta} t_\delta e^{-i\mathbf{k}\cdot\delta}. \tag{20}$$

It can easily be verified that the Hamiltonian does not break rotational symmetry. H^{MF} is readily diagonalized by the quasiparticle operators: $\alpha_{\mathbf{k}\alpha} = \cosh\theta_{\mathbf{k}}b_{\mathbf{k}\alpha} + \sinh\theta_{\mathbf{k}}b^\dagger_{-\mathbf{k}\alpha}$. Here, $\tanh(2\theta_{\mathbf{k}}) = -zQ\gamma_{\mathbf{k}}/\lambda$. Thus, the mean-field free energy is given by

$$\frac{1}{N}F^{\text{MF}} = \tfrac{1}{2}zQ^2 - \tfrac{1}{2}(2S+1)\lambda + \frac{1}{\beta}\int\frac{d^dk}{(2\pi)^d}\ln(2\sinh\tfrac{1}{2}\beta\omega_{\mathbf{k}}),$$
$$\omega_{\mathbf{k}} = \sqrt{\lambda^2 - z^2Q^2|\lambda_{\mathbf{k}}|^2}, \tag{21}$$

and the steepest descents equations are

$$\frac{1}{N}\frac{dF^{\text{MF}}}{d\lambda} = \int\frac{d^dk}{(2\pi)^d}\cosh(2\theta_{\mathbf{k}})(n_{\mathbf{k}}+\tfrac{1}{2}) - (S+\tfrac{1}{2}) = 0, \tag{22}$$

$$\frac{1}{N}\frac{dF^{\text{MF}}}{dQ} = -z\int\frac{d^dk}{(2\pi)^d}\gamma_{\mathbf{k}}\sinh(2\theta_{\mathbf{k}})(n_{\mathbf{k}}+\tfrac{1}{2}) + \frac{zQ}{J} = 0. \tag{23}$$

Here, $n_{\mathbf{k}}$ is the Bose occupation $[\exp(\omega_{\mathbf{k}}/T)-1]^{-1}$. We have shown[9] that the stable mean-field solution has [see Eq. (20)] $t_\delta = 1$. The structure factor for the SU(2) model (two Schwinger boson flavors) is defined as $S \equiv \langle S^z(\mathbf{q},\omega)S^z((-\mathbf{q},-\omega)\rangle$, and in the mean-field level is given by

$$S^{\text{MF}}(\mathbf{q},iq_n) = \frac{1}{2}\int_0^\beta d\tau\, e^{iq_n\tau}\sum_{\mathbf{k}\mathbf{k}'}\langle T[b^\dagger_{1,\mathbf{k}}(\tau)b_{1,\mathbf{k}+\tilde{\mathbf{q}}}(\tau) \times b^\dagger_{1,\mathbf{k}'+\tilde{\mathbf{q}}}(0)b_{1,\mathbf{k}'}(0)]\rangle, \tag{24}$$

where q_n is a Matsubara frequency. Here it is convenient to measure the reduced momentum with respect to the antiferromagnetic vector π, i.e., $\tilde{\mathbf{q}} \equiv \mathbf{q} - \pi$. In (24) we have exploited the decoupling of the different Schwinger bosons at the mean-field level.

It is convenient to parametrize the dispersion $\omega_{\mathbf{k}}$ in terms of the spin-wave velocity $c = \sqrt{8}Q$, and a parameter κ such that $c\kappa = 2\sqrt{\lambda^2 - (4Q)^2}$. The dispersion is then given by $\omega_{\mathbf{k}} = c\sqrt{(\kappa/2)^2 + 2(1-\gamma^2_{\mathbf{k}})}$. We note that $\kappa/2$ serves as a cutoff in the momenta integrations in Eqs. (22) and (23). Our spin waves are therefore "massive" when κ is finite. In Fig. 2, we plot $\omega_{\mathbf{k}}$ in one dimension. It is possible to write down the (unrenormalized) projected variational ground state corresponding to the mean-field approximation

$$|\Psi_0\rangle = \mathscr{P}_S\exp\left(\sum_{m,n,\alpha}u(\mathbf{n})b^\dagger_{m,\alpha}b^\dagger_{m+1,\alpha}\right)|0\rangle, \tag{25}$$

where

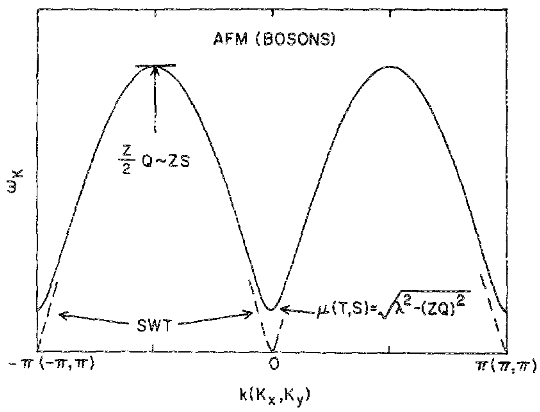

FIG. 2. Schwinger boson excitation spectrum in the mean-field theory for the antiferromagnetic chain. See Eq. (7). Gapless spin waves of the naïve spin-wave theory (SWT) are marked with dashed lines.

$$u(\mathbf{n}) = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tanh\theta_k e^{i\mathbf{k}\cdot\mathbf{n}},$$

$$\tanh\theta_k = \left(\sqrt{1-\eta^2\gamma_{\mathbf{k}}^2}-1\right)/\eta\gamma_{\mathbf{k}}. \qquad (26)$$

Here, $\eta^2 = (1+\kappa^2/4)^{-1}$. $\mathscr{P}_S$ is the "Gutzwiller" projector, which eliminates all components with the wrong number (not equal to NS of Schwinger bosons at any site. It is easy to see that $u(\mathbf{n})$ vanishes unless $e^{i\pi\cdot\mathbf{n}} = -1$, i.e., $\mathbf{n}$ connects one sublattice to the other. Now if we consider the $N=2$ model and reverse our sublattice rotation, we find

$$|\Psi_0\rangle = \mathscr{P}_S \exp\left(2\sum_{\substack{\mathbf{m}\in\mathscr{A}\\ \mathbf{m}+\mathbf{n}\in\mathscr{B}}} u(\mathbf{n})(a^\dagger_{\mathbf{m}} b^\dagger_{\mathbf{m}+\mathbf{n}} - b^\dagger_{\mathbf{m}} a^\dagger_{\mathbf{m}+\mathbf{n}})\right)|0\rangle, \qquad (27)$$

where the sum on $\mathbf{m}$ is over the $\mathscr{A}$ sublattice only. Thus, $|\Psi_0\rangle$ is a sum of states each of which is constructed by successive application of the (rotationally invariant) composite operator $\mathscr{C}^\dagger_{\mathbf{m},\mathbf{m}+\mathbf{n}} = (a^\dagger_{\mathbf{m}} b^\dagger_{m+n} - b^\dagger_{\mathbf{m}} a^\dagger_{\mathbf{m}+\mathbf{n}})$ between sites on alternate sublattices, the amplitude for each configuration in the sum given by the product of the associated $u(\mathbf{n})$'s. The presence of $\mathscr{P}_S$ means that only configurations where $2S$ such "bonds" emanate from *each* site are present. If we consider the $S=1$ chain and take $\eta\to 0$, ignoring for the moment the fact that η is a function of S through Eqs. (22) and (23) (one could imagine adding terms to H which would lead to an S-independent variation of η), then from Eq. (26) we obtain

$$u(n) = -\tfrac{1}{4}\eta(1+\tfrac{3}{16}\eta^2)\delta_{n,\pm 1} - \tfrac{1}{64}\eta^3\delta_{n,\pm 3} + \dots, \quad (28)$$

which says, to lowest order in η, that the projected mean-field state contains only those configurations in which the composite operators $\mathscr{C}^\dagger_{m,m+n}$ connect nearest-neighbor sites. Thus, for an infinite chain of integer spin, the state (27) becomes the ground state of the recently studied AKLT model![14,15] This observation adds support to the belief that the AKLT model has similar properties to the Heisenberg model which does not have the AKLT biquadratic term.

The mean-field equations (22) and (23) lead to the following results for c and κ:

One dimension: At $T=0$, we expand the solution to leading powers of $1/S$. Restoring the unit of energy which is given by the Heisenberg exchange coupling J, we obtain:

$$c = [S+1-2/\pi]\sqrt{2}J/\hbar[1+\mathscr{O}(1/S)],$$

$$\kappa = 16\exp[-\pi(S+\tfrac{1}{2})[1+\mathscr{O}(1/S)]. \qquad (29)$$

The antiferromagnetic correlations were shown to decay as $\exp(-\kappa R)$. Since true magnon excitations carry spin 1, they are bound states of pairs of Schwinger boson excitations. Thus we expect their gap to be $\Delta = c\kappa$ at the zone edge. This assignment is confirmed in the position of the peaks of the structure factor.[10] It is also a manifestation of the underlying Lorentz symmetry of the quantum antiferromagnet. The gap has the asymptotic form $\Delta\sim S\exp(-\pi S)$, which should be compared with the result $\Delta\sim S^2\exp(-\pi S)$, obtained from the two-loop order calculation of the $(1+1)$-dimensional sigma model.[5] It is remarkable that our simple mean-field theory reproduces the asymptotic S dependence of the Haldane gap. All is not well, however, because our mean-field theory is unable to discern the topological terms responsible for the gaplessness of all half-odd-integer antiferromagnetic chains. Alternatively stated, the Lieb–Schultz–Mattis theorem,[7] which exploits the differing properties of integer and half-odd-integer spins under SU(2) rotations is violated at the mean-field level, since it requires that all half-odd-integer Heisenberg antiferromagnetic chains must have either degenerate ground states or gapless excitations in the thermodynamic limit. We stress that the bosonic mean field theory *is* applicable to any model in which the ground state is ordered. In particular, this applies to the $S=\frac{1}{2}$ model in two dimensions, as argued in the following discussion.

Two dimensions: For $T<JS(S+1)$ the solutions of Eqs. (22) and (23)

$$c = Z_c\sqrt{8}JS/\hbar \qquad (30)$$

and

$$\kappa = \exp[-Z_\kappa 2\pi S(S+1)J/T]. \qquad (31)$$

The solutions to the renormalization factors $Z_c(T,S)$, and $Z_\kappa(T,S)$ are obtained numerically for small values of S in Table I. It was also previously shown[9] that Eq. (22) ensures that Z_κ has a finite $T=0$ limit for all $S>S_c\sim 0.2$, and is only weakly T dependence for $T<JS(S+1)$. For large S, $\lim_{S\to\infty}Z_\kappa = 1$, and Eq. (29) agrees, to one loop order, with the renormalization group calculation of the classical Heisenberg model.[11,13] Since $Z_\kappa(S=1/2)=0.246$, it is apparent that quantum fluctuations drastically reduce the correlation length at finite temperatures from its classical value. On the other hand, κ^{-1} still diverges at $T=0$, which implies that this system has a Néel-ordered ground state, in agreement with numerical results for finite-size systems.[16,17] Hirsch and Tang[18] have recently shown that by taking $T\to 0$ limit on a *finite* system, for $S>S_c$ Bose condensation occurs, since the $\mathbf{k}=(0,0)$, (π,π) modes get macroscopically occupied in order to satisfy Eq. (22). $S(\pi,\pi)$ was shown to agree with Anderson's spin-wave calculation[3] of the ground-state staggered magnetization squared. For $T>JS(S+1)$, a breakdown of the mean-field theory occurs, and no solution

TABLE I. Results of the Schwinger boson mean field theory (SBMFT) compared to spin-wave theory (Ref. 19) (SWT), and to the sigma model calculation (Ref. 26) (CHN). Z_c, Z_χ, and Z_κ are the $T\to 0$ limit of the renormalization constants of the spin-wave velocity, susceptibility, and correlation length exponent, respectively. [See (30), (32), and (31).]

Theory	Coefficient	$S=1/2$	$S=1$
SBMFT	Z_c	1.159	1.079
SWT	$Z_c = 1 + 0.158/2S$	1.158	1.079
SBMFT	Z_χ	0.53 ± 0.01	0.73 ± 0.01
SWT	$Z_\chi = 1 - 0.552/2S$	0.448	0.724
SBMFT	$JS(S+1)(dZ_\chi/d_T$	$0.22 + 0.01$	0.27 ± 0.01
SBMFT	Z_κ	0.232	0.442
RGSWT	$Z_\kappa = \hbar c Z_c Z_\chi / a\sqrt{8}(S+1)$	0.200	0.421
SBMFT	$\delta = C_v[T/S(S+1)J]^{-2}$	1.3 ± 0.05	1.2 ± 0.05

for κ and c is found. This upper temperature does not correspond to a true phase transition [except perhaps for the large N SU(N) model], but to the breakdown of coherence between neighboring antiferromagnetically aligned spins.

Since, in effect, spin-wave theory is a continuation of the SBMFT to the broken symmetry phase, it is interesting to compare our $N=2$ results to those obtained by the Hartree–Fock approximation of SWT.[19] (1) The values of the spin-wave velocity renormalization Z_c, Eq. (30), agree well for $S=\frac{1}{2}$ and $S=1$, as shown in Table I. Since κ vanishes at $T=0$, our quasiparticle dispersion matches the spin-wave result. (2) The expression for $F^{\rm MF} - E_c$ (where E_c is the classical energy) is twice that of SWT. This is the same situation we have previously encountered in our results for the ferromagnetic chain in Sec. II. Here, however, we have not yet attempted to compute the Gaussian corrections. (3) The spin correlation function $S^{\rm MF}$ (3.7) is exactly $\frac{3}{2}$ times the *rotationally averaged* expression of SWT. It can be easily verified that the susceptibility sum rule yields $\Sigma_{\mathbf{q},q_n} S^{\rm MF} = S(S+1)/2$, which is also $\frac{3}{2}$ too large.

Therefore, in order to obey the condition that our theory for $N=2$ should match SWT at $T=0$, for large S, and also obey the sum rule, we correct our free energy and correlation functions by $F=\frac{1}{2}F^{\rm MF}$, and $S(\mathbf{q},\omega)\equiv\frac{2}{3}S^{\rm MF}$. We suggest without proof that this normalization partly compensates for the missing fluctuation effects (which enforce the Gutzwiller projection).[20]

In Fig. 3, we plot the dynamical structure factor[10] in the positive $(\omega,\tilde{q})$ quadrant, where $\tilde{q}\equiv|\tilde{\mathbf{q}}|$. For $(\omega, \tilde{q}) < (T, T/c)$ there is a reflection symmetry on both energy and momentum axis. Two distinct regimes are observed: (a) $(\omega,\tilde{q}) \leqslant (c\kappa, \kappa)$, and (b) $(\omega, \tilde{q}) \gg (c\kappa, \kappa)$. Region (a) is a quasielastic peak, which increases, and narrows with decreasing κ. This peak turns into the magnetic Bragg peak at $T=0$, and its width reflects the overdamped nature of the spin waves with wavelength longer than the coherence length. In region (b), the structure factor becomes asymptotically proportional to the naïve spin-wave theory result, which predicts spin-wave peaks at energy $c|\tilde{\mathbf{q}}|$. We note that there is a gap between the normal $(\omega < c\tilde{q})$ and the anomalous $(\omega > c\sqrt{\kappa^2 + q^2})$ contributions of the Schwinger boson scattering. We suspect, however, that this structure is an artifact of the static mean-field approximation, and that it might be washed out by fluctuations in λ and Q. Other groups[21,22] have arrived at a similar general structure, although with the following notable differences: (i) The aforementioned gap does not appear. (ii) In Ref. 21 the characteristic energy scale differs from ours by a prefactor of $T^{1/2}$. (iii) The spin-wave linewidth in both Refs. 21 and 22 is substantially larger than the SBMFT which is of order $c\kappa$. Reference 22, in fact, find that it is of order $T^2 \gg c\kappa$. Such damping

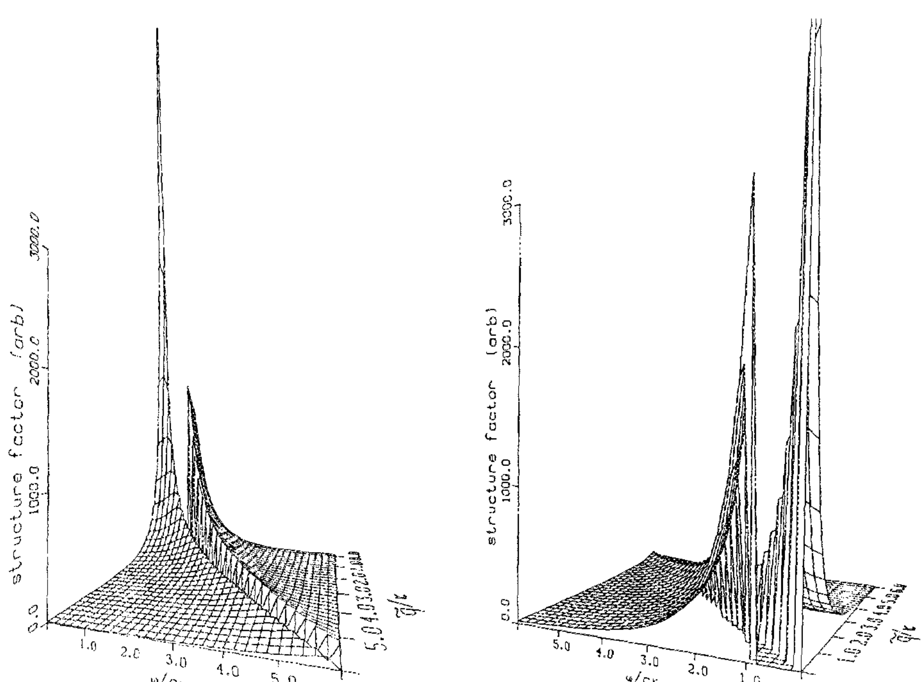

FIG. 3. Two views of the structure factor of the square lattice antiferromagnet at $T \ll JS(S+1)$. $\tilde{q}$ is the distance from the antiferromagnetic wave vector (π, π). κ is the inverse correlation length given in Eq. (31), and c is the spin-wave velocity Eq. (30). For low frequencies $\omega \ll T$, the structure factor is symmetric under reflection on both $\tilde{q}$ and ω axis.

effects are expected to be manifested in the higher-order (Gaussian) corrections to the SBMFT.

It is easy to compute the uniform susceptibility,[10] which is given by

$$\chi = \frac{1}{T} S(\mathbf{q}=0) = \frac{1}{3T} \int \frac{d^d k}{(2\pi)^d} n_{\mathbf{k}} (n_{\mathbf{k}}+1) \equiv \frac{1}{8J} Z_\chi . \tag{32}$$

In Fig. 4 we plot our $\chi(T)$ (for the applicable range of T) and show how it interpolates between the rotationally averaged SWT result[19] and the high-temperature series (HTS) expansion of Rushbrooke and Wood.[23] It is also important to note slight discrepancies for the value of Z_χ between our result and that of Oguchi as seen in Table I. Singh,[24] using a high-order series expansion from the Ising limit, has obtained values consistent with the SBMFT for both Z_c and Z_κ. Monte Carlo calculations[25] of χ on finite lattices have found our theory (Fig. 4) to be in good agreement with the numerical results also for T¿0. We also present our result for the specific heat T^2 coefficient δ in Table I.

Chakravarty, Halperin, and Nelson[26] (CHN) have evaluated the temperature-dependent correlation length κ^{-1} of the 2D Heisenberg model by studying the nonlinear sigma model in a slab of finite thickness. They applied Oguchi's spin-wave theory[19] (SWT) to determine the appropriate value of the sigma model coupling g for the nearest-neighbor model of $S=\frac{1}{2}$. They found that $g<g_c$, and that the correlation length has renormalized classic behavior which agrees to the one-loop order with our result, Eq. (31). In Table I we compare CHN's renormalization constant to ours. The discrepancy in its numerical value could be traced back to their use of the Z_χ from spin-wave theory. CHN have demonstrated the consistency of their $\kappa(T)$ with the experimentally determined correlations.[27] The regime of validity of this field theory, without the Berry phases measuring the hedgehog configurations, is identical to that of the SBMFT. These terms were shown to lead to destructive interference between different topological sectors, and thus to quasidegeneracy of the disordered ground states.[8] Read and Sachdev[28] have calculated the *dynamical* effects of the Q_{ij} *phase* fluctuations, and the λ_i constraint field, which were held static in the SBMFT approximation. They mapped the Heisenberg action [Eq. (18)] to that of compact $(d+1)$-dimensional quantum electrodynamics with additional topological Berry phase terms and couplings to the staggered *amplitude* bond variables. Appealing to established results of compact QED, they concluded that the *rotationally* disordered ground states of $S=1 \bmod 4$ have a spin-Peierls (dimerization) order parameter which breaks the *lattice* fourfold degeneracy.

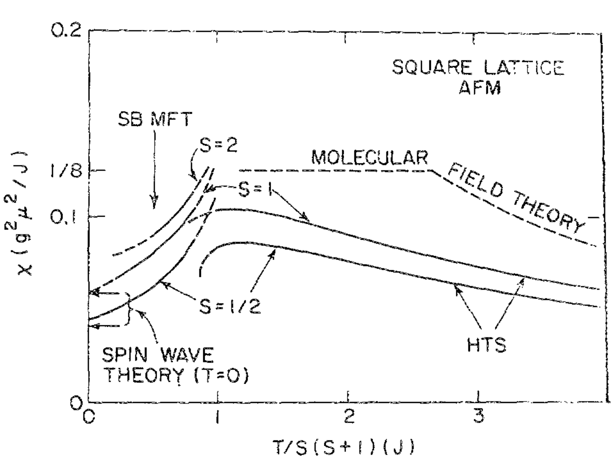

FIG. 4. Uniform magnetic susceptibility for different spin value S. High-temperature series (HTS) and the molecular-field theory are given in Ref. 23. The spin-wave theory (Ref. 19) susceptibility is rotationally averaged. The Schwinger boson mean field theory (SBMFT) interpolates between the two regimes.

ACKNOWLEDGMENTS

This work was supported in part by National Science Foundation Grant No. DMR-8914045. A. A. is an Alfred P. Sloan fellow.

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