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A path decomposition expansion proof for the method of images

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Abstract. The path decomposition expansion for the path integral is used to derive the method of images.

1. Introduction

Hard walls present a problem in path integration [1] since the path integral is then difficult to evaluate in the discretized time step form. A manifestation of this problem occurs when the contributions of classical paths are summed in the semiclassical approximation. A path that bounces off a classical turning point in a continuous potential appears with a phase factor of $\pi/2$, while if it bounces off a hard wall the correct prescription is to introduce a phase factor of π by hand.

There is a technique, the path decomposition expansion (PDX) [2], that enables us to surmount this difficulty by proving the method of images (MI). In this method, which is often used in electrostatics, the potential is reflected on the other side of the surface, Σ, of an infinite hard wall, and image charges are introduced to mimic the reflections from Σ.

2. Method of images from PDX

We are given a potential in N-dimensional configuration space, $V(\boldsymbol{x})$, the mass m, and a *locally flat* surface Σ that closes at infinity, thus dividing the configuration space into an inside and outside, where

$$V(\boldsymbol{x}) = \begin{cases} \infty & \text{if } \boldsymbol{x} \text{ is inside } \Sigma \\ < \infty & \text{if } \boldsymbol{x} \text{ is outside } \Sigma. \end{cases} \tag{1}$$

Clearly this implies Dirichlet boundary conditions for the Green's function, G^r, of this problem. Thus G^r is a restricted Green's function, which in the path integral representation is given by

$$G^r(\boldsymbol{x}_1, \boldsymbol{x}_2, T) = \int_{\boldsymbol{x}(0)=\boldsymbol{x}_1}^{\boldsymbol{x}(T)=\boldsymbol{x}_2} \mathcal{D}^r[\boldsymbol{x}] \mathrm{e}^{\mathrm{i}S[\boldsymbol{x}]/\hbar} \tag{2}$$

$$S = \int_0^T \mathrm{d}t [\tfrac{1}{2} m \dot{x}^2 - V(\boldsymbol{x})] \tag{3}$$

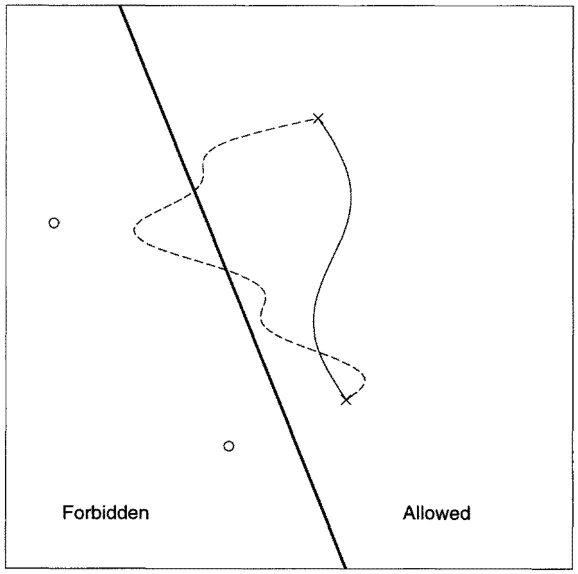

Figure 1. Paths appearing in G^u and G^r. The heavy line is Σ. The full curve is a typical path in both G^u and G^r. The broken curve is a path appearing in G^u–G^r (equation (4)). The 'x'-symbols are the path endpoints and the circles are their reflected images.

where $\mathcal{D}^r$ is the restricted integration over all paths that go from x_1 to x_2 in time T, never crossing the surface Σ. (Paths that only touch the surface have zero measure.) Let us replace $V(x)$ by

$$V^u(\boldsymbol{x}) = \begin{cases} V(\boldsymbol{x}) & \text{if } x \text{ is outside } \Sigma \\ V(\boldsymbol{R}(\boldsymbol{x})) & \text{if } x \text{ is inside } \Sigma \end{cases} \tag{4}$$

where $\boldsymbol{R}(x)$ is the mirror image of x reflected through Σ and $V^u(x)$ defines the Green's function G^u. All paths that appear in G^r are contained in G^u with the same weight (action). We call the path integral over paths that belong to G^u and not to G^r, G^{cross},

$$G^{\text{cross}}(\boldsymbol{x}, \boldsymbol{x}', T) = G^u(\boldsymbol{x}, \boldsymbol{x}', T) - G^r(\boldsymbol{x}, \boldsymbol{x}', T). \tag{5}$$

By its definition, all paths in G^{cross} must have crossed the surface Σ at some time $0 < t < T$. The PDX formula factors G^{cross} into the sum of all paths from x_1 to x_σ at time t, and a sum of all restricted paths from x_σ at time $T - t$, and then integrates over t and x_σ, with the appropriate Jacobian [2]. (x_σ is the position of the last crossing of Σ by the path.) Thus we find that

$$G^{\text{cross}}(\boldsymbol{x}_1, \boldsymbol{x}_2, T) = \int_\Sigma \mathrm{d}\sigma \int_0^T \mathrm{d}t\, G^u(\boldsymbol{x}_1, \boldsymbol{x}_\sigma, t) \left(\frac{-\mathrm{i}\hbar}{2m}\right) (\boldsymbol{n}_\sigma \cdot \nabla_{\boldsymbol{x}}) G^r(\boldsymbol{x}, x_2, T - t)|_{\boldsymbol{x}=\boldsymbol{x}_\sigma} \tag{6}$$

where $\boldsymbol{n}_\sigma$ is normal to the surface at $\boldsymbol{x}_\sigma$. Equation (6) is graphically illustrated in figure 1.

By equation (4), and since Σ is flat, G^u possesses the reflection symmetry,

$$G^u(\boldsymbol{x}_1, \boldsymbol{x}_\sigma, T) = G^u(\boldsymbol{R}(\boldsymbol{x}_1), \boldsymbol{x}_\sigma, T). \tag{7}$$

Therefore we can apply the PDX formula again (in reverse) and equate the right-hand side of equation (6) with G^u, where the surface now lies between the two endpoints, $\boldsymbol{R}(\boldsymbol{x}_1)$ and

$\boldsymbol{x}_2$. Using equation (6) we directly obtain the MI expression for G^r

$$G^r(\boldsymbol{x}_1, \boldsymbol{x}_2, T) = G^u(\boldsymbol{x}_1, \boldsymbol{x}_2, T) - G^u(\boldsymbol{R}(\boldsymbol{x}_1), \boldsymbol{x}_2, T). \quad (8)$$

Equation (8) is useful for evaluating restricted Green's functions, which often appear in problems involving the PDX (e.g. multidimensional tunneling [2]).

3. Concluding remarks

The sorting of paths for the PDX is similar to that in [3], and as we have shown, the general formalism of PDX can be brought to bear on the one-dimensional hard wall problem discussed in [3].

Note added in proof. This paper was written in 1984 (authors' addresses at the time: AA, SUNY Stony Brook, LSS, Technion), but not published. Requests for this work (which had been cited elsewhere) in the intervening years have induced the authors to publish. A good discussion of PDX can be found in [4].

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