

AN ANALYSIS OF TRANSVERSE FLUCTUATIONS IN MULTIDIMENSIONAL TUNNELING

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We examine the asymptotic behaviour of the ground state tunnel-splitting of the multidimensional double well, with non-quadratic minima, where instanton techniques are inapplicable. We apply the recently developed path decomposition expansion for two model problems; the important effects of the transverse degrees of freedom are explored. In particular we discuss tunneling in the presences of a vacuum valley, which describes features exhibited in SU(2) gauge theories in a finite volume.

1. Introduction

The ground state tunnel-splitting of the multidimensional symmetric double well has been a subject of extensive study in recent years [1–3], since it serves as a model for a large range of non-perturbative physical phenomena. If g is the dimensionless quantum parameter of order \hbar , the leading order contribution in g , called the semiclassical approximation, was evaluated using the instanton formalism [1], the multidimensional WKB [2], or the functional methods of ref. [3]. In all these cases the potentials were limited to ones with quadratic minima.

This paper will discuss a unified approach to derive the asymptotic expression for multidimensional tunneling problems and bounds on the error. We will concentrate on situations where instanton techniques breakdown and discuss two particular types of multidimensional problems, for each of which we solve one example explicitly. These problems are inspired by SU(N) gauge fields on a torus [4]. Solving the tunneling problems for this case allows one to calculate the energy of electric flux in weak coupling.

The first problem concerns a double-cone potential in 3 dimensions:

$$H = -\frac{1}{2}g^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \left[(|x| - 1)^2 + y^2 + z^2 \right]^{1/2}. \quad (1.1)$$

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Standard instanton techniques break down in this example for two reasons. First the instanton path, still well defined, only spends a short time near the classical minima $x = (\pm 1, 0, 0)$, secondly the transverse fluctuations diverge while approaching these minima. The expression for the ground state energy splitting will turn out to be:

$$\Delta E = Ag^{5/3}\exp(-Sg^{-1} + T\epsilon g^{-1/3})(1 + O(g^{1/3})), \quad (1.2)$$

where A , S , T and ϵ are constants to be specified later. This result exhibits two rather interesting deviations from the canonical result for potentials with quadratic minima. First the correction ($\exp(T\epsilon g^{-1/3})$) to the leading result ($\exp(-S/g)$) is exponential, which is related to the fact that the instanton path only spends a short time near the classical minima. This feature is also present for analogous one-dimensional problems [5]. We will discuss one such one-dimensional problem, namely the W -potential:

$$H = -\frac{g^2}{2} \frac{\partial^2}{\partial x^2} + ||x| - 1|. \quad (1.3)$$

(Note the equality of this potential with that of eq. (1.1) along the x -axis.) It gives rise to an energy splitting:

$$\Delta E = \tilde{A}g^{2/3}\exp(-Sg^{-1} + T\tilde{\epsilon}g^{-1/3}). \quad (1.4)$$

This brings us to the second deviation of eq. (1.2) from the canonical result, that is the powers of g in ΔE . Although it would be too naive to expect the canonical result $g^{1/2}$, one would expect the same power as in eq. (1.4). That is, $g^{2/3}$ instead of $g^{5/3}$. We will show how to understand this deviation in terms of transverse fluctuations which, as remarked before, diverge when approaching the minima.

The second problem we will analyse in detail, is that of a vacuum valley *not* related to a symmetry of the hamiltonian. This means that the shape of the potential transverse to the vacuum valley depends on the parameters describing this valley. Generically, but not generally, the potential is quadratic in the transverse direction. Quantizing the transverse degrees of freedom leads to an effective potential V_0 along the vacuum valley. V_0 is just the zero-point energy for the transverse fluctuations; by assumption depending on the parameters of the vacuum valley. If V_0 is again of a double-well shape we would have reduced the problem to a lower dimensional one, but this is only true if the effective potential V_0 leads to an accurate approximation for the full problem. To verify this, one has to invoke the adiabatic approximation. A situation where this approximation will break down is when the potential is quadratic in the transverse direction except for isolated points, where it is quartic. The important observation is that for the aforementioned problems of gauge fields on a torus, the presence of a vacuum valley is related to the topology of

the torus and cannot be avoided. On the other hand, the nature of gauge groups guarantees the presence of quartic points, once a vacuum valley is present.

The simplest toy model which exhibits a vacuum valley with quartic points is:

$$H = -\frac{1}{2}g^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2g^2} (x^2 - 1)^2 y^2; \quad (1.5)$$

it is identically zero along the x -axis and the zero-point energy for the quadratic fluctuations in the y -direction is given by:

$$V_0(x) = \frac{1}{2}|x^2 - 1|. \quad (1.6)$$

As mentioned before, the breakdown of the adiabatic approximation (close to $x = \pm 1$) prevents one from calculating the ground state energy splitting using the effective one-dimensional hamiltonian:

$$H = -\frac{g^2}{2} \frac{\partial^2}{\partial x^2} + V_0(x). \quad (1.7)$$

Eq. (1.7) would lead to:

$$\Delta E = A'g^{2/3}\exp(-S'g^{-1} + T'\epsilon'g^{-1/3}). \quad (1.8)$$

Also, a direct application of the multidimensional WKB method is not possible since the classically allowed region is connected for all positive energies. Nevertheless we will show how one still can do a semiclassical computation yielding a result:

$$\Delta E = \tilde{A}'g^{2/3}\exp(-S'g^{-1} + T'\tilde{\epsilon}'g^{-1/3}), \quad (1.9)$$

where only \tilde{A}' and $\tilde{\epsilon}'$ differ from A' and ϵ' in eq. (1.8) determined by the one-dimensional problem of eq. (1.7). \tilde{A}' and $\tilde{\epsilon}'$ are determined by the *perturbative* part of the full two-dimensional problem.

There are other places in field theory where vacuum valleys (flat potentials) arose, namely in supersymmetric models. The analogue of V_0 vanishes in all orders in perturbation theory due to (unbroken) supersymmetry, which is a blessing for the hierarchy and cosmological constant problem [6], but a curse for the evaluation of the Witten index in supersymmetric QCD on a torus [7]*. Our techniques are not applicable to these problems.

* A problem still not resolved satisfactorily, is associated with the fact that in weak-coupling (for zero magnetic field) one cannot separate the spatially constant diagonal modes from the off-diagonal ones when expanding around $A = 0$.

We will use the recently developed path decomposition expansion (PDX) [8] for the two model problems, eqs. (1.1) and (1.5). The PDX gives in theory an expression for the energy split, accurate up to exponential corrections and possibly provides a framework in which our results can be made rigorous in the sense of ref. [3]. In practical applications the PDX provides a natural and straightforward expression for the connection formula, but is further equivalent to a suitable generalization of the WKB methods of ref. [2].

The rest of this paper is organized as follows: sect. 2 describes the PDX method and results, and the role of the decomposition surfaces in controlling the higher order corrections. Sect. 3 applies this method, as a simple pedagogical exercise, to the one-dimensional quartic and “ W ” potential ($V(x) = ||x| - 1|$). Sect. 4 derives and discusses the result for the 3-dimensional double-cone potential (1.1) and sect. 5 gives a careful analysis for the 2-dimensional vacuum valley (1.5). We conclude with a brief summary of the results. An appendix contains some details on the error estimates. This paper replaces an earlier version entitled: “Transverse fluctuations in multidimensional tunneling: non-adiabatic effects and induced potentials”.

2. The path decomposition expansion [8]

The path decomposition expansion (PDX) is a Green function identity, which relates the solutions of the Schrödinger equation on the full configuration space, to the restricted Green functions, which are defined on parts of configuration space. These restricted Green functions satisfy Dirichlet boundary conditions on the so-called decomposition surfaces which break up configuration space. This enables one to use separate, and different, approximation schemes for each region and then use the PDX as essentially a multidimensional connection formula. In ref. [8] the semiclassical evaluation of the tunnel-splitting was carried out by a multiple steepest-descent approximation for the decomposition surface integrals (see eq. (2.2)) of the wave functions and the restricted Green function defined in the classically forbidden region. The latter was determined by the semiclassical approximation (cmp. refs. [2] and [9]). Here we shall briefly review only the necessary steps leading to the determination of the ground state energy splitting $\Delta(g)$ for the hamiltonian H with a symmetric non-negative potential V :

$$H = -\frac{g^2}{2} \frac{\partial^2}{\partial \mathbf{x}^2} + V(\mathbf{x}),$$

$$V(\pm \mathbf{e}_1) = 0, \quad \mathbf{e}_1 = (1, 0, 0, \dots). \quad (2.1)$$

V has the reflection symmetries $V(P_i \mathbf{x}) = V(\mathbf{x})$, $i = 1, 2, \dots, N$, where $P_i \mathbf{x} = (x_1, \dots, -x_i, \dots, x_N)$ and we assume that $V(\mathbf{x})$, for each fixed x_1 , is minimal at $x_2 = x_3 = \dots = x_N = 0$. This implies that the tunneling path (instanton) lies on the x_1 axis. A straight tunneling path greatly facilitates the calculation of the transverse

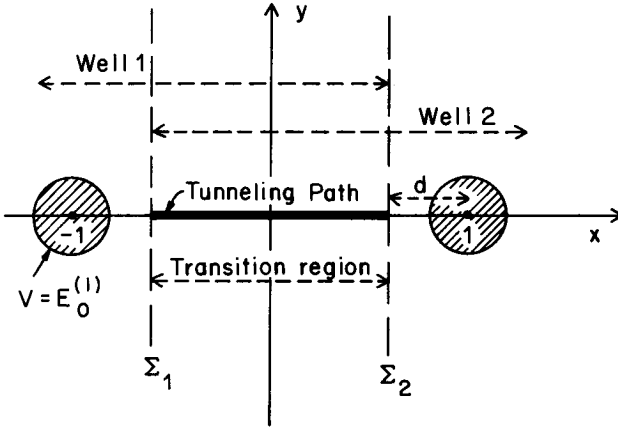


Fig. 1. The elements of the path decomposition expansion for a double-well potential. The wave functions $\psi^{(i)}$ are defined in the regions “Well i ”; the shaded areas correspond to the classically allowed regions. The contribution to the transition Green function is dominated by the tunneling path.

fluctuations. The applications we have in mind do satisfy this property, whereas a generalization to curved tunneling paths is straightforward [8], without altering the main conclusions of this paper.

We begin by breaking configuration space into two overlapping “wells” bounded by surfaces Σ_1 and Σ_2 , each of which contains *one* minimum and the barrier region (see fig. 1). We define restricted states and energies $(\psi_n^{(i)}, E_n^{(i)})$ $i = 1, 2$, such that $\psi_n^{(i)}$ satisfies vanishing (i.e. Dirichlet) boundary conditions on Σ_2 (similarly $\psi_n^{(2)}$ vanishes on Σ_1). We choose $\Sigma_1 = \{x|x_1 = -(1-d)\}$ and $\Sigma_2 = \{x|x_1 = (1-d)\}$ and thus $\Sigma_2 = P_1\Sigma_1$. This implies to all orders in g a degeneracy, $E_n^{(1)} = E_n^{(2)}$. This degeneracy is lifted only by tunneling, which is given by the coupling between the two restricted hamiltonians and is expressed in terms of the transition Green function G^{tr} , which vanishes both on Σ_1 and Σ_2 and satisfies the Schrödinger equation in the transition region (see fig. 1):

$$M_{nn'}(E) = -\frac{1}{4}g^2 \int \int d_{\Sigma_1}x d_{\Sigma_2}y \psi_n^{(1)}(x) \psi_{n'}^{(2)}(y)^* \times \partial_{n_1} \partial_{n_2} G^{tr}(x, y; E). \quad (2.2)$$

In this equation $d_{\Sigma_i}x$ is the surface element of Σ_i and ∂_{n_i} its normal derivative. G^{tr} is the energy Green function defined by the sum of paths, restricted to the transition region, which excludes the minima. To be able to apply the standard semiclassical techniques to evaluate G^{tr} one uses the method of images [10] to find:

$$G^{tr}(x, y; E) = \sum_{k \in \mathbf{Z}} (-1)^k G^u(R_k x, y; E), \quad (2.3)$$

where we defined the k th multiple image of \mathbf{x} by:

$$R_k \mathbf{x} = (2k(1-d) + (-1)^k x_1 - x_1) \mathbf{e}_1 + \mathbf{x} \quad (2.4)$$

and G^u is the unrestricted Green function obtained by taking as potential $V^u(\mathbf{x})$ the periodic extension of $V(\mathbf{x})$:

$$V^u(\mathbf{x} + 2k(1-d)\mathbf{e}_1) = V^u(\mathbf{x}), \quad V^u(\mathbf{x}) = V(\mathbf{x}) \quad \text{for} \quad |x_1| \leq (1-d). \quad (2.5)$$

To verify if G^u in eq. (2.2) satisfies the proper boundary conditions one uses the properties

$$\begin{aligned} G^u(P_1 \mathbf{x}, P_1 \mathbf{y}; E) &= G^u(\mathbf{x}, \mathbf{y}; E), \\ G^u(R_1 \mathbf{x}, R_1 \mathbf{y}; E) &= G^u(\mathbf{x}, R_{i+1} \mathbf{y}; E). \end{aligned} \quad (2.6)$$

Furthermore one fixes an overall constant by requiring that G^u satisfies the Schrödinger equation with the appropriate δ -function singularity.

The E -dependent hamiltonian on the *overcomplete* basis $\{\psi_n^{(1)} + \psi_m^{(2)}\}$ is given by:

$$H(E) = \begin{pmatrix} E^{(1)} & gM(E) \\ gM^+(E) & E^{(2)} \end{pmatrix}, \quad (2.7)$$

with $E^{(2)} = E^{(1)} = \text{diag}(E_0^{(1)}, E_1^{(1)}, E_2^{(1)}, \dots)$. By choosing d such that $E \ll V$ for the whole transition region, we can guarantee that $M(E)$ is exponentially small in g and to first order in $M_{00}(E_0^{(1)}) \sim \exp(-\text{const}/g)$ we find for the energy split of the even and odd ground states:

$$\Delta(g) = 2g |M_{00}(E_0^{(1)})|. \quad (2.8)$$

Substituting (2.3) in (2.2) we find

$$\Delta(g) = 2g^3 \left| \int \int d_{\Sigma_1} \mathbf{x} d_{\Sigma_2} \mathbf{y} \psi_0^{(1)}(\mathbf{x}) \psi_0^{(2)}(\mathbf{y})^* \partial_{n_1} \partial_{n_2} G^u(\mathbf{x}, \mathbf{y}; E) \right| \quad (2.9)$$

up to *exponentially small* relative errors. A factor 4 w.r.t. eq. (2.2) is familiar from hard wall boundary conditions and also reflects the fact that in eq. (2.3) only 4 terms are of the same order; the other terms are exponentially suppressed.

The power of the PDX technique lies in that it allows us to calculate $\psi_0^{(i)}$ and G^u using separate approximations. G^u is only accurately known deep enough in the transition region, that is for d large enough. $\psi_0^{(i)}$ and $E_0^{(i)}$ are only known up to a

few orders in perturbation theory, which in particular means for d too large, that $\psi_0^{(i)}|_{\Sigma_i}$ is not accurately described by the truncated perturbative series. In the examples, we will show that there exists an optimal choice of d (or $\alpha > 0$, with $d = g^\alpha$), for which the relative error in $\Delta(g)$ due to the above described approximations is a positive power of g , giving an *upper bound* for the actual higher order corrections. In particular we will truncate the perturbative expansion for $\psi_0^{(i)}$ and $E_0^{(i)}$ at its lowest non-trivial order. To be precise, let $(\psi_0^{[0]}, E_0^{[0]})$ be the ground state for the lowest order single-well potential $V_{[0]}$, given by the power-law behaviour of V near its minimum:

$$V_{[0]}(x) = \lim_{r \rightarrow 0} r^{-\beta} V(rx - e_1), \quad (2.10)$$

with β such that $V_{[0]}$ is neither zero nor infinite*. We can replace ψ_0 in eq. (2.9) by $\psi_0^{[0]}$ if we can choose d such that, for η small and perpendicular to e_1 , the single-well error given by:

$$e_{sw} \equiv \ln(\psi_0^{(1)}((d-1)e_1 + \eta) / \psi_0^{[0]}(de_1 + \eta)) \quad (2.11)$$

vanishes as some power of g . For the appropriate choice of $d = g^\alpha$ the relative errors e_{sc} in the semiclassical approximation for $\partial_{n_1} \partial_{n_2} G^u(x, y; E)$ and the steepest-descent approximations in the surface integrals should also vanish as some power of g .

3. One-dimensional examples

As a purely pedagogical exercise we will demonstrate the PDX for some simple one-dimensional examples. For the ones we will discuss, $\Delta(g)$ can actually be determined up to exponential corrections, but this is in general not possible for non-separable higher dimensional problems. Thus we want to use these simple examples to explain the method of the error estimates.

The semiclassical expression for a one-dimensional potential with decomposition points at $\pm(1-d)$ is given by [8]:

$$\begin{aligned} \Delta(g) = & 2g\sqrt{2(V(1-d) - E_0)} |\psi_0^{(1)}(1-d)|^2 \\ & \times \exp\left(-\frac{1}{g} \int_{d-1}^{1-d} \sqrt{2(V(x) - E_0)} dx\right) (1 + O(e_{sc})). \end{aligned} \quad (3.1)$$

Eq. (3.1) is equivalent to using the WKB method with the correct connection

* We assume that at least the ground state wave function for V is localized, otherways the notion of tunneling is ill-defined.

formula (depending on β , see eq. (2.10)). One can easily verify the well known results of, for example the double harmonic oscillator [11] $V(x) = \frac{1}{2}(|x| - 1)^2$.

Let us examine the quartic double-well potential $V(x) = \frac{1}{8}(x^2 - 1)^2$ in somewhat more detail. The single-well potential is given by $V_{|0\rangle}(x) = \frac{1}{2}x^2$ and when we substitute $x = -1 + g^{1/2}\hat{x}$ we find:

$$H = g \left(-\frac{1}{2} \frac{\partial^2}{\partial \hat{x}^2} + \hat{V}(\hat{x}) \right) = g \left(-\frac{1}{2} \frac{\partial^2}{\partial \hat{x}^2} + \frac{1}{2}\hat{x}^2 - \frac{1}{2}g^{1/2}\hat{x}^3 + \frac{1}{8}g\hat{x}^4 \right). \quad (3.2)$$

This determines the perturbative series

$$E_0 = \frac{1}{2}g + O(g^2), \quad \psi_0(x) = g^{-1/4}\pi^{-1/4}e^{-1/2x^2} + O(g^{3/4}), \quad (3.3)$$

i.e.

$$E_0^{[0]} = \frac{1}{2}g, \quad \psi_0^{[0]}(x) = g^{-1/4}\pi^{-1/4}e^{-x^2/2g}. \quad (3.4)$$

As was discussed by de Vega et al. [2] we choose d such that $\hat{d} \equiv dg^{-1/2} \gg 1$, to ensure accuracy of the WKB approximation (in this case giving $O(\hat{d}^{-2})$ relative errors), and $\hat{d}^3g^{1/2} = d^3g^{-1} \ll 1$ so that the potential can be approximated by $V_{|0\rangle}(x)$. Hence one can choose in this problem $d = g^\alpha$ with α in between $\frac{1}{3}$ and $\frac{1}{2}$. In the appendix we argue that this indeed implies control over the error in $\Delta(g)$. We will consider the general situation specified by the positive parameters β and γ (cf. eq. (2.10)) such that:

$$V(-1+x) = |x|^\beta(1 + O(|x|^\gamma)). \quad (3.5)$$

(An overall constant is irrelevant in our discussion and will always be put equal to 1.)

Let us give here an heuristic argument. For distances large compared with the classical turning point and small w.r.t. the decomposition surface at $1-d$, $\hat{\psi}^{(1)}$ behaves as (\hat{x}_{cl} the classical turning point):

$$\hat{\psi}(\hat{x}) = \frac{C(g)(1 + O(\hat{x}^{-(\beta+2)/2}))}{(2(\hat{V}(\hat{x}) - \hat{E}_0))^{1/4}} \exp\left(-\int_{\hat{x}_{cl}}^{\hat{x}} \sqrt{2(\hat{V}(\hat{x}) - \hat{E}_0)} d\hat{x}\right). \quad (3.6)$$

This g -dependent constant C enters in the expression for $\Delta(g)$ and continuity of $C(g)$ in $g=0$ would give the required result. To make sense, $C(0)$ should correspond to the constant $C^{[0]}$ defined in a similar way for the single-well potential. The main reason to worry about the validity of this argument is of course that the only (obviously) information we have is that $\int(|\psi^{(1)}(x)|^2 - |\psi^{[0]}(x)|^2) dx \rightarrow 0$ for $g \rightarrow 0$, allowing for relative errors growing exponentially in the tail of the wave function.

The appendix will basically quantify our intuition and we derive the bound $O(gd^{-(\beta+2)/2}) + O(g^{-1}d^{\gamma+1+\beta/2})$ for $e_{sc} + e_{sw}$ and $C^{[0]}/C(g) - 1$.

For the quartic double-well $d = g^{2/5}$ yields an upperbound of $e_{sc} + e_{sw} = g^{1/5}$ for the total relative error, which suffices to claim control over higher order corrections. Note that the actual higher order correction is $O(g)$ [1]; our discussion did not aim at reproducing this but rather to provide an analysis which can easily be generalized to more complicated situations, especially where instanton techniques are not available.

It is also instructive for comparison with the multidimensional problems in the subsequent sections, to solve the tunneling problem for the “ W ” potential given by:

$$V_w(x) = ||x| - 1|. \tag{3.7}$$

In this case $(\psi_0^{(i)}, E_0^{(i)})$ and $(\psi_0^{[0]}, E_0^{[0]})$ coincide (perturbatively) and $\psi_0^{[0]}$ is given by the Airy function $\text{Ai}(x)$ [12] with $E_0^{[0]} = 2^{-1/3}z_0'g^{2/3}$, where $z_0' (= 1.018\dots)$ is the first zero of $\text{Ai}'(-x)$. We find easily:

$$\Delta_w(g) = \frac{2^{-4/3}g^{2/3}}{\pi z_0' \text{Ai}(-z_0')^2} \exp\left(-\frac{4}{3}\sqrt{2}g^{-1} + z_0'2^{7/6}g^{-1/3}\right)(1 + O(g^{1/3})), \tag{3.8}$$

which can be verified by writing down an exact transcendental equation for E and expanding in g to the relevant order. The W -potential provides an example where the instanton technique breaks down; it would predict $\Delta(g) = Ag^{1/2}\exp(-\frac{4}{3}\sqrt{2}g^{-1})$. The term of order $g^{-1/3}$ in the exponent of eq. (3.8) comes from $g^{-1}E_0 \int_{-1}^1 (2V(s))^{-1/2} ds$, which is related to the time duration of the tunneling solution. It converges for potentials with $V_{[0]} \sim |x|^\beta$ and $0 < \beta < 2$, whereas for $\beta \uparrow 2$ it becomes logarithmically divergent explaining the extra power $g^{-1/2}$ in $\Delta(g)$ for quadratic minima [5]. Since for these non-quadratic minima ($0 < \beta < 2$) the classical paths do not spend an infinite time in the minima regions, the dilute gas approximation of the standard instanton methods will breakdown.

Finally for later purpose we consider a slight modification of the W -potential:

$$V(x) = \frac{1}{2}|x^2 - 1|. \tag{3.9}$$

This has $V_{[0]}(x) = |x|$ and the rescaled hamiltonian is obtained by putting $x = -1 + g^{2/3}\hat{x}$, yielding:

$$H = g^{2/3} \left(-\frac{1}{2} \frac{\partial^2}{\partial \hat{x}^2} + \left| \hat{x} - \frac{1}{2}g^{2/3}\hat{x}^2 \right| \right), \tag{3.10}$$

from which one can estimate the errors $e_{sw} + e_{sc}$ to be $O(gd^{-3/2} + g^{-1}d^{5/2})$; hence we choose $d = g^{1/2}$ yielding $e_{sw} + e_{sc} = g^{1/4}$, although the actual higher order term

will most certainly be $O(g^{1/3})$, like in (3.8). In this case the result for $\Delta(g)$ is:

$$\Delta(g) = g^{2/3} \frac{2^{-4/3}}{\pi z_0^2 \text{Ai}(-z_0')^2} \exp\left(-\frac{1}{2}\pi g^{-1} + 2^{-1/3}\pi z_0' g^{-1/3}\right) \times (1 + O(g^\kappa)), \quad (3.11)$$

with $\frac{1}{4} < \kappa \leq \frac{1}{3}$. Note that the prefactor is the same as for eq (3.8).

4. The double-cone problem

The 3-dimensional double-cone potential is given by:

$$V_{\text{DC}}(\mathbf{x}) = \left[(|x_1| - 1)^2 + x_2^2 + x_3^2 \right]^{1/2}. \quad (4.1)$$

Along the tunneling path (minimum action path: $-1 < x_1 < 1$, $x_2 = x_3 = 0$) we find the same barrier as for the W -potential. However, the transverse degrees x_2 and x_3 , are not separable in the Schrödinger equation. We will be able to solve explicitly the transverse fluctuation equations, which enter the semiclassical approximation. If the adiabatic fluctuation approximation (AFA) [8] would be applicable these transverse fluctuations would lead to a $O(g)$ correction of the potential. This would only modify the prefactor in comparing Δ for the double cone and the “ W ” potential. But for the double cone the AFA turns out to be wrong, as we demonstrate below.

We use the results of ref. [8] to evaluate $\Delta_{\text{DC}}(g)$:

$$\begin{aligned} \Delta_{\text{DC}}(g) &= 2g \sqrt{2(V((1-d)\mathbf{e}_1) - E_0)} |\psi_0^{[0]}(d\mathbf{e}_1)|^2 \\ &\times \exp\left[-\frac{1}{g} \int_{d-1}^{1-d} \sqrt{2(V(s\mathbf{e}_1) - E_0)} ds\right] \Lambda(d)(1 + O(g^\kappa)), \end{aligned} \quad (4.2)$$

where $\psi_0^{[0]}$ is the ground state for the “single-cone” potential $V_{[0]}(\mathbf{x}) = |\mathbf{x}|$:

$$\psi_0^{[0]}(\mathbf{x}) = \frac{2^{1/6} \text{Ai}(2^{1/3} g^{-2/3} (|\mathbf{x}| - E_0^{[0]}))}{g^{1/3} (4\pi)^{1/2} \text{Ai}'(-z_0) |\mathbf{x}|}. \quad (4.3)$$

Here $z_0 (= 2.338\dots)$ is the first zero of $\text{Ai}(-x)$ and $E_0^{[0]} = 2^{-1/3} z_0 g^{2/3}$ (equal to the perturbative part of $E_0^{(i)}$). The contribution from the transverse degrees of freedom (x_2 and x_3) are contained in $\Lambda(d)$, given by:

$$\Lambda(d) = 2\pi g [\dot{q}(T) + \phi''_{\perp}(d) q(T)]^{-1}. \quad (4.4)$$

The fluctuation $q(t)$ is a solution of the “stability equation” ($\dot{q}(t) = (d/dt)q(t)$):

$$\ddot{q}(t) = \Omega^2(t) q(t), \quad \Omega^2(t) = \frac{\partial^2 V}{\partial x_2^2}(\mathbf{x}(t)), \quad (4.5)$$

with the initial conditions:

$$q(0) = 1,$$

$$\dot{q}(0) = \phi''_{\perp}(d) = \lim_{g \rightarrow 0} -g \frac{\partial^2}{\partial x_2^2} \ln(|\psi_0^{[0]}(\mathbf{x})|) \Big|_{\mathbf{x} = d\mathbf{e}_1} = \sqrt{2/d}. \quad (4.6)$$

$\mathbf{x}(t)$ is the classical path $\mathbf{x}(t) = x(t)\mathbf{e}_1$, which obeys the euclidean equations of motion:

$$\dot{x} = \sqrt{2(V(x\mathbf{e}_1) - E_0)}, \quad (4.7)$$

with the boundary conditions

$$x(0) = d - 1, \quad x(T) = 1 - d, \quad (4.8)$$

where T is obtained from eq. (4.7):

$$T = \int_{d-1}^{1-d} \frac{ds}{\sqrt{2(V(s\mathbf{e}_1) - E_0)}}. \quad (4.9)$$

We will drop the E_0 dependence since we will only need $\Lambda(d)$ to leading order (introducing an extra relative error of $O(g^{2/3})$ in $\Delta_{\text{DC}}(g)$, which is irrelevant).

The DC potential provides an example of a rapidly varying transverse frequency:

$$\begin{aligned} \Omega^2(t) &= 2\left(t_0 + \frac{1}{2}T - |t - \frac{1}{2}T|\right)^{-2}, \\ t_0 &= \sqrt{2d}, \quad T = 2(\sqrt{2} - t_0). \end{aligned} \quad (4.10)$$

The parameter which governs the adiabatic approximation, $|\dot{\Omega}\Omega^{-2}| = 2^{-1/2}$ is not small and we need the exact solution to eq. (4.5) to evaluate $\Lambda(d)$. We find:

$$q(t) = \begin{cases} (1 + t/t_0)^2, & t \leq \frac{1}{2}T, \\ \frac{4}{3}d^{-3/2}(1 + (T-t)/t_0)^{-1} - \frac{1}{3}(1 + (T-t)/t_0)^2, & t \geq \frac{1}{2}T. \end{cases} \quad (4.11)$$

Using eqs. (4.4) and (4.6) we obtain (up to $O(g^{2/3})$):

$$\Lambda(d) = \frac{1}{2}\pi\sqrt{2} d^2 g. \quad (4.12)$$

The final result for $\Delta_{\text{DC}}(g)$ then becomes:

$$\Delta_{\text{DC}}(g) = g^{5/3} \frac{2^{1/6}}{8\pi \text{Ai}(-z_0)^2} \exp\left(-\frac{4}{3}\sqrt{2}g^{-1} + 2^{7/6}z_0g^{-1/3}\right) \times (1 + O(g^{1/3})). \quad (4.13)$$

The extra power of g , relative to the result for the W -potential (3.7), is therefore due to the breakdown of the adiabatic approximation for the fluctuation equation.

Note that (4.13) is d -independent up to the displayed order, which is a consistency check on our approximations. In many cases a three-dimensional potential is spherically symmetric at its minima; for these potentials it is then useful to note that always $\Lambda(d) = \text{const } g d^2$. This will be proven in a future publication. For these spherically symmetric $V_{|0\rangle}(\mathbf{x})$, $\psi_0^{[0]}(\mathbf{x}) = (1/|\mathbf{x}|)\hat{\psi}_0^{[0]}(|\mathbf{x}|)$ and hence we see that the d^2 in $\Lambda(d)$ cancels against the explicit $1/d^2$ in $|\psi_0^{[0]}(d\mathbf{e}_1)|^2$.

5. The vacuum valley problem

The following problem deals with a case where the transverse fluctuations determine the exponential contribution to the tunnel-splitting. We will study the following two-dimensional hamiltonian as the simplest example:

$$H = -\frac{1}{2}g^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2g^2}(x^2 - 1)^2 y^2. \quad (5.1)$$

The essential features of this toy model are the vacuum valley at $y=0$ and the breakdown of the quadratic approximation for the transverse direction (y) at distinctive points in this vacuum valley (namely at $x = \pm 1$). This model exhibits certain features of gauge fields on a torus [4].

For $g \rightarrow 0$ the potential reproduces in the classical case the constraint $y=0$, leaving a free motion in the x -direction. It is well known that in quantum mechanics, the dynamics depend on the form of the constraining potential [13]. The easiest way to see this is to derive the effective potential along the vacuum valley in the adiabatic approximation,

$$V_0(x) = \frac{1}{2}|x^2 - 1|, \quad (5.2)$$

by “integrating out” the y -coordinates. This potential has two degenerate minima at $x = \pm 1$, and we expect a non-perturbative splitting of the even and odd ground state energies due to tunneling through this effective potential barrier.

From the V-shape singularity of V_0 at $x = \pm 1$ one sees that the adiabatic approximation will breakdown sufficiently close of these points and V_0 *cannot* be used to derive the single well ground state energy and wave function in lowest order. Nevertheless it does indicate that the perturbative ground state is obtained by expanding the potential in eq. (5.1) around $x = \pm 1$, $y = 0$. The single-well potential is given by

$$V_{|0\rangle} = 2g^{-2}x^2y^2 \quad (5.3)$$

and the rescaled single-well hamiltonian (with $x = -1 + g^{2/3}\hat{x}$, $y = g^{2/3}\hat{y}$) becomes:

$$H = g^{2/3} \left(-\frac{1}{2} \left(\frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{y}^2} \right) + 2\hat{x}^2\hat{y}^2 - 2g^{2/3}\hat{x}^3\hat{y}^2 + \frac{1}{2}g^{4/3}\hat{x}^4\hat{y}^2 \right). \quad (5.4)$$

The hamiltonian $-\frac{1}{2}(\partial^2/\partial\hat{x}^2 + \partial^2/\partial\hat{y}^2) + 2\hat{x}^2\hat{y}^2$ has a discrete spectrum with localized wave functions [14].

In contrast, if we would use the hamiltonian

$$H_a = -\frac{1}{2}g^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2g^2} \left((x^2 - 1)^2 + a^2 \right) y^2, \quad (5.5)$$

the effective potential in the adiabatic approximation would be completely regular: $V_0(x; a) = \frac{1}{2}\sqrt{(x^2 - 1)^2 + a^2}$, and the whole calculation of $\Delta(g)$, up to the order we are interested in, reduces to a one-dimensional problem for the hamiltonian

$$H = -\frac{g^2}{2} \frac{\partial^2}{\partial x^2} + V_0(x; a). \quad (5.6)$$

The aim of the present discussion is to show that when the adiabatic approximation breaks down near the perturbative regions the exponential contribution to $\Delta(g)$ is still governed by the adiabatically approximated effective hamiltonian along the vacuum valley; which in the present case becomes:

$$H_{\text{ad}} = \frac{-g^2}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2}|x^2 - 1|, \quad (5.7)$$

but one needs to use the full two-dimensional wave function and energy to compute the prefactor. Essential is that our analysis will also provide control over the error due to the breakdown of the adiabatic approximation.

We will still choose the decomposition surfaces as in fig. 1, but the transition region will always contain part of the classically allowed region for $E > 0$. Hence, a naive semiclassical approximation for G^{tr} is impossible. But the adiabatic approximation will be applicable in the transition region and the larger we choose d the better the approximation becomes. Our derivation therefore stays as close as possible to the notion of an effective potential V_0 by writing the hamiltonian in a mixed "transverse energy" and coordinate representation:

$$H_{nm}(x) = -\frac{1}{2}g^2 \left(\delta_{nm} \frac{\partial}{\partial x} - A_{nm}(x) \right)^2 + \delta_{nm} V_n(x). \quad (5.8)$$

The wave function $\psi(x, y)$ is expressed in this mixed representation by the expansion:

$$\psi(x, y) = \sum_n \phi^{(n)}(x) \chi_{[x]}^{(n)}(y), \quad (5.9)$$

where $\chi_{[x]}^{(n)}(y)$ are the “frozen- x ” normalized eigenfunctions for the hamiltonian H , i.e. we define:

$$\left(\frac{-g^2}{2} \frac{\partial^2}{\partial y^2} + \frac{1}{2g^2} (x^2 - 1)^2 y^2 \right) \chi_{[x]}^{(n)}(y) = V_n(x) \chi_{[x]}^{(n)}(y). \quad (5.10)$$

Finally, the antisymmetric matrix $A_{nm}(x)$, playing the role of a “gauge field” [15], is given by:

$$A_{nm}(x) = \int \chi_{[x]}^{(m)}(y)^* \frac{\partial}{\partial x} \chi_{[x]}^{(n)}(y) dy. \quad (5.11)$$

Note that $V_0(x)$ in eq. (5.10) coincides with the adiabatic approximation of the effective potential.

Next we want to compute the transition Green function which, as in sect. 2, can be expressed in an unrestricted Green function by extending $V_n(x)$, $A_{nm}(x)$ and $\chi_{[x]}^{(n)}(y)$ periodically in x . G^u will have the following path integral form:

$$G^u(x, x', E) = \int_0^\infty dT \int_{x(0)=x}^{x(T)=x'} Dx \exp \left[\frac{i}{g} \int_0^T \left(\frac{1}{2} \dot{x}^2 + E \right) dt \right] K(y, y', \{x(t)\}). \quad (5.12)$$

The kernel or propagator K is given by:

$$K(y, y', \{x(t)\}) = \int_{y(0)=y}^{y(T)=y'} Dy \exp \left[\frac{i}{g} \int_0^T \left(\frac{1}{2} \dot{y}^2 - \frac{(x(t)^2 - 1)^2}{2g^2} y^2 \right) dt \right]. \quad (5.13)$$

This is purely quadratic in y for our simple toy model and actually allows for an “exact” solution [17], to which we will come back further on. In the mixed representation we have:

$$G_{mn}(x, x', E) = \int dy dy' \chi_{[x']}^{(m)*}(y') G^u(x, x'; E) \chi_{[x]}^{(n)}(y), \quad (5.14a)$$

$$\begin{aligned} K_{mn}(\{x(t)\}) &= \int dy dy' \chi_{[x(T)]}^{(m)*}(y') K(y, y'; \{x(t)\}) \chi_{[x(0)]}^{(n)}(y) \\ &= P \exp \left[- \int_0^T \left(\frac{i}{g} \hat{V}_n(x(t)) \delta_{mn} + \dot{x}(t) \hat{A}_{mn}(x(t)) \right) dt \right]. \end{aligned} \quad (5.14b)$$

Note that G_{mn} satisfies the equation:

$$\hat{H}_{mn}(x)G_{nk}(x, x'; E) = -g\delta_{mk}\delta(x - x'), \quad (5.15)$$

where the hat denotes the above described periodic extension. It is defined independently of the path integral representation.

Substituting (5.9), (5.10) and (5.14) in eq. (2.9) gives the following expression for $\Delta(g)$:

$$\Delta(g) = 2g^3 \left| \phi_0^{(n)}(x)\phi_0^{(m)}(x')^* D_{nk}(x)D_{ml}(x')G_{kl}(x, x'; E_0) \right|, \quad (5.16)$$

with $x = -x' = 1 - d$, $E_0 = E_0^{(1)} = E_0^{(2)}$ and $D_{mn}(x)$ the covariant derivative:

$$D_{nm}(x) = \delta_{nm} \frac{\partial}{\partial x} - A_{nm}(x). \quad (5.17)$$

The aim is to show that in eq. (5.16) only the $n = m = k = l = 0$ term survives and that $G_{00}(x, x'; E_0)$ is determined by \hat{H}_{ad} in eq. (5.7). We want to be careful and keep track of the errors involved, and will give two separate accounts. The first is heuristic and based on analyzing the situation for eq. (5.5) with $a \neq 0$. The second discusses the exact evaluation of K [17]. We will also discuss a relation with the adiabatic theorem, without elaborating on it in detail.

Let us note that we can treat the hamiltonian of eq. (5.5) in exactly the same way, now with V_n and A_{nm} depending on a . It is actually not hard to explicitly calculate these quantities:

$$V_n(x; a) = \left(n + \frac{1}{2} \right) \sqrt{(x^2 - 1)^2 + a^2},$$

$$A_{nm}(x; a) = \frac{x(x^2 - 1)}{2[(x^2 - 1)^2 + a^2]} \left(\sqrt{n(n-1)} \delta_{n-2, m} - \sqrt{m(m-1)} \delta_{m-2, n} \right). \quad (5.18)$$

One could ask for no better illustration for the breakdown of the adiabatic approximation when $a \equiv 0$. In that case $A_{nm}(x; 0)$ becomes infinite at $x = \pm 1$. It is therefore impossible to obtain perturbative results from an adiabatic approximation. To see this most dramatically, let us consider $a \neq 0$ and make a weak coupling expansion. The perturbative expansion for E_0 is given by

$$E_0(a) = \frac{1}{2}a + g/\sqrt{2a} \cdots = \frac{1}{2}a(1 + \sqrt{2}ga^{-3/2} + \cdots), \quad (5.19)$$

which behaves distinctly different from the result for $a = 0$, obtained from eq. (5.4);

$$E_0(0) = \epsilon_0 g^{2/3} + O(g^{4/3}). \quad (5.20)$$

Another way of phrasing the same problem is: one cannot interchange the $a \rightarrow 0$ and $g \rightarrow 0$ limits.

The proper expansion parameter in eq. (5.17) is $ga^{-3/2}$, which becomes bad for $a \rightarrow 0$, hence we should identify $e_{na} = ga^{-3/2}$ as the error due to the breakdown of the adiabatic approximation. For the case $a \neq 0$ it is thus reasonable to expect the following result:

$$\begin{aligned} \Delta(g; a) &= 2g|\phi_0^{(0)}(1-d; a)|^2 \sqrt{2(V_0(1-d; a) - E_0(a))} \\ &\quad \times \exp\left(-g^{-1} \int_{d-1}^{1-d} dx \sqrt{2(V_0(x; a) - E_0(a))}\right) \\ &\quad \times (1 + O(ga^{-3/2}) + O(gd^{-2}a^{1/2})). \end{aligned} \quad (5.21)$$

($O(gd^{-2}a^{1/2})$ is the relative error due to the semiclassical approximation.) Since this involves (through eq. (5.14)) only V_n and A_{nm} for $|x^2 - 1| \geq d$, these functions for $a = 0$ behave as if $a = d$. This then would yield the following result for the hamiltonian of eq. (5.1):

$$\begin{aligned} \Delta(g) &= 2g|\phi_0^{(0)}(1-d)|^2 \sqrt{2(V_0(1-d) - E_0)} \\ &\quad \times \exp\left(-g^{-1} \int_{d-1}^{1-d} dx \sqrt{2(V_0(x) - E_0)}\right) (1 + O(gd^{-3/2})). \end{aligned} \quad (5.22)$$

We will now give a more careful derivation and again (for illustrative purposes) consider the hamiltonian of eq. (5.5), with arbitrary a . Hence we evaluate ($g\omega(t) = |x(t)^2 - 1|$):

$$\begin{aligned} K(y, y', \{\omega(t)\}) &= \int_{y(0)=y}^{y(T)=y'} Dy \exp\left[\frac{i}{g} \int_0^T \frac{1}{2}(\dot{y}^2 - \omega^2(t)y^2) dt\right], \\ \bar{\omega} &= \omega(0), \quad \bar{\omega}' = \omega(T), \quad x = x(0), \quad x' = x(T). \end{aligned} \quad (5.23)$$

The result [17] can be formulated in terms of the solution $s(t)$ of the equation:

$$\begin{aligned} \ddot{s}(t) + \omega^2(t)s(t) - \bar{\omega}^2 s^{-3}(t) &= 0, \\ s(0) &= 1, \quad \dot{s}(0) = 0. \end{aligned} \quad (5.24)$$

This equation is equivalent to the complex equation:

$$\ddot{u}(t) + \omega^2(t)u(t) = 0, \quad u(0) = 1, \quad \dot{u}(0) = i\bar{\omega}, \quad (5.25)$$

where $u(t)$ can be expressed in terms of $s(t)$ by:

$$\begin{aligned} u(t) &= s(t)\exp(i\bar{\omega}\tau(t)), \\ \tau(t) &= \int_0^t s^{-2}(t') dt'. \end{aligned} \quad (5.26)$$

If we introduce the following parameters:

$$\begin{aligned} \bar{T} &= \tau(T), \quad c = s(T), \quad \hat{c} = (\bar{\omega}'/\bar{\omega})^{1/2}c, \\ b &= \frac{d}{dt} [\ln s(t)]_{t=\tau}, \end{aligned} \quad (5.27)$$

one finally gets [17]:

$$\begin{aligned} K(y, y', \{\omega(t)\}) &= \left(\frac{\bar{\omega}}{2\pi c g \sin(\bar{\omega}\bar{T})} \right)^{1/2} \exp(i b y'^2 / 2g) \\ &\times \exp \left[\frac{i\bar{\omega}}{2g \sin(\bar{\omega}\bar{T})} \left\{ ((y'/c)^2 + y^2) \cos(\bar{\omega}\bar{T}) - 2y(y'/c) \right\} \right] \\ &= c^{-1/2} \exp(i b y'^2 / 2g) K_{\bar{\omega}}(y, y'/c; \bar{T}). \end{aligned} \quad (5.28)$$

In here, $K_{\omega}(z, z'; T)$ is the harmonic oscillator propagator for fixed ω . One can therefore derive the following identity:

$$\begin{aligned} \int K(y, y', \{\omega(t)\}) \chi_{[x]}^{(n)}(y) dy &= \exp(-i(2n+1)\frac{1}{2}\bar{\omega}\bar{T}) \\ &\times \exp(i b y'^2 / 2g) \hat{c}^{-1/2} \chi_{[x']}^{(n)}(y'/\hat{c}). \end{aligned} \quad (5.29)$$

As it should be, the r.h.s. is normalized to 1. We can now easily compute $K_{mn}(\{x(t)\})$:

$$K_{mn}(\{x(t)\}) = \exp(-i(2n+1)\bar{\omega}\bar{T}) P_{mn}(\{x(t)\}), \quad (5.30)$$

with:

$$\begin{aligned} P_{mn}(\{x(t)\}) &= \frac{2^{-(n+m)/2}}{(n!m!\hat{c}\pi)^{1/2}} \int_{-\infty}^{\infty} dz H_m(z) H_n(z/\hat{c}) e^{-\lambda z^2}, \\ \lambda &= \frac{1}{2}(1 + \hat{c}^{-2} - i b/\bar{\omega}'). \end{aligned} \quad (5.31)$$

Before we continue, we should investigate the expressions for b , \hat{c} and \bar{T} more closely.

If we exhibit the implicit g -dependence of ω by defining:

$$\nu(t) = g\omega(t), \quad \bar{\nu} = g\bar{\omega}, \quad \bar{\nu}' = g\bar{\omega}', \quad (5.32)$$

then we can solve $s(t)$ iteratively:

$$s_{n+1}(t) = \left[\frac{\nu^2(t)}{\bar{\nu}^2} + \frac{g^2 \ddot{s}_n(t)}{\bar{\nu}^2 s_n(t)} \right]^{-i/4}. \quad (5.33)$$

This establishes the expansion for the various parameters; to lowest order one finds:

$$\begin{aligned} \frac{1}{2}\bar{\omega}\bar{T} &= \frac{1}{g} \int_0^T V_0(x(t)) dt + O(\epsilon), \\ \frac{b}{\bar{\omega}'} &= O(\epsilon), \quad \hat{c} = 1 + O(\epsilon^2), \\ \epsilon &= \frac{g\dot{\nu}}{\nu^2} = \frac{g\dot{x}(x^2-1)}{((x^2-1)^2+a^2)^{3/2}}. \end{aligned} \quad (5.34)$$

We leave it to the true perfectionist to evaluate $P_{mn}(\{x(t)\})$ exactly by using generating function techniques. We will be satisfied with the following result:

$$\begin{aligned} P_{00}(\{x(t)\}) &= \left[\frac{1}{2}\hat{c}(1 + \hat{c}^{-2} - ib/\bar{\omega}') \right]^{-1/2}, \\ P_{nm} &= 0 \quad \text{for } n+m \text{ odd}, \end{aligned} \quad (5.35)$$

which is exact. Furthermore we have the estimates:

$$\begin{aligned} P_{nn}(\{x(t)\}) &= P_{00}(\{x(t)\})^{2n+1}(1 + nO(\epsilon^2)), \\ \frac{P_{n+2i,n}}{P_{n,n+2i}} &= P_{00}(\{x(t)\})^{2n+1} 2^i \left(\frac{(n+2i)!}{n!(i!)^2} \right)^{1/2} O(\epsilon^i), \end{aligned} \quad (5.36)$$

It is important to observe that for any $\{x(t)\}$:

$$\lim_{g \rightarrow 0} P_{nm}(\{x(t)\}) = \delta_{nm}, \quad (5.37)$$

but this convergence is far from uniform; we will come back to this in a moment.

We have from eqs. (5.12), (5.13), (5.14), and (5.30):

$$G_{mn}(x, x'; E_0) = \int_0^\infty dT \int_{x(0)=x}^{x(T)=x'} Dx \times \exp \left[\frac{i}{g} \int_0^T \left(\frac{1}{2} \dot{x}^2 - V_n(x) + E_0 \right) dt \right] W_{mn}(\{x(t)\}), \quad (5.38)$$

where

$$W_{mn}(\{x(t)\}) = \exp \left(\frac{i}{g} \int_0^T V_n(x(t)) dt \right) K_{mn}(\{x(t)\}),$$

$$\lim_{g \rightarrow 0} W_{mn}(\{x(t)\}) = \delta_{mn}. \quad (5.39)$$

For $W_{mn}(\{x(t)\})$ the same remarks w.r.t. convergence can be made as for $P_{mn}(\{x(t)\})$.

Obviously for $a \neq 0$ eq. (5.34) will lead to $\varepsilon \leq 0(ga^{-3/2})$. For $a = 0$ and d such that $V_0(1-d) \gg E_0$ (note that $V_n(x) > V_0(x)$) we can use a steepest-descent approximation for each m and n in eq. (5.38), and because $\ln W_{mn}$ is a power series correction in g to the action $\frac{1}{2} \dot{x}^2 - V_n(x)$, the saddle point can be obtained by a series expansion, where the next to leading order introduces a $O(g^2)$ correction and can therefore be neglected (i.e. W_{mn} does not upset the saddle-point approximation.) One now easily derives eq. (5.22). The error includes an estimate for the non-derivative term in $D_{02}(x')$, occurring in the expression for $\Delta(g)$ (eq. (5.16)). Note that K_{mn} is exponentially suppressed w.r.t. K_{m0} for $n \neq 0$. The estimates for the P_{m0} , $m \neq 0$ guarantee that $\sum_{m \neq 0} P_{m0} \phi_0^{(m)}(1-d)$ converges and is bounded by $O(\varepsilon)$.

We will now briefly comment on the relation to the adiabatic theorem. This theorem studies the hamiltonian of eq. (5.10) under slow variations of its parameter x . We will follow the discussion of Messiah [16]. One has:

$$\varepsilon_n(s) = V_n(x(s)), \quad (5.40a)$$

$$\phi_n(s) = \int_0^s \varepsilon_n(t) dt/g, \quad (5.40b)$$

$$\bar{K}_{nm}(s) = \exp(i\phi_n(s)) A_{nm}(s) \exp(-i\phi_m(s)) \frac{dx(s)}{ds}, \quad (5.40c)$$

$$W_{nm}(s) = W_{nm}(\{x_s(t)\}). \quad (5.40d)$$

In (5.40d) $x_s(t)$ simply means $x(t)$ restricted to the interval $[0, s]$. We now easily

derive (using eqs. (5.14)) and (5.39)):

$$\frac{dW_{mn}(s)}{ds} = i\bar{K}_{mk}(s)W_{kn}(s), \quad W_{mn}(0) = \delta_{nm}. \quad (5.41)$$

Or if one introduces [16]:

$$F_{mn}(s) = \int_0^s \exp(i(\phi_m(t) - \phi_n(t))) A_{mn}(t) \frac{dx(t)}{dt} dt, \quad (5.42)$$

we find:

$$W_{mn}(s) = \delta_{mn} + F_{mk}(s)W_{kn}(s) - i \int_0^s F_{mk}(t) A_{kl}(t) W_{ln}(t) \frac{dx(t)}{dt} dt. \quad (5.43)$$

It is easily seen that A and F are not bounded matrices.

We can therefore not apply the standard argument of ref. [16] to conclude that $W_{mn} \approx \delta_{mn}$. This standard argument for bounded A guarantees *uniform* convergence of W_{mn} to δ_{mn} . But we have previously seen in an explicit way, that this cannot be true for the harmonic oscillator. However with the hindsight of convergence, eq. (5.43) with the standard estimates of F_{mn} [16], is consistent with $W_{mn} - \delta_{mn} = O(\epsilon)$. It should be possible to derive the properties of W_{mn} from eq. (5.43), but we will not dwell upon this here.

From the behaviour of the Green function G^u one also derives the following asymptotics for $\phi_0^{(0)}$.

$$\phi_0^{(0)}(1-d) = C(g)g^{-1/6}(d^2 - 2d - 2E_0)^{-1/4} \exp\left(-\frac{1}{g} \int_{\sqrt{1+2E_0}}^{1-d} \sqrt{x^2 - 1 - 2E_0} dx\right). \quad (5.44)$$

The explicit power of $g^{-1/6}$ was extracted from the scaling behaviour as indicated above eq. (5.4). Combining this with eq. (5.22) yields:

$$\Delta(g) = 2g^{2/3}|C(g)|^2 \exp\left(-\frac{1}{2}\pi g^{-1} + \epsilon_0 g^{-2/3}\pi\right) \left(1 + O(gd^{-3/2}) + O(g^{1/3})\right). \quad (5.45)$$

It is now very likely that also in this case $C(g)$ has a smooth limit for $g \rightarrow 0$ and that $C \equiv C(0)$ is given by the single-well problem (but we have not yet bothered to prove this). In conclusion one replaces $C(g)$ in eq. (5.45) by C , where C and ϵ_0 are defined by the ground state for the single-well hamiltonian (this will only introduce

a relative error which vanishes for $g \rightarrow 0$):

$$C = \lim_{x \rightarrow \infty} \frac{(2x)^{1/2}}{\pi^{1/4}} \exp\left(\frac{1}{3}(2x - 2\epsilon_0)^{3/2}\right) \int dy \tilde{\psi}^{[0]}(x, y) e^{-xy^2},$$

$$\left\{ -\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2x^2 y^2 \right\} \tilde{\psi}^{[0]}(x, y) = \epsilon_0 \tilde{\psi}^{[0]}(x, y),$$

$$\int |\tilde{\psi}^{[0]}(x, y)|^2 dx dy = 1. \tag{5.46}$$

We want to stress that in many higher dimensional problems like eq. (5.46), $\psi^{[0]}$ cannot be solved exactly, in which case ϵ_0 and C have to be calculated numerically. Of these, ϵ_0 is most easily accessible.

We want to end this section by remarking that V_0 need only be known to lowest order in g . Our example, eq. (5.1), is special in the sense that $V_0 = \frac{1}{2}|(x^2 - 1)|$ is exact, which is due to the fact that y occurs quadratically in the hamiltonian. In the field theory terminology, the one-loop approximation for V_0 is exact. In general however we can have higher powers of y , e.g. $g^2 V(x, y) = \frac{1}{2}(x^2 - 1)^2 y^2 + y^4$. In that case $V_0(x) = \frac{1}{2}|x^2 - 1|^2 + g^2/(4|x^2 - 1|) + O(g^4/|x^2 - 1|^2)$. If we write $\Delta(g)$ as follows:

$$\Delta(g) = Ag^\gamma \exp(g^{-1}(-S + E_0 T))(1 + O(g^\kappa)), \tag{5.47}$$

it is easily seen that the higher loop corrections to V_0 do not influence S, T, γ . But since $V_{[0]}$ changes drastically, E_0 and C are rather different.

6. Summary

We have shown that the path decomposition expansion is well suited to handle tunneling problems with non-quadratic minima. When these minima are isolated (and in particular rotationally symmetric in lowest order) the analysis is a generalization of ref. [2] and we find in d dimensions*:

$$\Delta(g) = Ag^{(d-1)/2 + 2\beta/(\beta+2)} \exp(-g^{-1}S + \epsilon_0 T g^{(\beta-2)/(\beta+2)})(1 + O(g^\kappa)), \tag{6.1}$$

where $S = \int_{-1}^1 \sqrt{2V(xe_1)} dx$, $T = \int_{-1}^1 (1/\sqrt{2V(xe_1)}) dx$, $V_{[0]}(x) = |x|^\beta$, $\kappa > 0$ and $\frac{2}{3} < \beta < 2$. Furthermore ϵ_0 is the ground state energy for the single-well hamiltonian $-\frac{1}{2}\partial^2/\partial x^2 + |x|^\beta$ and A can be calculated from the transverse fluctuations and the asymptotics of the normalized single-well ground state wave function. For $\beta \uparrow 2$

* For the one-dimensional case part of this result (terms proportional to S and T) was derived by B. Simon [5].

there is a discontinuity in this formula as explained previously (T diverges) and for $\beta < \frac{2}{3}$, E_0 behaves as $\epsilon_0 g^\nu$, with $\nu = 2\beta/(\beta + 2) < \frac{1}{2}$, so that there will appear extra terms in the exponent, that can no longer be absorbed in the relative error $O(g^\kappa)$.

As the most important and novel result we consider the vacuum valley problem where the adiabatic approximation fails in the perturbative regions. We propose the following algorithm for evaluating $\Delta(g)$:

(i) Calculate the 1-loop effective potential in the adiabatic limit along the vacuum valley and suppose it is $O(1)$, and has degenerate *isolated* minima.

(ii) Calculate the ground state energy $E^{[0]}$ and wave function $\psi^{[0]}$ for the single-well potential centered around the aforementioned minima to extract the constant ϵ_0 and wave function $\phi_{[0]}^{(0)}(\mathbf{x})$, where $\phi_{[0]}^{(0)}(\mathbf{x})$ is defined as for eqs. (5.8) and (5.9), but now using $V_{[0]}(\mathbf{x}, y)$. \mathbf{x} denotes the vacuum valley parameters (m in number).

(iii) Calculate $\Delta(g)$ for the m -dimensional hamiltonian with as potential the 1-loop approximation for $V_0(\mathbf{x})$, but use for the energy and wave function, E_0 and $\phi_{[0]}^{(0)}(\mathbf{x})$, as determined from the full problem (step (ii)).

We proved this for the particular 2-dimensional example, but are confident that a generalization is straightforward.

Especially for the vacuum valley problem the path decomposition expansion is of significant help in separating the perturbative and tunneling contribution and providing the appropriate connection formula. The methods described are being applied [4] to $SU(N)$ gauge fields in a finite cubic volume, where $\Delta(g)$ gives the energy of electric flux, to be used to probe a possible string tension. In that example the vacuum valley is 3-dimensional and the one-loop approximation for $V_0(\mathbf{x})$ is of the conic shape near its minima.

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Appendix

This appendix will deal with some of the details for the error estimates of $\Delta(g)$ in the one-dimensional case. At the end we comment on the higher dimensional situation. For a potential of the form of eq. (3.5) we get the following scaling properties

$$\begin{aligned} 1 + x &= \hat{x} g^{2/(2+\beta)}, & E &= \hat{E} g^{2\beta/(2+\beta)}, \\ V(x) &= g^{2\beta/(2+\beta)} \hat{V}(\hat{x}), & \psi(x) &= g^{-1/(2+\beta)} \hat{\psi}(\hat{x}). \end{aligned} \quad (\text{A.1})$$

For the purpose of both establishing control over the error and independence of d within this error we introduce a range of d -values, specified by

$$d_1 \equiv g^{\alpha_1} \leq d \leq g^{\alpha_2} \equiv d_2, \quad 0 < \alpha_2 < \alpha_1. \tag{A.2}$$

The constraint for α_1 comes from requiring that d should be far inside the classically forbidden region:

$$\alpha_2 < \alpha_1 < \frac{2}{2 + \beta}. \tag{A.3}$$

A non-zero lower bound on α_2 will follow after we have discussed all the appropriate requirements.

Wherever we do not specify a suffix (1) or [0] the expressions are supposed to be valid for both the single-well and restricted problems, with g sufficiently small ($g < g_0$). We will rewrite the Schrödinger equation in a WKB inspired fashion. We choose $\eta(\hat{x})$ such that $u^2(\hat{x})$ is positive definite, where:

$$u^2(x) = 2(\hat{V}(\hat{x}) - \hat{E} + \eta(\hat{x})) \geq u_0^2. \tag{A.4}$$

Apart from this implicit constraint on the function η we further require it to be g -independent (for $g < g_0$) and have compact support contained in the disk D_1 of radius d_1 centered at the minima of V (and sharing the symmetries of V). As usual one introduces the coordinate y and wave function $\chi(y)$:

$$y(\hat{x}) = \int_0^{\hat{x}} u(x) dx, \tag{A.5}$$

$$\chi(y) = u^{1/2}(\hat{x}) \hat{\psi}(\hat{x}). \tag{A.6}$$

We choose the positive root for $u(x)$, and $\chi(y)$ satisfies the Schrödinger equation:

$$-\frac{d^2\chi}{dy^2} + (1 + 2W(y))\chi = 0, \tag{A.7}$$

where the bounded potential W is given by:

$$W = \frac{1}{2\sqrt{u}} \frac{d^2\sqrt{u}}{dy^2} - \frac{\eta}{u^2}. \tag{A.8}$$

Especially for the single-well potential W is very well behaved. By choosing η judiciously it can be made monotonic and negative definite, behaving asymptotically

as y^{-2} . Therefore:

$$\chi^{[0]}(y) = C^{[0]}(1 + O(|y|^{-1}))e^{-|y|}. \quad (\text{A.9})$$

We observe that $\chi^{(1)}(y)$ cannot satisfy an identical asymptotic expression due to the presence of the decomposition surface Σ_2 where the wave function has to vanish. However this effect will be negligible far enough removed from Σ_2 , that is for $d_2 \ll 1$, which is guaranteed by $\alpha_2 > 0$. We will define the equivalent of eq. (A.9) in terms of:

$$C^{(1)}(g) \equiv e^{y(\hat{d})}\chi^{(1)}(y(\hat{d})). \quad (\text{A.10})$$

Then $\chi^{(1)}(y)$ behaves as eq. (A.9) in the annulus $D_2 - D_1$. (Apart from the $O(|y|^{-1})$ it also contains $O(|y(\hat{d})|^{-1})$ corrections). It is also clear that up to exponential corrections all the probability of the wave function $\chi^{(1)}$ is contained in the disk D_1 (and certainly in the disk D_2 of radius d_2). The same is obviously true for $\chi^{[0]}$. So for y_0 in $(D_2^{(1)} - D_1^{(1)}) \cap (D_2^{[0]} - D_1^{[0]})$ we have due to the normalization of χ :

$$|C^{[0]}|^2 \int_{D_0} \left| \frac{\chi^{[0]}}{C^{[0]}} \right|^2 = |C^{(1)}|^2 \int_{D_0} \left| \frac{\chi^{(1)}}{C^{(1)}} \right|^2, \quad (\text{A.11})$$

up to exponential corrections (D_0 is of course the disk of radius y_0 in y -space).

This trivial observation is instrumental in concluding equality of $|C^{[0]}|^2$ and $|C^{(1)}|^2$. For consider the functions $f_{\pm}(y)$ defined on \mathbb{R}^+ in terms of χ and C by

$$f_{\pm}(y) = \chi(\pm y)e^y/C. \quad (\text{A.12})$$

Then due to eqs. (A.9) and (A.10) $f_{\pm}(y)$ behaves as $1 + \text{const}/y$ in $D_2 - D_1$. The equation for f_{\pm} is given by

$$-\frac{d^2 f_{\pm}(y)}{2dy^2} + \frac{df_{\pm}(y)}{dy} + W(\pm y)f_{\pm}(y) = 0. \quad (\text{A.13})$$

If we can show that not only in $D_2 - D_1$ but in the whole of D_0 , $f_{\pm}^{[0]}$ and $f_{\pm}^{(1)}$ are bounded and coincide up to a positive power of g , then this guarantees by eq. (A.11) the desired result.

Actually this is sufficient to claim control over the error for $\Delta(g)$ since in terms of $C^{(1)}$ one easily derives

$$\begin{aligned} \Delta(g) &= 2g^{2\beta/(2+\beta)}|C^{(1)}|^2(1 + O(y(d)^{-1})) \\ &\times \exp\left(-\frac{2}{g} \int_0^1 \sqrt{2(V-E + g^{2\beta/(2+\beta)}\eta)} dx\right) \end{aligned} \quad (\text{A.14})$$

and all what remains to be calculated is $C^{[0]}$ from the single-well asymptotics. (Note: expanding the integral in powers of g , η will lead to a $O(g)$ contribution due to its support of $O(g^{2/(2+\beta)})$ in x -space – this contributes to the prefactor!)

Let us now study solutions to eq. (A.13) more carefully. We will only consider the $y > 0$ sector, the analysis for the $y < 0$ sector is identical. Since χ behaves asymptotically as $e^{\pm y}$, the general solution of eq. (A.13) can be written as:

$$f(y) = c_1 h(y) + c_2 k(y), \tag{A.15}$$

$$\lim_{g \rightarrow 0} h(y(g^\alpha)) = 1, \quad \lim_{g \rightarrow 0} e^{-2y(g^\alpha)} k(y(g^\alpha)) = 1. \tag{A.16}$$

Since W is regular we furthermore have that h , dh/dy , $e^{-2y}k$ and $e^{-2y}dk/dy$ are bounded uniformly on D_0 by a g -independent constant $M > 1$. This will be sufficient to prove $\lim_{g \rightarrow 0} \sup_{D_0} |f^{[0]}(y) - f^{(1)}(y)| = 0$.

Let u be the vector:

$$u(y) = \left(f(y), \frac{df}{dy}(y) \right); \tag{A.17}$$

it satisfies the first-order equation

$$\frac{du(y)}{dy} = A(y)u(y); \quad A(y) = \begin{bmatrix} 0 & 1 \\ 2W(y) & 2 \end{bmatrix}. \tag{A.18}$$

Its solutions are given by:

$$u(y) = \Phi(y)\Phi^{-1}(y_0)u(y_0), \quad \Phi(y) = \begin{bmatrix} h(y) & k(y) \\ \frac{dh}{dy}(y) & \frac{dk}{dy}(y) \end{bmatrix}. \tag{A.19}$$

Subtracting the equation for $u^{[0]}$ from that for $u^{(1)}$ we find the following expression for $\Delta u = u^{(1)} - u^{[0]}$ ($\Delta W \equiv W^{(1)} - W^{[0]}$), with $y < y_0$:

$$\begin{aligned} \Delta u(y) &= \Phi^{[0]}(y)\Phi^{[0]-1}(y_0)\Delta u(y_0) \\ &+ 2 \int_{y_0}^y \Phi^{[0]}(y)\Phi^{[0]}(\zeta)^{-1} \begin{bmatrix} 0 & 0 \\ \Delta W(\zeta) & 0 \end{bmatrix} u^{(1)}(\zeta) d\zeta. \end{aligned} \tag{A.20}$$

Using the fact that the wronskian for two solutions of eq. (A.7) is constant we derive:

$$\frac{d}{dy} (e^{-2y} \det(\Phi^{[0]}(y))) = 0. \tag{A.21}$$

Substituting the asymptotics for the equations we easily derive:

$$\det(\Phi^{[0]}(y)) = 2e^{2y}, \quad (\text{A.22})$$

and hence for $y < y_0$:

$$|\Phi^{[0]}(y)\Phi^{[0]}(y_0)^{-1}| \leq 4M^2. \quad (\text{A.23})$$

Therefore we get the following bound on $\Delta u(y < y_0)$:

$$|\Delta u(y)| \leq 4M^2 \left(|\Delta u(y_0)| + 4M \int_0^{y_0} |\Delta W(y)| dy \right). \quad (\text{A.24})$$

Since $|\Delta u(y_0)| = O(y_0^{-1})$ (which also equals the relative error (e_{sc}) in G^{tr} due to the semiclassical approximation) vanishes as a positive power of g for α_2 as in eq. (A.3), it suffices to restrict D_2 such that $\int_0^{y_0} |\Delta W(y)| dy$ vanishes ($y_0 \equiv y_{[0]}(d_2)$). It is natural to further restrict d_2 , such that $\hat{V}_{[0]}(\hat{x}) - \hat{V}(\hat{x})$ and $y_{[0]}(\hat{x}) - y_{(1)}(\hat{x})$ is small on D_2 ; this guarantees the required property for $\int_0^{y_0} |\Delta W| dy$. Recall that the corrections to $\hat{V}(\hat{x})$ are determined by $\gamma > 0$ in eq. (3.5), this also fixes

$$\hat{E}_{(1)} - \hat{E}_{[0]} \leq O(g^{2\gamma/(2+\beta)}), \quad (\text{A.25})$$

and we leave it to the reader to verify:

$$\sup_{\hat{x} \in D_2} |y_{[0]}(\hat{x}) - y_{(1)}(\hat{x})| \leq O(g^{2\gamma/(2+\beta)} \hat{d}_2^{\gamma+1+\beta/2}). \quad (\text{A.26})$$

This also bounds $\int_0^{y_0} |\Delta W(y)| dy$. The lower bound on α_2 is therefore

$$\alpha_2 > \frac{2}{2 + \beta + 2\gamma}. \quad (\text{A.27})$$

As required this implies vanishing $\hat{V}(\hat{d}_2) - \hat{V}_{[0]}(\hat{d}_2)$ for $\beta \leq 2$, to which we restrict ourselves in this paper. Furthermore one easily verifies that:

$$||C^{[0]}|^2 - |C^{(1)}|^2| \leq e_{sw} \leq O(g^{-1} d_2^{\gamma+1+\beta/2}) + O(g d_2^{-(\beta+2)/2}). \quad (\text{A.28})$$

This error is minimal for:

$$\alpha_2 = \frac{2}{2 + \gamma + \beta}, \quad (\text{A.29})$$

and leads to:

$$e_{sc} = e_{sw} = O(g^{\gamma/(2+\gamma+\beta)}). \quad (\text{A.30})$$

This completes the one-dimensional analysis.

In higher dimensions we will consider the case of isolated minima, with spherically symmetric $V_{[0]}$:

$$V(-\mathbf{e}_1 + \mathbf{x}) = |\mathbf{x}|^\beta (1 + O(|\mathbf{x}|^\gamma)). \quad (\text{A.31})$$

We can choose the disks D_1 and D_2 similar as in the one-dimensional case, especially the bounds on α_1 and α_2 we choose identical. Again one can prove (use results of ref. [3]) that up to exponential corrections the wave function is contained in the disk D_2 . One can now show that as in the one-dimensional case:

$$\sup_{\hat{x} \in D_2} |\psi^{(1)}(\hat{x})/\psi^{[0]}(\hat{x})| \leq O(g^{-1}d_2^{\gamma+1+\beta/2}) + O(gd_2^{-(\beta+2)/2}), \quad (\text{A.32})$$

we will not give details here. Basically one splits the wave function into a spherical and non-spherical part. From perturbation theory we know that up to $O(g^{2\gamma/(2+\beta)})$ all of the wave function is contained in the spherical part, which then can be treated as in one dimension. The only thing we still have to worry about is the surface integrals for the decomposition surfaces, but this is easily seen to give a relative error smaller than eq. (A.32).

In summary, one chooses a disk D around the minimum $\mathbf{x} = -\mathbf{e}_1$, which contains the wave function with probability 1 up to exponential corrections, but is still far separated from the decomposition surface Σ_2 . This disk shrinks to the minima for $g \rightarrow 0$ in the original coordinates \mathbf{x} , and in it the single-well and restricted wave functions satisfy the same Schrödinger equation up to small perturbations in the potential. Extracting the exponentially decaying part of the wave function leaves one (up to a constant C) with a bounded function h , which tends to 1 on the boundary of D . Stability of this bounded solution in the interior of D guarantees that the difference in h tends to 0 for $g \rightarrow 0$ uniformly on D , where the error is determined by the difference in the boundary condition and (an accumulation of) the difference in the potentials. Using the fact that both wave functions are normalized to 1 also guarantees that the constants C become equal for $g \rightarrow 0$. This then implies $e_{\text{sw}} \rightarrow 0$ for $g \rightarrow 0$.

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