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Landau-level spin waves and Skyrmion energy in the two-dimensional Heisenberg antiferromagnet

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A continuum perturbative analysis and the Schwinger-boson mean-field theory are used to evaluate the ground-state energy of the two-dimensional Heisenberg antiferromagnet in the presence of background topological density. The spin-wave spectrum is bunched into Landau levels. The gauge charge of the Schwinger bosons is half the charge of the Holstein-Primakoff spin waves. The mean-field Skyrmion creation energy vanishes with the ordered moment at the disordering transition.

The stable topological excitations, called Skyrmions, in the two-dimensional Heisenberg (isotropic) magnet have inspired researchers in condensed-matter and in high-energy physics.[1,2] Haldane has recently shown[3] that in the two-dimensional quantum antiferromagnet "hedgehog events," which describe Skyrmion creation and annihilation processes, give rise to quantum interference between their associated Berry phases. The abundance of hedgehog events affects the ground-state symmetry and excitations in the disordered phase.[3-5] It is the purpose of this paper to investigate the quantum corrections to the Skyrmion energy in the long-range-ordered phase near the transition.

Our Hamiltonian is the quantum Heisenberg model (QHM) on a bipartite square lattice,

$$H^{\mathrm{QHM}}=\sum_{ij} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j,\quad \mathbf{S}^2=S(S+1)\,, \tag{1}$$

where $J_{ij}>0$ for nearest neighbors (ij). If frustrating longer-range exchanges are not too strong, the long-wavelength properties of (1) are described by the Euclidean $(2+1)$-dimensional nonlinear σ model[3] (NLSM)

$$\mathcal{L}^{\sigma}=\frac{1}{2g}\int d^3x(\partial_\mu \mathbf{n})^2+i\sum_{\mathbf{x}}\zeta_{\mathbf{x}},\quad |\mathbf{n}|=1\,. \tag{2}$$

$\mathbf{n}$ is the $O(3)$ Néel field, g is the coupling constant, and $x^\mu=(x,y,ct)$, where c is the spin-wave velocity. Our units are chosen such that $\hbar=1$ and the lattice constant a is the unit of length. Summation over repeated indices is assumed. Haldane has calculated the Berry phases $\zeta_{\mathbf{x}}$ which are integer multiples of $(2S\,\mathrm{mod}\,4)\pi/2$. They count hedgehog tunneling events centered at plaquettes labeled by $\mathbf{x}$.

The continuous field $\mathbf{n}(\mathbf{x},t)$ defines a topological density in two dimensions given by

$$\rho(\mathbf{x},t)=\tfrac{1}{2}\,\partial_x\mathbf{n}\times\partial_y\mathbf{n}\cdot\mathbf{n}=\partial_x\mathbf{A}_y-\partial_y\mathbf{A}_x\,, \tag{3}$$

which defines the Néel vector potential $\mathbf{A}(\mathbf{x},t)$ (up to a gauge transformation). For periodic boundary conditions, the corresponding topological "flux" is quantized as

$$p(t)=\frac{1}{2\pi}\int d^2\mathbf{x}\,\rho(\mathbf{x},t)=0,\pm 1,\pm 2,\ldots\,. \tag{4}$$

p is the Pontryagin integer, which is conserved if $\mathbf{n}$ has no discontinuities. Hedgehog events are singular events in which $p\rightarrow p\pm 1$. For $g<g_c$, the NLSM ground state is long-range ordered and $\langle p\rangle=0$. Hedgehog events can connect the ground state only to higher-energy sectors and are therefore rare and unimportant. It is interesting to investigate how the different Pontryagin sectors become degenerate at the disordering transition.

The static configurations $\mathbf{n}^p$ with topological charge p and which minimize the spatial part of $\mathcal{L}^\sigma$ have the classical energy,

$$\mathcal{L}^{2d}=\frac{\gamma}{2}\int d^2x|\nabla\mathbf{n}^p|^2=4\pi\gamma p\,, \tag{5}$$

where $\gamma=c/g$ is the stiffness constant. $\gamma\sim JS^2$ for the large S near neighbor QHM. $\mathbf{n}^p$ are multi-Skyrmion configurations, or Belavin-Polyakov solitons, whose explicit form is given in Ref. 1. *Classically*, Skyrmions do not interact. Dimensional analysis of Eq. (5) shows that their energy is scale invariant.

Here we study the quantum energy of Eqs. (1) and (2) in the presence of an infinitesimal uniform topological charge $\rho(\mathbf{x},t)=\bar{\rho}$. This background represents a noncoplanar twist in the Néel order parameter, as experienced near the center of a large Skyrmion. We use two complementary approaches: First, the low-order spin-wave corrections to the classical energy are calculated perturbatively in the NLSM, Eq. (2). Although the numerical prefactors are cutoff dependent, this approach allows us to determine the first nonlinear contribution analytically, which would tell us something about Skyrmion interactions. The second approach will apply the Schwinger-boson mean-field theory (SBMFT) for the nearest-neighbor lattice model, Eq. (1). The SBMFT will be solved numerically in the range $0\leq 1/S\leq 5$. This regime, within the mean-field theory, corresponds to the weak-coupling ordered phase $g\leq g_c$ of the NLSM.

CONTINUUM MODEL: LINEAR SPIN WAVES

The small fluctuations of the Néel field are parametrized by

$$\mathbf{n}=\mathbf{n}^p(1-|\psi|^2)^{1/2}+\psi^1\hat{\mathbf{e}}_\theta+\psi^2\hat{\mathbf{e}}_\phi\,, \tag{6}$$

where $\psi=\psi_1+i\psi_2$. The unit vectors $\hat{\mathbf{e}}_\theta[\mathbf{n}^p],\hat{\mathbf{e}}_\phi[\mathbf{n}^p]$ describe the coordinate convention on the sphere at the point

$\mathbf{n}^p$. Since $\mathbf{n}^p$ is a classical configuration, the linear variations of $\mathcal{L}^\sigma$ vanish. Following Polyakov,[6] we expand the action to quadratic order about $\mathbf{n}^p$, and obtain

$$\mathcal{L}^\sigma \approx \mathcal{L}^\sigma[\mathbf{n}^p] + \mathcal{S}^{(2)} ,$$

$$\mathcal{S}^{(2)} = \frac{1}{2g}\int d^3x |(\partial_\mu + i2A_\mu)\psi|^2 + B_\mu^\alpha B_\mu^\beta[\psi^\alpha\psi^\beta - \delta_{ij}|\psi|^2] + (\partial_\mu \mathbf{n}^p)^2 + O(\psi)^3 . \quad (7)$$

Here $\mathbf{A}$ is the gauge field

$$A_\mu = \tfrac{1}{2}\hat{\mathbf{e}}_\theta \partial_\mu \hat{\mathbf{e}}_\phi , \quad (8)$$

and $B_\mu^\alpha \equiv \mathbf{n}^p \partial_\mu \hat{\mathbf{e}}_\alpha$ satisfies $(B_\mu^\alpha)^2 = (\partial_\mu \mathbf{n}^p)^2$.

$\mathbf{A}$ of Eq. (8) is an explicit construction of the Néel gauge field since it satisfies Eq. (3). Thus, the Néel gauge field has a geometrical interpretation. The spin-wave spectrum is that of two-dimensional relativistic bosons, $\psi(x)$, *of charge two,* moving in a "magnetic field" of magnitude ρ. (The plaquette has unit area.) A change in the transverse coordinate system on the sphere (e.g., changing the position of the north pole) is a gauge transformation under which (7) is invariant.

Integrating out the fluctuations ψ^*, ψ we obtain

$$E^p = \mathcal{L}^{2d}[\mathbf{n}^p] + \lim_{\beta\to\infty}\frac{1}{\beta}\sum_n \mathrm{Tr}\ln|(i\omega_n)^2 + (\partial_\mu + i2\mathbf{A}_\mu)^2 - (\partial_\mu \mathbf{n}^p)^2| + O(g) . \quad (9)$$

$\omega_n = 2n\pi/\beta$ are the Bose-Matsubara frequencies. In the Néel background $\mathbf{n}^p = \hat{\mathbf{z}}$, Eq. (9) recovers the standard spin-wave result. In (8) we find that the background fields $\mathbf{n}^p$, and $\mathbf{A}$ affect the perturbative correction to the energy in two ways: (i) a renormalization of γ, the classical stiffness constant, obtained by expanding $(\partial_\mu \mathbf{n}^p)^2$ out of the Gaussian determinant, and (ii) a "magnetic field" contribution to the spin waves zero-point energy.

Both contributions depend on the short-wavelength details (cutoff). Here we choose a circular Brillouin zone (with two degenerate spin-wave modes) and normalize the spin-wave sum such that the total number of states is one per site. Let us recall the standard solution to the problem of a two-dimensional nonrelativistic particle moving in a transverse magnetic field in the continuum. If the strength of the field is $\bar{\rho}$ then the energy levels in our dimensionless units are[7] $E_n = 4n\bar{\rho}$ with a degeneracy of $1/2\pi l^2$ for each Landau level, where the magnetic length is defined by $l = 1/(2\rho)^{1/2}$. Going back to Eq. (8), we perform the Matsubara sum in the usual way to obtain the energy per site as

$$e(\bar{\rho}) = 2\gamma\bar{\rho} + \frac{c\bar{\rho}}{\pi}\sum_{n=0}^{\pi/2\bar{\rho}-1}(4n\bar{\rho})^{1/2} . \quad (10)$$

We use the Euler-Maclaurin formula to convert the sum into an integral,

$$\sum_{n_1}^{n_2} F(n) = \int_{n_1}^{n_2} F(n)dn + \frac{F(n_1)+F(n_2)}{2} + \sum_1^\infty \frac{B_{2n}}{2n!}[F^{(2n-1)}(n_2) - F^{(2n-1)}(n_1)] . \quad (11)$$

The final result for the free energy per site is

$$e(\bar{\rho}) = c[2\bar{\rho}/g + \sqrt{2\pi}/3 - (2\pi)^{-1/2}\bar{\rho} - 0.13208\bar{\rho}^{3/2} + O(\bar{\rho})^2 + O(g)] . \quad (12)$$

The quantum spin fluctuations have two effects. They renormalize the coefficient of the term linear in $\bar{\rho}$, which is the zero-density chemical potential of the Skyrmions, and they also introduce higher-order terms in $\bar{\rho}$. This is indicative of the fact that though the Skyrmions are noninteracting classically, quantum fluctuations induce interactions. Of particular interest is the $(-\bar{\rho}^{3/2})$ term, which gives the system a negative compressibility at low Skyrmion density. The topological density would therefore tend to phase separate. This will result in clumping into localized regions of high topological density.

SCHWINGER-BOSON MEAN-FIELD ENERGY

For larger values of the quantum parameter, we use the Schwinger-boson mean-field theory,[8,9] which is the large-N theory of the SU(N) antiferromagnet, applied to the $N=2$ system. In the presence of background topological density, the mean-field theory is given by a quadratic Bose Hamiltonian,

$$H^{\mathrm{MF}} = \sum_{\langle ij\rangle, m=1,2} [\lambda(a_{im}^\dagger a_{im} + a_{jm}a_{jm}^\dagger) + Q(e^{i\theta_{ij}}a_{im}^\dagger a_{jm}^\dagger + e^{-i\theta_{ij}}a_{im}a_{jm})]^\dagger \mathcal{N}\lambda(2S+1) + \mathcal{N}4Q^2/J , \quad (13)$$

where $\mathcal{N}$ is the number of sites, and λ, Q are real variational parameters. $1/S$ serves as the quantum parameter. The quantitative correspondence between $1/S$ and g depends on the short-wavelength details of the model.

Read and Sachdev[4] have mapped the fluctuations about the mean-field saddle point to the large-N expansion of the CP^{N-1} field theory. Consequently, they have been able to derive an important correspondence between the phases θ_{ij} and the CP^{N-1} gauge field (for a lattice constant of unity):

$$\theta_{ij} \Rightarrow (-1)^{\mathbf{i}}\mathbf{A}\cdot(\mathbf{i}-\mathbf{j}) . \quad (14)$$

For the physical model $N=2$, $\mathbf{A}$ is none other than the Néel gauge field defined in Eq. (3). *Note in (13) that the charge of the Schwinger bosons with respect to* $\mathbf{A}$ *is unity.*

Equation (13) is diagonalized by following Hofstadter's solution of the tight-binding square lattice in a uniform perpendicular magnetic field.[10] To simplify our calculation, we choose the topological flux to be commensurate with a large unit cell, and the gauge field is taken to be

$$(\mathbf{A}_x, \mathbf{A}_y) = (0, x\bar{\rho}_q) = (0, 2\pi x/q), \quad q = \text{integer} . \quad (15)$$

The mean-field free energy per site of (13) is given by

$$e^{\mathrm{MF}} = \frac{1}{q}\sum_n \int_{-1}^{1} d\gamma\, n(\gamma)\omega_n(\gamma) - \lambda(2S+1) + \frac{4Q^2}{J} + 2JS^2 . \quad (16)$$

Here $n(\gamma)=\mathcal{N}^{-1}\sum_{\mathbf{k}}\delta(\gamma-\gamma_{\mathbf{k}})$, and $\gamma_{\mathbf{k}}=\frac{1}{2}(\cos k_x+\cos k_y)$. The spin-wave energies are $\omega_n(\gamma)=\{\lambda^2-[Q\xi_n(\gamma)]^2\}^{1/2}$, where $\xi_n(\gamma)$ is the nth eigenvalue of the Hermitian Harper's matrix $H_{ij}^{(q)}(\gamma)$, whose nonzero elements are given by

$$H_{jj}^{(q)}=-2\cos\left[k+\frac{2\pi}{q}j\right],\quad j=1,2,\ldots,q\,,$$
$$H_{j,j+1}^{(q)}=-e^{ik},\quad j=1,2,\ldots,q-1,\quad H_{1,q}^{(q)}=e^{-ik}\,, \tag{17}$$

where $k=\cos^{-1}\gamma$. The matrix elements below the diagonal are determined by the Hermiticity condition.

The variational parameters $\lambda(\bar{\rho},S)$, and $Q(\bar{\rho},S)$ are determined by two equations:[8] the constraint equation

$$\frac{1}{q}\sum_n\int d\gamma\, n(\gamma)\frac{\lambda}{\omega_n(\gamma)}=2S+1\,, \tag{18}$$

and the spin-wave velocity equation

$$\frac{1}{q}\sum_n\int d\gamma\, n(\gamma)\frac{\xi_n(\gamma)^2}{\omega_n(\gamma)}=\frac{8}{J}\,. \tag{19}$$

Equations (12), (14), and (15) can be combined to obtain the mean-field energy per site

$$e^{\mathrm{MF}}=-4Q^2/J+2JS^2\,. \tag{20}$$

We use the following numerical procedure to solve the mean-field equations: For each value of $q=16,32,64$, we diagonalize the $q\times q$ Harpers matrix (17) on a grid of γ values. The left-hand sides of Eqs. (18) and (19) are computed to obtain $S(\lambda,\bar{\rho}_q)$, and $Q(\lambda,\bar{\rho}_q)$, respectively. The Skyrmion chemical potential is given by

$$\mu^{\mathrm{MF}}(\bar{\rho},S)=4\pi\frac{\partial e^{\mathrm{MF}}}{\partial\bar{\rho}}\,. \tag{21}$$

In Fig. 1 we plot our results for Eq. (21) in the ordered-phase regime. The solid line is the mean-field ordered moment m_0 which goes as $m_0/S=1-S_c/S$, where $S_c=0.196\,60$ is the "critical spin." By extrapolation, Fig. 1 shows that for vanishing topological density, μ, or the creation energy of Skyrmions, is proportional to the ordered moment.

Note, it is known that for $\bar{\rho}=0$, the SBMFT energy yields *twice* the $O(1/S)$ correction as the Holstein-Primakoff (HP) spin-wave result calculated by Anderson:[11]

$$e^{\mathrm{MF}}=-JS^2[2+0.632/S+(0.315/S)^2]\,,$$
$$e^{\mathrm{HP}}=-JS^2[2+0.316/S+O(1/S)^2]\,. \tag{22}$$

This discrepancy is due to the summation over *two Schwinger bosons* per site, while there is only *one HP boson site.* The error in the SBMFT is in relaxing the constraint in the large-N approximation. This error, however, is compensated by the factor-of-2 difference between the *charge with respect to* **A** of the spin waves and the Schwinger bosons. [Compare Eqs. (9) and (13).] Therefore, μ^{MF} and μ^{HP} *agree up to order* $1/S$.

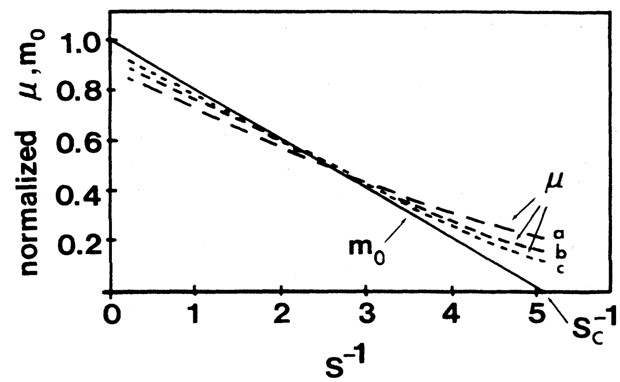

FIG. 1. The normalized mean-field chemical potential for Skyrmions $\mu^{\mathrm{MF}}(S)/\mu^{\mathrm{MF}}(\infty)$, for several background uniform topological densities $\bar{\rho}_q$ equal to: curve a, $\pi/8$; curve b, $\pi/16$; and curve c, $\pi/32$. The solid line is the mean-field normalized ordered moment m_0/S, which vanishes linearly at the critical spin $S_c=0.196\,60$.

To summarize, we have carried out two separate calculations which explicate different aspects of the energy of Skyrmions in the ordered phase of the quantum antiferromagnet. The spin-wave calculation is valid for small g (large spin) and shows that quantum fluctuations induce the Skyrmions to clump together in a spatially nonuniform manner. The numerical calculation is based on mean-field theory. It shows that the chemical potential for Skyrmions vanishes continuously at the disordering transition. This confirms the large-N analysis of the CP^{N-1} model in the massive (disordered) phase,[1,4] where the gauge field energy goes as ρ^2. Since mean-field theory does not treat properly the critical fluctuations near the transition, we cannot trust our critical exponent (one), and we can regard the correspondence between $m_0(g)$ and $\mu(g)$ as merely suggestive. The analogy between the spin waves in the background of a topological density and the magnetic-field problem might be further explored in other Heisenberg systems, such as magnetic bubbles in the two-dimensional ferromagnet and Heisenberg spin glasses.

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