

Quantized Hall insulator: Transverse and longitudinal transport

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(Received 1 October 1996)

We model the insulator neighboring the $1/k$ quantum Hall phase by a random network of puddles of filling fraction $1/k$. The puddles are coupled by weak tunnel barriers. Using Kirchoff's laws we prove that the macroscopic Hall resistivity is quantized at kh/e^2 and independent of magnetic field and current bias, in agreement with recent experimental observations. In addition, for $k > 1$ this theory predicts a nonlinear longitudinal response $V \sim I^\alpha$ at zero temperature and $V/I \sim T^{1-1/\alpha}$ at low bias. α is determined using Renn and Arovas's theory for the single junction response [Phys. Rev. B **51**, 16 832 (1995)] and is related to the Luttinger liquid spectra of the edge states straddling the typical tunnel barrier. The dependence of $V(I)$ on the magnetic field is related to the typical puddle size. Deviations of $V(I)$ from a pure power are estimated using a series/parallel approximation for the two-dimensional random nonlinear resistor network. We check the validity of this approximation by numerically solving for a finite square lattice network. [S0163-1829(97)03615-1]

I. INTRODUCTION

The "Hall insulator" defines a peculiar insulating state in which the longitudinal resistivity ρ_{xx} diverges in the limit of zero temperature and frequency yet the Hall resistivity ρ_{xy} remains *finite*. Such a behavior of ρ_{xy} has been argued to be a quite generic property of disordered single-electron models,¹⁻⁴ which follows from the relation $\sigma_{xy} \sim \sigma_{xx}^2$ in the limit of vanishing conductivities σ_{xx} and σ_{xy} ,⁵ that is to say,

$$\lim_{\sigma_{xx} \rightarrow 0} \rho_{xy} = \sigma_{yx} / (\sigma_{xx}^2 + \sigma_{xy}^2) < \infty. \quad (1)$$

Experimentally, several groups have observed a Hall insulating behavior in strong magnetic fields both in three-dimensional samples⁶ and in the quantum Hall (QH) regime.^{7,8} In the global phase diagram of Kivelson, Lee, and Zhang⁹ the entire insulating phase surrounding the QH liquid phases is predicted to exhibit Hall insulating behavior with $\rho_{xy} \sim B/nec$, as in a *classical* Hall conductor, in agreement with the data of, e.g., Ref. 7. (Here B is the perpendicular magnetic field and n is the electron density.)

Confusion has been compounded recently by measurement of the Hall voltage near the transition between a $1/3$ QH liquid and the insulator,⁸ which indicates a different behavior of ρ_{xy} in the insulating phase: quite remarkably, it preserves the quantized value $3h/e^2$ over a finite range of the magnetic field (the parameter that drives the transition) *beyond* the critical point. Moreover, the Hall voltage $V_H(I)$ is linear in the low current range where the longitudinal voltage $V(I)$ exhibits an insulatinglike nonlinear dependence. A deviation from the quantized Hall resistance, approaching a linear rise as a function of B , is observed only deeper in the insulating phase. This persistence of the QH plateau cannot

be explained by means of transport models based on hopping between strongly localized single-electron sites, as such models do not pose any particular restriction on the *value* of the finite Hall coefficient.

In this paper we propose a transport model that reflects the prominent phenomenology of the exotic insulating state described above, hereon dubbed "a quantized Hall insulator" (QHI). It is clear that one needs to take into account both electron interactions, which are responsible for the fractional QH effect, and the random potential. This task is manageable in the limit of slow potential variations with respect to the magnetic length. The incompressibility of the electron liquid at magic densities creates puddles of QH liquid at these densities in the shape of the equipotential contours.

Thus we extend Chalker and Coddington's network model¹⁰ from the integer to the fractional QH regime. In place of the semiclassical single-electron orbits, the transport here is conducted through a random network of edge states surrounding the puddles of $1/k$ QH liquid defined to be of density $n = B/k\phi_0$, where ϕ_0 is a flux quantum. The edge states are connected to each other by tunnel barriers. As we show below, if the percolating network of edge states belongs to a *single fraction* $1/k$, ρ_{xy} of the entire network acquires the quantized value $k\hbar/e^2$.¹¹ This quantization is *not sensitive* to the details of the dissipative part of the network's response.

Since edge states tunneling between a fractional QH liquid involves a power law density of states, the longitudinal current-voltage characteristic of the system $V(I)$ is generally nonlinear and $dV/dI \rightarrow \infty$ for $I \rightarrow 0$ in the weak tunneling limit. Below we calculate the longitudinal response of the edge states network and relate its properties to the total carrier density, the magnetic field, and the potential fluctuations.

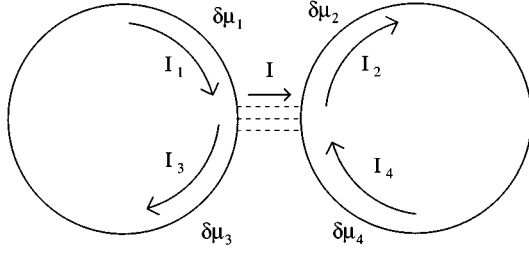


FIG. 1. Single junction between puddles.

In the framework of the present paper we do not elaborate on the justification of the puddle model; rather we focus on its consequences on transport properties, some of which are yet to be confronted with experiment. It is worthwhile pointing out, though, that the true ground state of certain realistic systems is quite likely to be well imitated by such a model. In restricted regions of the sample, where considerable variations in the disorder potential occur over length scales much larger than the magnetic length, the formation of puddles of incompressible liquid is energetically favorable. Since transport inside such a puddle is dissipationless, a current carrying path that crosses the entire sample is expected to be dominated by channels that “hop” between neighboring puddles at places of minimal separation.

In Sec. II we prove that the Hall resistance of a $1/k$ puddle network is quantized at $\rho_{xy} = k\hbar/e^2$. In Sec. III we calculate the nonlinear longitudinal response $V(I)$ of the network. Deviations from a pure power-law behavior are estimated in Appendix A; the parameters of the model are related to the magnetic field and potential fluctuations (using the theory of Renn and Arovas¹² for single QH tunnel junctions) in Appendix B. In Sec. IV we summarize our main results and point out some open questions and suggestions for experimental tests of our model.

II. THE HALL RESISTANCE OF A PUDDLE NETWORK

Consider a random two-dimensional network, combined of the basic elements schematically depicted in Fig. 1. Circles denote the “puddles”; each couple of puddles is separated by a tunnel junction that involves four edge currents I_1 – I_4 . By current conservation the tunneling current I is given by

$$I = I_1 - I_3 = I_2 - I_4. \quad (2)$$

The macroscopic theory of a Hall liquid in a confining potential yields a fundamental relation between the excess chemical potentials at the edges $\delta\mu_i$ and the edge currents¹³

$$\delta\mu_i = \text{sgn}(B) \frac{\hbar}{e^2} k I_i, \quad (3)$$

where I_i 's are positive in the clockwise direction around the puddle. Equations (2) and (3) yield a simple proportionality between the Hall voltage and the tunnel current

$$V_H \equiv \delta\mu_1 - \delta\mu_3 = \delta\mu_2 - \delta\mu_4 = \text{sgn}(B) \frac{\hbar}{e^2} k I. \quad (4)$$

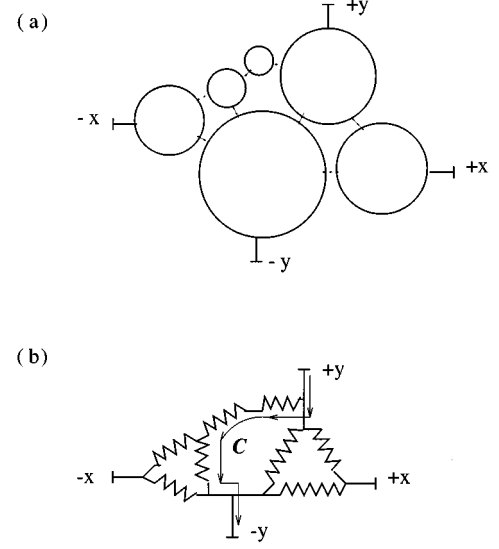


FIG. 2. (a) Typical puddle network (with $N_v = 6$ and $N_b = 8$). (b) Corresponding equivalent circuit. Path C is denoted by arrows.

The relation between the *longitudinal* voltage drop and the tunneling current through the barrier is

$$V(I) \equiv \delta\mu_1 - \delta\mu_2 = \delta\mu_3 - \delta\mu_4, \quad (5)$$

which is, in general, a nonlinear function. Here we assume symmetry under reversal of the magnetic field $V(I, B) = V(I, -B)$, which is expected for dissipative current transport across a narrow channel.

We now consider the response of a random network of puddles and tunnel barriers with two current leads at $-x$ and $+x$ and two voltage leads at $-y$ and $+y$. The network is described by a general two-dimensional graph with N_v vertices at the locations of the puddles and N_b bonds for each of the tunnel barriers (see, e.g., Fig. 2). The two-dimensional layout of the puddles network ensures that bonds do not cross.

Henceforth we shall assume that all quantum interference effects take place within the tunnel barrier length scales L_{ij} beyond which dissipation due to low-lying edge excitations destroys coherence between tunneling events. Thus the response of the puddles network is given by classical Kirchoff laws. First, current conservation at each vertex (puddle) i is given by

$$\sum_{j \in \{(ij)\}} I_{ij} = 0, \quad i = 1, \dots, N_v, \quad (6)$$

where $\{(ij)\}$ denotes the set of bonds emanating from i . Second, the sum of voltage differences around each plaquette p is given by

$$\sum_{(ij) \in p} V_{(ij)}(I_{ij}) = 0, \quad p = 1, \dots, N_p, \quad (7)$$

where $V_{(ij)}(I_{ij})$ is the nonlinear function of Eq. (5). A total current I is forced through the network through a lead coming from $-x$ and leaving toward $+x$. There are no currents flowing through external leads in the $\pm y$ directions. It is easy to prove the following:

Lemma. The currents I_{ij} in the network are completely determined by I .

The proof uses Euler's theorem for two-dimensional graphs¹⁴

$$N_v + N_p - N_b = 1. \quad (8)$$

Thus the number of Kirchoff equations (6) and (7) is $N_v + N_p$, which exceeds the number of unknown currents N_b by one. The additional equation determines that the current flowing out of the $+x$ lead must be, of course, I . Q.E.D.

As shown above, the Hall relations (4) have no effect on the currents I_{ij} . The total transverse voltage V_y is given by choosing any path of bonds \mathcal{C} that connects the $-y$ lead to the $+y$ lead (see Fig. 2) and summing the voltages

$$V_y = \sum_{i \in \mathcal{C}} V_{i,i+1}(I_{i,i+1}, |B|) + \text{sgn}(B) k \frac{\hbar}{e^2} \sum_{i \in \mathcal{C}, (ij)'} I_{i,j}, \quad (9)$$

where $(ij)'$ denote all currents entering vertex i from $-x$. By global current conservation, the second term is proportional to the total current. Defining the Hall voltage V_H to be the antisymmetric component of V_y we thus obtain

$$V_H = \text{sgn}(B) k \frac{\hbar}{e^2} I, \quad (10)$$

which yields a quantized Hall resistance of $\rho_H = k(\hbar/e^2)$ that is completely independent of B and I .

This relation should hold as long as the network does not involve appreciable contributions from edge states of puddles of different k values. The width of the QHI regime therefore depends on the relative abundance of different density puddles, which depends in turn on the distribution of potential fluctuations. As the magnetic field increases, a wide distribution of potential minima will create mixed phases with puddles of different densities. Relation (4) does not apply for tunneling between different $1/k$ QH liquids and thus the above analysis fails for the mixed phase.

III. THE NONLINEAR LONGITUDINAL TRANSPORT

The dissipative response in the model introduced above is associated with the longitudinal transport through the tunnel barriers. The barriers connect edge states of neighboring puddles of density $n = B/k\phi_0$, and we assume henceforth that k is the same in all puddles.

A nonlinear current-voltage relation for a tunnel junction between $1/k$ QH liquids was proposed by Wen,¹⁵ who mapped the fractional QH edge states to chiral Luttinger liquids. For small currents, the relation is a power law^{15,16}

$$I \sim \text{sgn}(V) V^{2g-1}, \quad (11)$$

where $g = k$ is the Luttinger liquid interaction parameter (and is equal to unity for the integer Hall liquid).

Renn and Arovas¹² (RA) have extended Wen's result to long tunnel barriers following Giamarchi and Schulz's renormalization-group equations for disordered Luttinger liquids.¹⁷ They consider the "disordered antiwire" geometry, i.e., a barrier of length with n_t tunnel couplings of av-

erage magnitude t . In the small current limit they obtained that g gets renormalized $g \rightarrow \tilde{g} > k$ and the longitudinal response is

$$V_{\text{RA}}(I) \approx V_0 \text{sgn}(I) \left(\frac{|I|}{I_0 D} \right)^{1/(2\tilde{g}-1)}, \quad (12)$$

where

$$V_0 = \frac{\hbar v}{e l},$$

$$I_0 = \frac{e v}{2 \pi l k},$$

$$D \approx \frac{n_t |t^2|}{2 \pi v^2 \hbar^2}. \quad (13)$$

Here v is the edge state velocity and $l = \sqrt{\hbar c / e B}$ is the magnetic length.

Here we consider a network of RA's junctions and assume that the dephasing time is short enough that the tunneling events through consecutive junctions in the network are incoherent (coherent backscattering effects are included in RA's calculation of the single junction). Our model consists of a random network of classical nonlinear resistors, each characterized by a power α_n and a conductance prefactor D_n ,

$$\frac{V_n}{V_0} = \text{sgn}(I) \left(\frac{|I_n|}{I_0 D_n} \right)^{\alpha_n}. \quad (14)$$

By Eq. (13) we assume that V_0 and I_0 are weakly dependent on the barrier height fluctuations and magnetic field, compared, e.g., to D . Thus, for simplicity, they are taken to be uniform in the entire network. The network of junctions with $D_n \leq 1$ is assumed to percolate through the sample. Thus we can choose D_n, α_n to be random variables whose distribution is bounded by

$$0 \leq D_n \leq 1,$$

$$(2k-1) \geq \alpha_n \geq 1/(2k-1). \quad (15)$$

In Appendix A we estimate the magnetic-field dependence of the average conductance prefactor to be

$$\bar{D}(B) \propto \exp \left[-k n_p \left(\frac{B - B_c}{2B_c} \right)^2 \right], \quad (16)$$

where n_p is the typical number of electrons in a puddle and B_c is the magnetic field at which the puddles percolate through the sample. The average power law is estimated using RA's renormalization-group equations. We find that (see Appendix A), in the limit of small D ,

$$\bar{\alpha}(B) \sim \frac{1}{2k-1} + \frac{k-3/2}{2k-1} \bar{D}(B), \quad (17)$$

and for $\bar{D} \rightarrow D_c = (2 \ln 2 - 1)$

$$\bar{\alpha} \approx \frac{1}{2 + 3 \sqrt{D_c - \bar{D}}}. \quad (18)$$

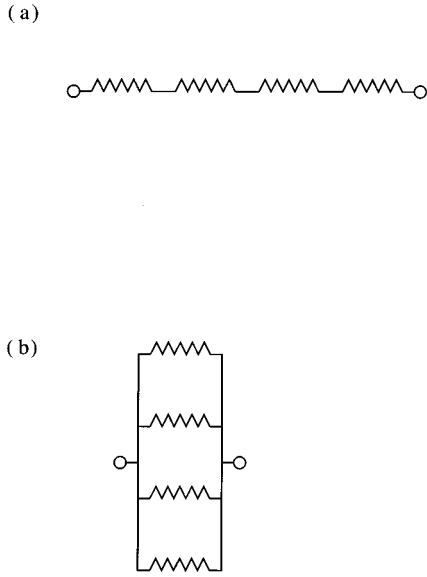


FIG. 3. (a) S -type element ($N_s=4$). (b) P -type element ($N_p=4$).

We have solved Eqs. (6) and (7) numerically, using a Levenberg-Marquardt algorithm, for square lattices of sizes up to 5×5 . The distributions of (D_n, α_n) were taken to be

$$P_1(D) = \Theta(D) - \Theta(D-1),$$

$$P_2(\alpha) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(\alpha - \bar{\alpha})^2}{2\sigma^2}\right\}. \quad (19)$$

We take the variance σ^2 according to our estimate in Appendix A to be 5–10 times smaller than the mean $\bar{\alpha}$. The numerical results in the regime $I_0/10 < I < I_0$, averaging over 5 realizations of disorder, can be summarized by the averaged network's I - V response

$$\frac{\ln V}{\ln I} \equiv \alpha_{\text{eff}} = \bar{\alpha} \pm \epsilon \quad (20)$$

where $\epsilon \sim 10^{-2} - 10^{-3}$. That is to say, in the moderate current regime *the total voltage-current relation is given quite well by the average power law*. In the extremely small current limit, one expects Eq. (20) to break down since due to the power-law resistors, the currents choose to flow through percolating networks of highest power laws. In this regime the numerical algorithm also fails to converge properly.

In order to better estimate the corrections to the pure power law, we examine a toy model dubbed the parallel-series (PS) network (see Appendix B for details). This model comprises a random combination of serial and parallel connections of elements P and S where P is composed of N_p resistors in parallel, and S is its dual, a linear chain of N_s resistors in series (see Fig. 3). The S and P components can be created from an ordinary two-dimensional network by a three-peaked distribution of D_n 's (shorts, disconnections, and resistors of $D_n = 1$). This model is symmetric on average with respect to exchange of the x and y directions and hence is an adequate description of macroscopically isotropic samples.

As we show in Appendix B, for currents that obey $\sigma^2 \ln(I/I_0) \ll \bar{\alpha}$,

$$\frac{V}{V_0} = \left(\frac{I}{I_0}\right)^{\alpha_{\text{eff}}}, \quad \alpha_{\text{eff}} = \bar{\alpha} + \left(\frac{\sigma^2 \ln(I/I_0)(1 - 1/\bar{\alpha})}{4}\right). \quad (21)$$

The deviation of α_{eff} from $\bar{\alpha}$ is positive for $\bar{\alpha} < 1$. This indicates that in the ‘‘insulating’’ regime, although serial and parallel connections are equally represented, parallel configurations dominate at low currents. The situation is reversed in the QH liquid side of the transition, where $\bar{\alpha} > 1$, while at the critical filling fraction (where $\bar{\alpha} = 1$) the S and P elements balance each other and $\alpha_{\text{eff}} = \bar{\alpha}$. We note that under a duality transformation, which exchanges each resistor in the network by a perpendicular resistor with (V_n/V_0) and (I_n/I_0) interchanged, the I - V characteristic of the whole network is inverted: $\bar{\alpha} \rightarrow 1/\bar{\alpha}$, $\sigma \rightarrow \sigma/\bar{\alpha}^2$, and consequently $\alpha_{\text{eff}} \rightarrow 1/\alpha_{\text{eff}}$, which is consistent with our requirement of macroscopic isotropy.

We also note that the deviations of the macroscopic I - V curve from a pure power law are at most *logarithmic* in the driving current. This gives us a sizable regime of current in which we can expect the curve to be well fitted by a pure power law ($\alpha_{\text{eff}} = \bar{\alpha}$):

$$I_0 e^{-1/\sigma^2(1-1/\bar{\alpha})} < I < I_0. \quad (22)$$

In comparing the results of the PS model to the square lattice simulations we find that the correction to a pure power law in the numerical results is smaller by at least a factor of 10 than the results of the PS model (21). We suggest that the difference arises due to the fact that the PS model assumes greater inhomogeneity in D_n as mentioned before. Equation (21) can therefore be regarded as an estimate of the upper limit on the discrepancy between the macroscopic α_{eff} and $\bar{\alpha}$ at moderate currents. The principal conclusion to be taken away from this calculation is that due to the self-averaging property, the macroscopic I - V is directly related to the physics of the single junction and the nonlinear tunneling response between fractional quantum Hall edge states.

IV. SUMMARY AND FINAL REMARKS

As demonstrated in the previous sections, the QHI phase observed in proximity to a QH liquid can be modeled by a network of puddles. Although similar in spirit to the semiclassical percolation description of Ref. 10, it naturally incorporates the electron interaction effects under the same assumption: smoothly varying potentials relative to the magnetic length l . The most important feature of this model is that, in contrast to models based on single-electron hopping, it yields a quantized Hall resistance. The quantization is not affected by the nonlinearity of the dissipative part of the response. The latter is studied for a $1/k$ QHI with $k > 1$, yielding a power-law behavior of the longitudinal I - V curve that is closely determined by the behavior of an average single junction between adjacent puddles. Deviations from a pure power law are at most of order $\sigma^2 \ln(I)$ (where σ is the

variance of the power distribution), estimated in Appendix A to be typically small.

The magnetic-field dependence of the average tunneling rate is Gaussian, as shown in Eq. (16), with a width defined by the inverse number of electrons in a typical puddle. Thus smaller puddle sizes allow a larger regime of the quantized Hall insulator phase. However, if these incompressible puddles are too small, it means that our assumption of slowly varying potential becomes invalid.

We note that the integer QH case of $k=1$ implies all $\alpha_n=1$ throughout the network. that is to say, the puddle model reduces naturally to a random Ohmic resistor network with conductances proportional to Eq. (16). Interference effects between junctions¹⁰ are ignored here since we assume an inelastic scattering length of the order of interjunction separation, an assumption that breaks down at low enough T .

Our analysis so far has concentrated on the nonlinear transport of tunnel junctions, applicable for large enough bias and low T . At finite T , transport in the junctions, and hence through the entire network, crosses over to linear response at sufficiently low bias $khi/e < k_B T$, where $i = I/N_y$ is the average current through single junction and N_y is the typical number of junctions across the sample. The linear conductivity is then predicted to vary as a power law of temperature,¹² i.e., $V \propto IT^{1-1/\bar{\alpha}}$. A temperature-dependent measurement of the resistance in the Ohmic regime can thus provide a further test of our model. In addition, for a given T the crossover from a linear to nonlinear $I(V)$ can provide an estimate of N_y .

One of the most interesting implications of our suggested puddle-network model is that the insulating phase, surrounding the fundamental QH liquids in the phase diagram of Ref. 9, is not a homogeneous phase. Restricted regions in the phase diagram that are in proximity to specific $1/k$ QH liquids are dominated by weakly coupled puddles of the corresponding liquid. It is therefore implied that a measurement of ρ_{xy} as a function of magnetic field at moderate disorder may exhibit plateaus at odd integer multiples of h/e^2 , even though the longitudinal transport indicates an insulatinglike behavior. The width of the plateaus is expected, however, to depend on details of the disorder potential in the sample. The width of the ‘‘mixed phases,’’ where ρ_{xy} rises with magnetic field between consecutive plateaus, is *not* expected to vanish for $T \rightarrow 0$ as in the QH liquid regime.

Finally, we would like to comment on an open problem with regard to a comparison of this theory with the experimental results of Ref. 8. The experiment has indicated a duality symmetry between I - V curves at opposite sides of the $1/3$ QH liquid-to-insulator transition. This phenomenon was interpreted in terms of charge-vortex duality or, equivalently, as particle-hole symmetry.⁸ In the puddle-network model, such duality would be observed if at $B > B_c$ each tunnel barrier with response $I = F(V)$ is related to a narrow channel formed at $B' < B_c$, such that $I' = F^{-1}(V')$. However, recent theories for a single scatterer in a narrow channel^{15,16} do not yield this relation. The multiple tunneling case¹² has only treated electron tunneling in the large barrier limit of $B > B_c$. Resolution of this point is left to further research.

ACKNOWLEDGMENTS

We thank D. Shahar for discussions that motivated this work and acknowledge useful conversations with D. Arovav, Y. Avron, D. Bergman, Y. Gefen, C. Henley, U. Sivan, S. Sondhi, and D. C. Tsui. This work was partly supported by the Technion–Haifa University Collaborative Research Foundation, the Fund for Promotion of Research at the Technion, and a grant from the Israeli Academy of Sciences.

APPENDIX A: DEPENDENCE OF DISTRIBUTIONS ON MAGNETIC FIELD

To facilitate a comparison with experiment, we must somehow relate the average and mean square deviations of D_n and α_n in Eqs. (19) to the external magnetic field B . Here we make substantial use of the results of Renn and Arovav,¹² which allow us to express D and α in terms of the semiclassical tunneling probability at the junction. The renormalization-group equations of Refs. 12 and 17 connect α to D as follows. In the insulating limit $D_n \rightarrow 0$,

$$\alpha_n = \frac{1}{2g_n - 1} \approx \frac{1}{2k - 1} + \frac{(k - 3/2)D_n}{2k - 1}; \quad (\text{A1})$$

in the regime $D_n \approx D_c \equiv (2\ln 2 - 1)$, we get

$$\alpha_n \approx \frac{1}{2 + 3\sqrt{D_c - D_n}}. \quad (\text{A2})$$

Equations (A1) and (A2) relate α_n to D_n in the range $1/(2k - 1) \leq \alpha_n \leq 1/2$; the analysis of Refs. 12 and 17 is not applicable closer to the QH liquid/insulator transition, where $1/2 < \alpha < 1$. Note that the effect of an increasing tunneling rate is to interpolate between the limits $\alpha_n = 1/(2k - 1)$ and $\alpha_n = 1/2$.

We assume that the potential fluctuations are bounded and have a characteristic length scale of fluctuations l_V . This length scale also represents the typical linear size of the puddles, which will turn out to be an important parameter in the following discussion.

Since the puddles are incompressible, a change in the magnetic field δB near the percolation field B_c will shrink the puddles by a linear distance δl , which is related to δB by

$$\frac{\delta l}{l_V} = \frac{\delta B}{2B_c}. \quad (\text{A3})$$

The tunneling rate of an electron in the lowest Landau level through a quadratic potential barrier $V(x, y) = \frac{1}{2} V''(-x^2 + y^2)$ is solved by mapping the problem to a one-dimensional Hamiltonian given by

$$H = \frac{1}{2m} p^2 - \frac{1}{2} V'' x^2, \quad (\text{A4})$$

where the ‘‘tunneling mass’’ is $m = \hbar^2/V''l^4$ (l being the magnetic length). Using the WKB expression for tunneling at energy $-V_B$,

$$\begin{aligned}
D &= D_0 \exp\left(-\frac{2\pi}{\hbar} V_B \sqrt{\frac{m}{V^n}}\right) \\
&= D_0 \exp\left[-\pi \left(\frac{\delta l}{l}\right)^2\right] \\
&= D_0 \exp\left[-\pi \left(\frac{\delta B}{2B_c}\right)^2 (l_V/l)^2\right]. \quad (\text{A5})
\end{aligned}$$

The factor $x = (l_V/l)^2$ is roughly k times the number of electrons in the puddle and it determines the field dependence of the tunneling rate near percolation. In a specific junction in the network, it is given by a random variable x_n with average $\langle x \rangle$ and variance σ_x such that

$$\begin{aligned}
D_n &\sim D_0 \exp\left[-x_n \left(\frac{\delta B}{2B_c}\right)^2\right], \\
\bar{D} &= D_0 \exp\{-\bar{x} (\delta B/B_c)^2\}, \\
\sigma_D &\approx \left(\frac{\delta B}{B_c}\right)^2 \sigma_x \bar{D}. \quad (\text{A6})
\end{aligned}$$

Employing Eqs. (A1) and (A2), we find that $\bar{\alpha}$ and σ of Sec. III are given by

$$\begin{aligned}
\bar{\alpha} &\approx \frac{1}{2k-1} + \frac{(k-3/2)\bar{D}}{2k-1}, \\
\sigma &\approx \frac{(k-3/2)\sigma_D}{2k-1} \quad (\text{A7})
\end{aligned}$$

for $\bar{D} \rightarrow 0$ and

$$\begin{aligned}
\bar{\alpha} &\approx \frac{1}{2+3\sqrt{D_c-\bar{D}}}, \\
\sigma &\approx \left(\frac{3}{2\bar{\alpha}^2} \frac{1}{\sqrt{D_c-\bar{D}}}\right) \sigma_D \quad (\text{A8})
\end{aligned}$$

for $\bar{D} \rightarrow D_c$. Substituting, e.g., $k=3$ and $\delta B/B_c < 10^{-1}$, we find that σ is typically 10 times smaller than $\bar{\alpha}$, which is the values we have used in the numerical simulations of the square lattice network.

APPENDIX B: DERIVATION OF $V(I)$ IN THE PS MODEL

Consider a system of S and P elements, which are serial and parallel connections of power-law resistors, respectively. We first derive the local current-voltage response of elements P and S separately. In P , the average current per parallel unit is related to the voltage V_p by

$$I_p = \left\langle \frac{1}{N_p} \sum_{n=1}^{N_p} I_n \right\rangle = I_0 \left\langle \frac{1}{N_p} \sum_{n=1}^{N_p} \left(\frac{V_p}{V_0}\right)^{1/\alpha_n} \right\rangle, \quad (\text{B1})$$

where the angular brackets denote averaging over the distribution $P_2(\alpha)$ [Eq. (19)], which yields

$$\frac{I_p}{I_0} = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} d\alpha \exp\left\{-\frac{(\alpha-\bar{\alpha})^2}{2\sigma^2}\right\} \left(\frac{V_p}{V_0}\right)^{1/\alpha}. \quad (\text{B2})$$

In the saddle-point approximation [valid for $\sigma^2 \ln(I_p/I_0) \ll \bar{\alpha}$],

$$\frac{V_p}{V_0} = \left(\frac{I_p}{I_0}\right)^{\alpha_p}, \quad \alpha_p = \bar{\alpha} - \frac{\sigma^2 \ln(I_p/I_0)}{2\bar{\alpha}}. \quad (\text{B3})$$

Note that the approximation breaks down in the limit of very small currents. Equation (B3) implies that the contribution of a purely parallel configuration to the network outweighs the significance of better conducting channels. This produces a positive shift of the power law that is enhanced at small currents. The response of a single S -type element indicates an opposite trend: similarly to Eq. (B3), the average voltage V_s (per serial unit) is related to the current I_s by

$$\frac{V_s}{V_0} = \left\langle \frac{1}{N_s} \sum_{n=1}^{N_s} \frac{V_n}{V_0} \right\rangle = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} d\alpha \exp\left\{-\frac{(\alpha-\bar{\alpha})^2}{2\sigma^2}\right\} \left(\frac{I_s}{I_0}\right)^{\alpha} \quad (\text{B4})$$

and hence

$$\frac{V_s}{V_0} = \left(\frac{I_s}{I_0}\right)^{\alpha_s}, \quad \alpha_s = \bar{\alpha} + \frac{\sigma^2 \ln(I_s/I_0)}{2}. \quad (\text{B5})$$

The negative shift of the effective power reflects the over-emphasis of the larger resistors in the chain, which is particularly pronounced at small currents.

We next consider the overall response of N serially connected elements, of type P and S alternately. Denoting $I = I_p = I_s$, we get

$$V = \left\langle \frac{1}{N} \sum_{n=1}^N V_n \right\rangle = \frac{V_0}{2} \left[\left(\frac{I}{I_0}\right)^{\alpha_p} + \left(\frac{I}{I_0}\right)^{\alpha_s} \right] \quad (\text{B6})$$

and thus, for $\sigma^2 \ln(I/I_0) \ll \bar{\alpha}$,

$$\frac{V}{V_0} = \left(\frac{I}{I_0}\right)^{\alpha_{\text{eff}}},$$

$$\alpha_{\text{eff}} = \frac{(\alpha_p + \alpha_s)}{2} = \bar{\alpha} + \left(\frac{\sigma^2 \ln(I/I_0)(1-1/\bar{\alpha})}{4}\right). \quad (\text{B7})$$

It is straightforward to show that a parallel connection of alternating P - and S -type elements yields the same effective power law. We therefore conclude that *any* configuration that involves serial and parallel connections of evenly distributed P - and S -type elements will have a current-voltage characteristic given by Eq. (B7).

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