$1/N$ Expansion and Long Range Antiferromagnetic Order

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The staggered magnetization of the Heisenberg antiferromagnet in two dimensions can be systematically approximated by a $1/N$ expansion. Cancellation between self-energy diagrams leads to a Luttinger-like theorem for the ground state. We prove (for a smooth enough self-energy) that the long range order of mean field theory ($N=\infty$) survives corrections to all orders of $1/N$. Divergences of this series provide a new route to the disordered phases of quantum antiferromagnets.

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In the study of quantum phase transitions, the order-to-disorder transitions of the Heisenberg antiferromagnet at zero temperature are particularly interesting. There is also hope that understanding such transitions may provide insight into the electronic correlations of lanthanum cuprates where under low doping, antiferromagnetism is replaced by superconductivity.

The ground state of the Heisenberg antiferromagnet in two dimensions can either have long range order or be disordered by quantum fluctuations [1]. Disordering can be induced by frustrating longer range interactions, or perhaps by slight hole doping as suggested by the phase diagram of lanthanum cuprates. In either case, there are theoretical difficulties in describing the transition itself. While semiclassical spin wave theory works well deep in the ordered phase, it assumes spontaneously broken symmetry, and fails when the staggered magnetization becomes small. The continuum approximation of the (2+1)-dimensional nonlinear sigma model relies on perfect short range antiferromagnetic correlations. Near the transition, however, the short range correlations deteriorate considerably. This complicates matters, since one needs to include field discontinuities (e.g., hedgehogs), and consider interference effects between their Berry phases [2].

The Schwinger boson (SB) large-N expansion [3] is a rotationally symmetric formulation, which in principle can treat both sides of the transition [4]. The mean field theory (MFT) describes the excitations as a free Bose gas of N decoupled flavors. Bose condensation in this system is equivalent to long range spin order [5]. However, MFT is strictly valid only at $N=\infty$, while the physically interesting system is at N=2. A connection between the two limits requires an understanding of the $1/N$ series. Higher order corrections involve interactions between SB which enforce the local constraints. However, finite N corrections to the staggered magnetization have not yet been evaluated. Before this could be done, it was necessary to place the $1/N$ expansion on firmer footing, i.e., to show that the higher order terms yield finite and sensible results, which do not immediately destroy the mean field ground state. Currently, we do not know whether the long range order found in the MFT survives for *any* $N<\infty$.

This paper specifically addresses this concern. We prove a theorem which establishes the $1/N$ expansion as a consistent approach for the ground state of finite N systems, starting from the MFT. Under a condition that the self-energy is sufficiently smooth at the ordering wave vectors, we prove that *if there is long range order in the MFT, the spontaneous staggered magnetization does not vanish to all orders of the $1/N$ expansion.* The proof uses a cancellation between self-energy diagrams and their tadpole counterparts, a feature special to the $1/N$ expansion.

The result is reminiscent of (but not equivalent to) Hugenholtz and Pines' self-energy condition for Bose condensed liquids [6]. It is closer in spirit to Luttinger's theorem for Fermi liquids [7]. The spontaneous staggered magnetization is analogous to Luttinger's Fermi surface. (Both appear as a discontinuity in the occupation number.) In Luttinger's theorem, under a similar condition on the self-energy, the Fermi surface discontinuity survives at each order in perturbation theory. Pushing this analogy further, we shall propose that the vanishing of the staggered magnetization at finite N may formally resemble one of the known Fermi surface instabilities.

For simplicity we discuss the nearest neighbor SU(N) Heisenberg antiferromagnet. The proof actually utilizes only general features of this model, and thus it is readily extendable to more general Hamiltonians. The spins are represented by N SB per site, $a^\dagger_{i,-m_{max}}\cdots a^\dagger_{i,m_{max}}$, where $m_{max}=(N-1)/2$, and the Hamiltonian is given by

$$\mathcal{H}=-\frac{J}{N}\sum_{\langle i,j\rangle}(a^\dagger_{im}a^\dagger_{jm})(a_{im'}a_{jm'})-h\sum_{im}ma^\dagger_{im}a_{im}. \quad (1)$$

$\langle i,j\rangle$ are nearest neighbor bonds on the square lattice. Summation over repeated indices is implicit, unless specified otherwise. The Hilbert space is constrained by the

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fixed SB number $a^\dagger_{im} a_{im} = Ns$, at each site. h is an infinitesimal ordering field. The generators of SU(N) are given by $a^\dagger_{im} a_{im'}$, where we use conjugated representations on opposite sublattices. For $N = 2$, Eq. (1) is equivalent (up to a constant) to the Heisenberg model $\mathcal{H} = J\sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j - h\sum_i (-1)^i S^z_i$ [3].

Following the standard procedure [3], the partition function can be written as a coherent states path integral. Hence one introduces real local fields λ_i to impose the constraints, and Hubbard Stratonovich fields Q_{ij} to decouple the quartic interactions.

After integrating out the SB field, we are left with

$$Z(h) = \int \mathcal{D}(\lambda Q)\ \exp\left[-N\mathcal{S}(\lambda, Q, h)\right]. \qquad (2)$$

The explicit expression for the action $\mathcal{S}$ can be found in Ref. [3]. Following [5], we consider the case of zero temperature and large, but finite volume. The staggered magnetization is given by

$$M = \lim_{h\to 0^+} \lim_{\beta\to\infty} \frac{1}{2\beta V} \mathrm{Tr}\ mG(k, m), \qquad (3)$$

where G is the full Green function of SB. It is given in the Nambu notation as a 2×2 matrix with normal and anomalous components $G_{nn'}(k, m) = \langle A_{nkm} A^\dagger_{n'km}\rangle$, where $A_{km} = (a_{km}, a^\dagger_{-km})$. β is the inverse temperature and V is the volume (number of lattice sites). We denote $k = (\mathbf{k}, \omega)$, where $\mathbf{k}$ and ω are lattice momentum and Matsubara frequency, respectively. Tr includes a trace over k, m, and the Nambu indices. G is evaluated by summing all one-particle diagrams generated by the large-N expansion of Eq. (2).

Let us briefly review the mean field results which were derived previously [3, 5]. At large N, (2) is dominated by the saddle point $\bar{\lambda}, \bar{Q}$, and G is approximated by the mean field Green function G_0,

$$G_0(k, m) = \begin{pmatrix} \bar{\lambda} - i\omega - hm & 4\bar{Q}\gamma_{\mathbf{k}} \\ 4\bar{Q}\gamma_{\mathbf{k}} & \bar{\lambda} + i\omega - hm \end{pmatrix}^{-1}, \qquad (4)$$

where $\gamma_{\mathbf{k}} = \frac{1}{2}(\cos k_x + \cos k_y)$. The poles of G_0 are at the SB frequencies

$$\omega_{\mathbf{k},m} = c\sqrt{\Delta_h^2 + \frac{\bar{\lambda}}{4\bar{Q}^2} h(m_{max} - m) + 2(1 - \gamma^2_{\mathbf{k}})}\ , \qquad (5)$$

where $c = \sqrt{8}\bar{Q}$ and $\Delta_h = c^{-1}\sqrt{\bar{\lambda}^2 - 16\bar{Q}^2 - 2h\bar{\lambda}m_{max}}$. $\omega_{\mathbf{k}}$ is minimized at the two points $\mathbf{k}_c = (\vec{0}, \vec{\pi})$. At those momenta, for $m = m_{max}$, the excitation gap is $c\Delta_h$. Solving the mean field equations yields $\Delta_h = \sqrt{2}/[NV(s - 0.1966\ldots)]$. In the thermodynamic limit, the SB with $m = m_{max}$ and $\mathbf{k} = \mathbf{k}_c$ contribute macroscopically to the momentum sum; i.e., they undergo *Bose condensation.* This condensate is the only term which survives the cancellation between positive and negative m's in (3), yielding the mean field staggered magnetization

$$M_0 = \frac{N-1}{2}\frac{\sqrt{2}}{V\Delta_h} = \frac{N(N-1)}{2}(s - 0.1966\ldots)\ , \qquad (6)$$

which for $N = 2$ agrees with spin wave theory [5].

The higher order $1/N$ corrections to G are described by diagrams which include lines for G_0 interacting via propagators D (defined later) which are depicted as wiggly lines. A diagram which involves L loops (traces of products of G_0) and P propagators is of order $(1/N)^p$ where $p = P - L$. One must exclude all diagrams which include the segments shown in Fig. 1. As shown in [3, 8], this leads to the fulfillment of the SB constraints at each order of $1/N$ separately.

As in MFT, a nonzero staggered magnetization is related to the divergence of the number of SB with $m = m_{max}$ at $\mathbf{k} = \mathbf{k}_c$. On the other hand, strictly at $h = 0$, MFT is SU(N) rotationally invariant, so the Bose condensation is equally shared among the different m flavors and the gap becomes $\Delta_0 = N\Delta_{h\neq 0}$. Henceforth we shall set $h = 0$, and have exact degeneracy between the different flavors $\omega_{\mathbf{k},m}$. Thus, long range order is associated with Bose condensation of all flavors at $\mathbf{k}_c$.

The self energy $\Sigma(k)$ (also a 2×2 matrix) is related to G by the Dyson equation $G^{-1} = G_0^{-1} - \Sigma$. In order to proceed we must make an important assumption on the smoothness of the self energy near the condensate momenta:

$$\lim_{k\to k_c} |\Sigma(k) - \Sigma(k_c)| = O\left(|k - k_c|^{2-\delta}\right), \quad \delta < 1\ , \qquad (7)$$

where $k_c \equiv (\mathbf{k}_c, 0)$. Σ should exhibit rotational symmetry about k_c as a consequence of the asymptotic "Lorentz invariance" of G_0 near k_c. (The SB dispersion vanishes linearly at $\mathbf{k}_c$.) We have verified that the leading order self-energy is smooth at k_c [i.e., obeys (7) with $\delta = 0$]. We argue that the smoothness assumption is plausible for models which have no pathology in the density of low excitations. For such models, the integrations in Σ are uniformly convergent for all external momenta.

However, we have not *proven* Eq. (7) to all orders in $1/N$, and we must regard it as an *assumption*; one which requires a separate justification for any particular model.

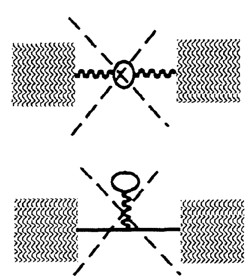

FIG. 1. Forbidden segments in $1/N$ expansion diagrams. Solid lines represent mean field Green functions G_0 and wavy lines, propagators of auxiliary fields D (see text).

The number of SB with momentum $\mathbf{k}_c$ is $n_{\mathbf{k}_c} = (2\beta)^{-1}\sum_\omega \mathrm{Tr}G(\mathbf{k}_c,\omega)$, where Tr traces over Nambu indices. This number diverges as $n_{\mathbf{k}_c} \sim V$ if $\det[G^{-1}(k_c)] \sim V^{-2}$. We use the Dyson equation and MFT relation $\Delta_0 \sim V^{-1}$ to state that

$$\Delta' \equiv \Sigma_{11}(k_c) - \gamma_{\mathbf{k}_c}\Sigma_{12}(k_c) = O(V^{-2}) \rightsquigarrow M \neq 0\,; \tag{8}$$

i.e., if the quantity Δ' vanishes rapidly enough in the thermodynamic limit, the ground state has long range order. It may be shown that $\Delta'(\vec{0}) = \Delta'(\vec{\pi})$.

Theorem: Under condition (7), Eq. (8) holds to all orders in the $1/N$ series.

Proof: The self-energy is decomposed into two parts,

$$\Sigma = \tilde{\Sigma} + \Sigma^{tad}\,, \tag{9}$$

where Σ^{tad} is the *single* tadpole diagrams (see Fig. 2), and $\tilde{\Sigma}$ are all the remaining diagrams. Although $\tilde{\Sigma}$ and Σ^{tad} are expected to be of $O(1)$ separately, we shall show that at $k = k_c$ the $O(1)$ contributions precisely cancel in (8) leaving us with terms of $O(V^{-2})$. *Note that in contrast to perturbation theory, the first and second terms in Fig. 2 have a different number of vertices, but are of the same order in* $1/N$. This enables the cancellation mechanism function at each order separately.

The rest of this discussion contains unavoidable technical details. The set of auxiliary fields is denoted by $(\lambda_j, \Re Q_{s,j}, \Re Q_{d,j}, \Im Q_{s,j}, \Im Q_{d,j})$. λ_j couples to the local boson density and $\Re(\Im)Q_{s(d),j}$ couple to the bilinear forms $\sum_{e=e_x,e_y}\eta_e^{s(d)}[a_j^\dagger a_{j+e}^\dagger + (-)a_j a_{j+e}]$, where $\eta^s_{e_x} = \eta^s_{e_y} = \eta^d_{e_x} = 1$ and $\eta^d_{e_y} = -1$. We define 2×2 vertices $\hat{v}^\alpha$ which connect between a field α and two G_0's. Thus a zero momentum field α is coupled to the form $\sum_{\mathbf{k}} A_k^\dagger \hat{v}^\alpha_{\mathbf{k}} A_k$, where $\hat{v}^1_{\mathbf{k}} = iI/2$, $\hat{v}^{2,3}_{\mathbf{k}} = \sigma^x(\cos k_x \pm \cos k_y)$, and $\hat{v}^{4,5}_{\mathbf{k}} = i\sigma^y(\cos k_x \pm \cos k_y)$. Using $\hat{v}^\alpha$, we can explicitly write $\Sigma^{tad}(k_c)$ (see Fig. 2) as

$$\Sigma^{tad}(k_c) = 2N\hat{v}^\alpha_{\mathbf{k}_c} D^{\alpha,\alpha'}(0)\sum_k \mathrm{Tr}[\hat{v}^{\alpha'}_{\mathbf{k}} G_0(k)R(k)G_0(k)]\,, \tag{10}$$

$$R = \tilde{\Sigma} + \Sigma G \Sigma\,,$$

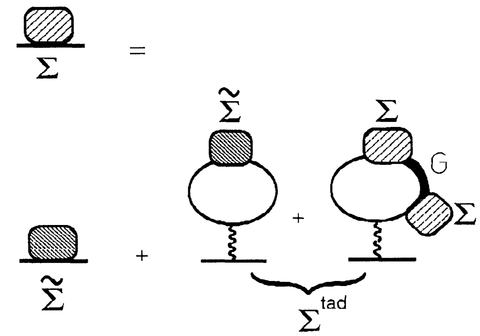

FIG. 2. Diagrammatic representation of Eqs. (9) and (10) for the self-energy. $\tilde{\Sigma}$ are all diagrams except the single tadpole diagrams. The cancellation between $\tilde{\Sigma}$ and the tadpole diagrams allows the staggered magnetization to survive finite N corrections.

where $\sum_k = \frac{1}{V}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}$. It may be seen that only $\alpha,\alpha' = 1,2$ give nonvanishing contribution to this formula.

$\Sigma(k)$ may be expanded as $\Sigma(k) = \Sigma_\alpha(k)\hat{u}^\alpha_{\mathbf{k}}$, where summation over α runs from 1 to 3, $\hat{u}^{1,2}_{\mathbf{k}} = \hat{v}^{1,2}_{\mathbf{k}}$ and $\hat{u}^3_{\mathbf{k}} = \sigma^z$. The coefficients of expansion satisfy the relation $\Sigma_\alpha(\vec{0},0) = \Sigma_\alpha(\vec{\pi},0)$ and $\Sigma_3(k_c) = 0$. The same expansion with $\hat{u}^\alpha$ holds for $R(k)$. Δ' can now be written as $\Delta' = f^\alpha\Sigma_\alpha(k_c)$, where $f^1 = i/2$, $f_2 = -2$, and $f^\alpha = 0$, $\alpha > 2$.

The propogator in the $1/N$ expansion is given by the matrix $D = \frac{1}{N}(\Pi_0 - \Pi)^{-1}$, where

$$\Pi_0^{\alpha,\alpha'}(q) = \delta_{\alpha,\alpha'}(1-\delta_{\alpha,1})\frac{4}{J}\,,$$
$$\Pi^{\alpha,\alpha'}(q) = 2\sum_k \mathrm{Tr}[\hat{v}^\alpha_{\mathbf{k},\mathbf{k}+\mathbf{q}} G_0(k+q)\hat{v}^{\alpha'}_{\mathbf{k}+\mathbf{q},\mathbf{k}} G_0(k)]\,. \tag{11}$$

Using the relation $ND\Pi = -1 + ND\Pi_0$, we find that at $k = k_c$, $\tilde{\Sigma}(k_c)$ is canceled on the right-hand side of Eq. (9) and we obtain

$$\Sigma_\alpha(k_c) = -[\Sigma(k_c)G(k_c)\Sigma(k_c)]_\alpha + \frac{4N}{J}D^{\alpha,2}(0)R_2(k_c)$$
$$+2ND^{\alpha,\alpha'}(0)\sum_k \mathrm{Tr}\left\{\hat{v}^{\alpha'}_{\mathbf{k}} G_0(k)\hat{u}^{\alpha''}_{\mathbf{k}} G_0(k)\left[\left(\frac{\Sigma(k)}{1-G_0(k)\Sigma(k)}\right)_{\alpha''} - \left(\frac{\Sigma(k_c)}{1-G_0(k_c)\Sigma(k_c)}\right)_{\alpha''}\right]\right\}\,, \tag{12}$$

where we have used the fact that $\Sigma^{tad}_{\alpha''}$ is independent of momentum.

The Bose condensation of $G_0(k_c)$ gives rise to the divergence of $\Pi(q=0)$. Extracting the volume divergences in (11) yields $\Pi^{\alpha,\alpha'}(0) = f^\alpha f^{\alpha'}(aV^2 + bV) + \tilde{P}^{\alpha,\alpha'}$ where a, b, and $\tilde{P}^{\alpha,\alpha'}$ are independent of volume. We see that both order V^2 and order V factorize, reflecting the emergence of a disconnected part in the correlation function due to Bose condensation. Denoting $P = \Pi_0 - \tilde{P}$ and inverting the polarization matrix we obtain the propagator to order V^{-2}:

$$ND^{\alpha,\alpha'}(0) = (P^{-1})^{\alpha,\alpha'} - \frac{(P^{-1})^{\alpha,\beta}f^\beta f^{\beta'}(P^{-1})^{\beta',\alpha'}}{f^\gamma(P^{-1})^{\gamma,\gamma'}f^{\gamma'}} - \frac{1}{aV^2}\frac{(P^{-1})^{\alpha,\beta}f^\beta f^{\beta'}(P^{-1})^{\beta',\alpha'}}{[f^\gamma(P^{-1})^{\gamma,\gamma'}f^{\gamma'}]^2}\,. \tag{13}$$

We now expand Σ in a power series of $1/N$: $\Sigma = \sum_p N^{-p}\Sigma^{(p)}$. We shall prove Eq. (8) by induction. Assume that Eq. (8) holds for $\Sigma^{(p)}, p \leq \bar{p}$. We take $\Sigma_\alpha^{(\bar{p}+1)}$ on the left-hand side of Eq. (12), and multiply both sides by the vector f^α. Using the Dyson equation for G, one can show that $f^\alpha[\Sigma G \Sigma]_\alpha$ is proportional to Δ', which, however, should be calculated with $\Sigma^{p\leq\bar{p}}$. Therefore, this term yields $O(V^{-2})$. Then, using (13), the terms of $O(1)$ in D get canceled by multiplying them on the left by f^α, leaving us with an overall factor of $O(V^{-2})$. We must still show that the second factor in the third term of (12) is not divergent. Since the summand diverges as $(k-k_c)^{-2}$ (2 powers of the phase space minus 4 powers from G_0), the momentum sum will converge if the self-energy obeys condition (7). Thus, we have shown that Eq. (8) holds to all orders in $1/N$. Q.E.D.

We note that it is crucial for the cancellation, described above, that the constraint has a *local* character and enforced by a fluctuating field. Indeed, this cancellation does not take place if the constraint is imposed only on average by a static chemical potential. On the other hand, our proof can be readily extended to different spin models with constrained Hilbert spaces. In particular it applies to the t'-J model, a semiclassical approximation to holes in the quantum antiferromagnet [9]. Also, a simpler version of this theorem applies to the long range order in resonating valence bond states [10], using a large-N expansion of the Gutzwiller projection [8].

In practical terms, this theorem sets the foundation for investigating the disordering transition using the $1/N$ expansion of the self-energy. We can propose two scenarios for the disordering mechanism at finite N: (i) Coupling of spins to soft charge fluctuations (holes) can give rise to violation of (7), i.e., a breakdown of our theorem and a destruction of long range order. This scenario is analogous to the one-dimensional Luttinger model where the Fermi surface discontinuity vanishes due to the large density of low excitations. (ii) The divergence of $V^2\sum_p N^{-p}\Sigma^{(p)} \to \infty$ may be detected in a partial resummation scheme. (Tadpole counterterms must be properly included, as shown above.) A divergence for example in nested diagrams, formally resembles the Cooper channel (superconductivity) instability in a Fermi liquid.

In summary, we have analyzed the corrections to the mean field ground state staggered magnetization of the two-dimensional antiferromagnetic Heisenberg model. We found an important cancellation mechanism between self-energy diagrams. This establishes that the $1/N$ expansion for the order parameter is a consistent asymptotic approach for finite N models. It is similar to perturbation theory about a noninteracting Fermi surface. We argue that the quantum disordering transition may be detected as a breakdown of the assumptions of this theorem, or a divergence in the $1/N$ series. These possibilities are worth further investigations.

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