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# Adiabatic Theorems for Dense Point Spectra*

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**Abstract.** We prove adiabatic theorems in situations where the Hamiltonian has dense point spectrum. The gap condition of the standard adiabatic theorems is replaced by an appropriate condition on the ineffectiveness of resonances.

## 1. Introduction

The prototype Adiabatic Theorem in quantum mechanics asserts that for operators with discrete spectra, in the adiabatic limit, the physical evolution takes an instantaneous eigenstate at $t = 0$ to the corresponding instantaneous eigenstate at a later time [3, 8]. More generally [1], with no assumptions about the nature of the instantaneous spectrum, but provided it has a gap for all times, the physical evolution in the adiabatic limit respects the splitting of the Hilbert space into spectral subspaces: A state in the subspace below the gap at $t = 0$ will evolve to a state that lies in the corresponding subspace below the corresponding gap at time $t$. While there are various kinds of adiabatic theorems that deal with the two settings above, (i.e. discrete spectra or gap conditions [1–3, 6, 8–11]), there are no results for situations that have no gaps, in particular in cases that involve dense point spectra. Our purpose here is to describe such results.

There is actually a good physical reason for the gap condition. A time-dependent Hamiltonian $H(t/\tau)$ with time scale $\tau$, can be thought of as describing a quantum system in an external time-dependent field, that for the sake of discussion we call photons. The photon field is switched on in the distant past and switched off in the distant future and in the limit $\tau \to \infty$ contains only soft photons with frequencies characterized by $1/\tau$. These cannot excite the system when it has gaps and that is why the evolution respects the spectral structure even on long time scales of order

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$\tau$[1]. In the absence of gaps no photons are soft. With dense point spectrum, if nearby eigenvalues correspond to eigenfunctions that are far apart then soft photons are again ineffective because matrix elements for the transition of states nearby in energy tend to be small.

We shall describe two adiabatic theorems for dense point spectra. The theorem of Sect. 4 is the analog of the case of discrete spectra since the spectral bundle is one dimensional. It applies to a setting where the instantaneous eigenvalues depend smoothly on time and the eigenvectors stay localized in some kind of configuration space. The theorem of Sect. 3 is the analog of the gap situation and applies in its simplest form to certain finite rank perturbations of dense point spectra where, in general, no continuously varying eigenvalue $\lambda(t)$ can be defined. Here the spectral bundle is infinite dimensional, since we choose the fibre to be the spectral subspace of $H(t/\tau)$ corresponding to a fixed interval of the real axis. The nonresonant condition is discussed in Sect. 3, and as we shall see, it endows infinitesimal gaps with the relevant properties of finite gaps.

## 2. A General Adiabatic Theorem

We shall now summarize the results of [1], in a slightly different formulation better adapted to our purposes. The need for reformulation, of course, is that the spectral gap of [1] is not present here. However, the proof of [1] goes through without essential change if we simply replace the gap condition by two of its consequences: existence of (a) a smooth reducing family of projection $P(s)$, and (b) a smooth solution of a commutator equation.

Throughout, we shall work with *scaled* time $s = t/\tau$ on the fixed interval $0 \leqq s \leqq 1$ rather than $t$ on $0 \leqq t \leqq \tau$. Thus instead of the evolution operator $U(t)$ satisfying the Schrödinger equation

$$i\frac{\partial}{\partial t}U(t) = H(t/\tau)U(t), \quad U(0) = 1$$

we shall work instead with the solution $U_\tau(s) = U(\tau s)$ of the *rescaled* equation,

$$i\frac{\partial}{\partial s}U_\tau(s) = \tau H(s)U_\tau(s), \quad U_\tau(0) = 1$$

We shall make the following assumptions:

(i) $H(s)$, $-\infty < s < \infty$, *is a family of selfadjoint operators on a Hilbert space* $\mathscr{H}$, *and having a s-independent domain* $\mathscr{D}$.

(ii) *The function* $(H(s)+i)^{-1}$, *with values in the Banach space* $\mathscr{L}(\mathscr{D}, \mathscr{H})$ *of linear operators, is twice strongly continuously differentiable.*

Under these conditions, $U_\tau(s)$ exists and is strongly differentiable.

[1] We are interested in conditions that give adiabatic evolution for time scales of order $\tau$. For shorter time scales, say of order unity, the evolution respects the spectral splitting (to order $1/\tau$) under weaker conditions, namely once the spectral bundle is differentiable

(iii) *$P(s)$ is a strongly $C^2$ family of orthogonal projections with*

$$H(s)P(s) = P(s)H(s)$$

*for each s.*

Our final assumption is the following condition:

(iv) *There exists a strongly $C^1$ family $\tilde{X}(s)$ satisfying*

$$[H(s), \tilde{X}(s)] = P(s)\dot{P}(s)Q(s) \tag{2.1}$$

*and*

$$\tilde{X}(s) = P(s)\tilde{X}(s)Q(s),$$

*where $Q(s) = 1 - P(s)$.*

Note that $\tilde{X}(s)^*$ satisfies (2.1) with the right-hand side replaced by $-Q(s)\dot{P}(s)P(s)$. As we shall see, the condition for inverting the commutator (2.1), i.e. for Eq. (2.2) below to make sense, can be interpreted as a condition on the mismatch of resonances. In [1], $\tilde{X}(s)$ is constructed as the Friedrichs integral

$$\tilde{X}(s) = -(2\pi i)^{-1} \int_{\Gamma} R(z,s)P(s)\dot{P}(s)Q(s)R(z,s)dz, \tag{2.2}$$

where $\Gamma$ encircles the part of the spectrum corresponding to $P(s)$, and $R(z,s) \equiv [H(s) - z]^{-1}$ is the resolvent. We shall use an adaptation of this construction below.

We can now state the adiabatic theorem [1].

**Theorem.** *If* (i)–(iv) *hold, then uniformly for* $0 \leqq s \leqq 1$,

$$U_\tau(s)P(0) - P(s)U_\tau(s) = O(\tau^{-1}). \tag{2.3}$$

The basic idea of the proof, which goes back to the classic paper [8] of Kato, is to approximate the dynamics generated by $H(s)$ by a dynamics that manifestly preserves the spectral splitting.

Given $H(s)$ and $P(s)$, there is a natural evolution that preserves the spectral splitting and is an approximation of the physical evolution $U_\tau(s)$. We denote this evolution by $U_A(s)$. It is the unitary given by:

$$\begin{aligned} i\partial_s U_A(s) &= \tau H_A(s)U_A(s), \\ U_A(0) &= 1 \end{aligned} \tag{2.4}$$

$$H_A(s) \equiv H(s) + \frac{i}{\tau}[\dot{P}(s), P(s)].$$

Respecting the spectral splitting means that

$$U_A(s)P(0) = P(s)U_A(s). \tag{2.5}$$

For a proof of (2.5) see [1].

On time scales where $t$ is of order one, the variable $s = t/\tau$ is of order $1/\tau$, so it is clear from the explicit form of $H_A$, that $U_A$ and $U_\tau$ are close, provided the spectral bundle is differentiable ($\dot{P}(s)$ is bounded). The adiabatic theorems are concerned with the comparison of $U_A(s)$ and $U_\tau(s)$ on long time scales of order $\tau$, where $s$ is of order one. Condition (iv) on the mismatch of resonances enters for $U_\tau$ and $U_A$ to agree on long stretches of time.

We shall now sketch the proof of Theorem 2.1. First note that $\Omega_\tau(s) \equiv U_A^*(s)U_\tau(s)$ satisfies the integral equation

$$\Omega_\tau(s) = 1 - \int_0^s K_\tau(s', P)\Omega_\tau(s')ds',$$

$$K_\tau(s, P) \equiv U_A^*(s)[\dot{P}(s), P(s)]U_A(s). \tag{2.6}$$

$\Omega_\tau(s)$ and $\dot{\Omega}_\tau(s)$ are clearly bounded ($\dot{\Omega}_\tau$ because of the integral equation). Writing

$$\Omega_\tau(s) - 1 = (Q(0)\Omega_\tau(s) - Q(0)) + (P(0)\Omega_\tau(s) - P(0)), \tag{2.7}$$

we will show that $Q(0)\,\Omega_\tau(s) - Q(0) = O(1/\tau)$ and similarly for $P(0)$. This implies (2.3). From the integral equation (2.6), the intertwining property (2.5), the commutator formula (2.1), and the evolution equations (2.4):

$$\begin{aligned}
Q(0)(\Omega_\tau(s) - 1) &= -\int_0^s Q(0)K_\tau(s', P)\Omega_\tau(s')ds' \\
&= -\int_0^s U_A^*(s')[P(s)\dot{P}(s)Q(s)]^* U_A(s')\Omega_\tau(s')ds' \\
&= \int_0^s U_A^*(s')[H(s), \tilde{X}(s)^*]U_A(s)\Omega_\tau(s')ds' \\
&= -\frac{i}{\tau}\int_0^s \partial_{s'}\{U_A^*(s')\tilde{X}(s')^* U_A(s')\}\Omega_\tau(s')ds' \\
&\quad + \frac{i}{\tau}Q(0)\int_0^s U_A^*(s')\{\dot{\tilde{X}}(s')^* - \dot{P}(s')P(s')\}U_A(s')\Omega_\tau(s')ds'.
\end{aligned} \tag{2.8}$$

Integrating the first term by parts, and using the boundedness of $\dot{\Omega}_\tau(s)$ shows that the first term is 0 $(1/\tau)$ if $\tilde{X}$ is bounded. The second term is bounded if $\dot{\tilde{X}}$ and $\dot{P}$ are.

## 3. Relatively Finite Perturbations

In this section, we shall consider situations of which the following is typical. Let $H$ be an operator with pure point spectrum, and $e_n$ a complete orthonormal set of eigenvectors:

$$He_n = \lambda_n e_n.$$

We consider a rank one perturbation of $H$:

$$H_\beta = H + \beta\langle\varphi, \cdot\rangle\varphi$$

with $\|\varphi\|^2 = 1$. As was shown in [7, 13], $H_\beta$ is pure point for a.e. $\beta$, provided that

$$\sum_n |\langle\varphi, e_n\rangle| < \infty. \tag{3.1}$$

If we now permit the coupling constant $\beta(s)$ to vary smoothly in time, we obtain a time-dependent family of Hamiltonians

$$H(s) = H + \beta(s)\langle \varphi, \cdot \rangle \varphi \tag{3.2}$$

for which we do not even know whether $H(s)$ has eigenvalues for all values of $s$ or not. Since we cannot follow a subspace corresponding to a single, continuously varying eigenvalue, we must find some other definition for the projection $P(s)$, if we are to prove an adiabatic theorem.

We solve this problem by choosing:

$$P(s) = E_s(a, b)$$

for a *fixed spectral interval* $(a, b)$, and assume, in analogy to the usual assumption of "no level crossings," that $a$ and $b$ are never in $\sigma_p(H(s))$ for any $s$; that is, no eigenvalues of $H(s)$ "leak out" of $(a, b)$. If, in addition, we assume that the endpoints $a, b$ belong to a certain set of "non-resonant" values, which is large in the sense of having full measure, then we are able to prove an adiabatic theorem for the family of projections $P(s)$.

It may seem an overly strong assumption that for *all* $s$, neither $a$ nor $b$ is in $\sigma_p(H(s))$. After all, the point spectrum is dense and keeps moving. In fact, general principles for rank one perturbations tell us that for any real $\lambda$, there is at most one $\beta = \beta_c(\lambda)$ so that $\lambda$ is an eigenvalue of $H_\beta$. Thus, the real line is divided into three open intervals where the conditions that $a, b \notin \sigma_p(H(s))$ is only that $\beta$ lies in only one of these intervals. (If $\beta_c(a) = \beta_c(b)$ there are two regions, not three.)

In order to define the non-resonant set precisely, we shall need the following result [7, 13].

**3.1. Proposition.** *Let* $p > 1$, $a_n > 0$ *and* $\sum_n a_n < \infty$. *Then the function*

$$F_p(\lambda) \equiv \sum_n |\lambda - \lambda_n|^{-p} a_n^p \tag{3.3}$$

*is finite for* a.e. $\lambda$, *regardless of the sequence* $\lambda_n$.

*Proof.* Let $S_n = \{\lambda \in J : |\lambda - \lambda_n| \geqq a_n\}$. Let $\chi_n(\lambda)$ be the characteristic function of $S_n$, and $\chi_n^c = 1 - \chi_n$. Write

$$F(\lambda) = \sum_n \frac{a_n^p}{|\lambda - \lambda_n|^p} \{\chi_n(\lambda) + \chi_n^c(\lambda)\}.$$

Integrate the first term:

$$\int_0^1 \sum_n \frac{a_n^p}{|\lambda - \lambda_n|^p} \chi_n(\lambda) d\lambda = \sum_n a_n^p \int_{S_n} \frac{d\lambda}{|\lambda - \lambda_n|^p} = \sum_n a_n^p \left[ \frac{(p-1)^{-1}}{|\lambda_n - \lambda|^{p-1}} \right]_{\partial S_n}$$

$$\leqq (p-1)^{-1} \sum_n a_n^p \cdot \frac{4}{a_n^{p-1}} = 4(p-1)^{-1} \sum_n a_n < \infty.$$

Hence, the first term converges a.e.

Write the second term as:

$$\sum_{n=1}^{N} \frac{a_n^p}{|\lambda - \lambda_n|^p} \chi_n^c(\lambda) + \sum_{n=N+1}^{\infty} \frac{a_n^p}{|\lambda - \lambda_n|^p} \chi_n^c(\lambda).$$

The first of these terms is finite except at the points of the sequence $\{\lambda_n\}$, and the last term is identically zero except on the set:

$$E_N = \bigcup_{n=N+1}^{\infty} S_n^c.$$

But $E_N$ has measure

$$\text{meas}(E_N) \leqq \sum_{n=N+1}^{\infty} \text{meas}(S_n^c) \leqq \sum_{n=N+1}^{\infty} 2a_n,$$

which is less than $\varepsilon$ if $N$ is large enough. □

Let $N(H, \varphi)$ be the set of all $\lambda$ such that

$$\sum_n |\lambda - \lambda_n|^{-4} |\langle \varphi, e_n \rangle|^2 = \infty.$$

By Proposition 3.1, $N(H, \varphi)$ has Lebesgue measure zero provided

$$\sum_n |\langle \varphi, e_n \rangle|^{1/2} < \infty. \tag{3.4}$$

If $\lambda \notin N(H, \varphi)$, then $(H - \lambda)^2 \varphi$ is a finite vector of $\mathscr{H}$.

**Theorem 3.2.** *Assume that $\beta(s)$ is $C^2$, and that (3.4) holds. Let $a, b \notin N(H, \varphi) \cup \sigma_p(H(s))$ for all s, and define*

$$P(s) = E_s(a, b). \tag{3.5}$$

*Then the adiabatic theorem holds for $P(s)$.*

Actually, we shall prove below a more general result suggested by [7]. If $A$ is bounded, and

$$\sum_n \| A e_n \|^{1/2} < \infty, \tag{3.6}$$

we shall denote by $N(H, A)$ the set of all $\lambda$ for which

$$\sum_n |\lambda - \lambda_n|^{-4} \| A e_n \|^2 = \infty.$$

By Proposition 3.1, $N(H, A)$ has measure zero. From the formula

$$(H - \lambda)^{-2} A = \sum_n (\lambda - \lambda_n)^{-2} \langle e_n, \cdot \rangle A e_n$$

it follows that $(H - \lambda)^{-2} A$ is a bounded operator if $\lambda \notin N(H, A)$. Note also that (3.6) implies that $A$ is compact (in fact, trace class).

We can now formulate the main theorem of this section. Let $H$ and $A$ be self adjoint, with $H$ pure point, and $A$ bounded. Let $W(s)$ be a strongly continuous family of self adjoint operators, uniformly bounded in operator norm. Define

$$H(s) = H + AW(s)A.$$

**Theorem 3.3.** *Assume that (3.6) holds, and that $W(s)$ is strongly $C^2$ with uniformly bounded derivatives. Let $a, b \notin N(H, A) \cup \sigma_p(H(s))$ for all s, and define*

$$P(s) = E_s(a, b).$$

*Then the adiabatic theorem holds for $P(s)$.*

*Remark.* Theorem 3.2 follows by choosing $A = A^2 = \langle \cdot, \varphi \rangle \varphi$ and $W(s) = \beta(s)\,1$.

*Proof.* We need to verify (iii) and (iv) of Theorem 2.1. In [1], this was done by writing, with $R(z,s) = (H(s) - z)^{-1}$,

$$P(s) = -\frac{1}{2\pi i} \int_\Gamma R(z,s)dz, \tag{3.7}$$

and

$$\tilde{X}(s) = -\frac{1}{2\pi i} \int_\Gamma R(z,s)dz,$$

where $\Gamma$ is a contour encircling the spectral set corresponding to $P(s)$ with positive orientation. Here, although the interval $(a,b)$ is not separated from the rest of the spectrum, we *nevertheless do exactly the same thing,* and take $\Gamma$ to be the positively oriented rectangle with corners $a \pm ic$ and $b \pm ic$ for some $c > 0$. Although $\Gamma$ runs right through the spectrum. We shall show that (3.7) and (3.8) make perfect sense and satisfy (iii) and (iv) of Sect. 2.

We first establish notation. Write $R_0(z) = (H - z)^{-1}$, so that by the second resolvent equation:

$$R(z,s) = R_0(z)T(z,s) = T^*(z,s)R_0(z), \tag{3.9}$$

where

$$T(z,s) = (I + AW(s)AR_0(z))^{-1} \tag{3.10}$$

and

$$T^*(z,s) = [T(\bar{z},s)]^* = (1 + R_0(z)AW(s)A)^{-1}. \tag{3.11}$$

The following is what makes the proof work.

**Lemma 3.4.** *$R_0(\zeta)$ $A$ and $R_0(z)$ $R_0(\zeta)$ $A$ are norm continuous on $\Gamma$ and $\Gamma \times \Gamma$ respectively.*

*Proof.* The only points of questionable continuity of $R_0(\zeta)A$ are $\zeta = a, b$. We have

$$[R(a+i\varepsilon) - R(a)]A\psi = \sum_{n=1}^{\infty} \frac{i\varepsilon}{(\lambda_n - a)(\lambda_n - a - i\varepsilon)} \langle Ae_n, \psi \rangle e_n$$

so that by Pythagoras' theorem

$$\begin{aligned} \|[R(a+i\varepsilon) - R(a)]A\psi\|^2 &= \varepsilon^2 \sum_{n=1}^{\infty} \frac{|\langle Ae_n, \psi \rangle|^2}{(\lambda_n - a)^2[(\lambda_n - a)^2 + \varepsilon^2]} \\ &\leqq \|\psi\|^2 \left\{ \sum_{n=N}^{\infty} \frac{\|Ae_n\|^2}{(\lambda_n - a)^2} + \varepsilon^2 \sum_{n=1}^{N-1} \frac{\|Ae_n\|^2}{(\lambda_n - a)^4} \right\}. \end{aligned}$$

Hence

$$\limsup_{\varepsilon \downarrow 0} \|R(a+i\varepsilon)A - R(a)A\|^2 \leqq \sum_{n=N}^{\infty} (\lambda_n - a)^{-2} \|Ae_n\|^2$$

for every $N$. But the series converges by choice of $a$.

A similar, slightly messier proof works for $R_0(z)R_0(\zeta)A$.

It follows that $AR_0(\zeta)=[R_0(\bar{\zeta})A]^*$, and $AR_0(\zeta)R_0(z)$ are norm continuous. Since $A$ is compact, they are all compact as well—trivially for $\zeta$, $z$ non-real, and by continuity otherwise. Moreover, $T(\zeta, s)$ is also norm continuous on $\Gamma$. For since $AW(s)AR_0(\zeta)$ is compact, the only question is whether $1+AW(s)AR_0(\zeta)$ is injective for $\zeta=a, b$. But if

$$[1+AW(s)AR_0(a)]\varphi=0,$$

then

$$\psi \equiv R_0(a)\varphi=-[R_0(a)A]W(s)[AR_0(a)\varphi]$$

is a finite vector, and

$$(H_0-a)\psi=\varphi=-AW(s)A\psi,$$

so that $H(s)\psi=a\psi$, and $a\in\sigma_p(H(s))$, contrary to hypothesis. As a function of $s$, $T(\zeta, s)$ is strongly $C^2$ since $W(s)$ is.

With these remarks, we proceed to the proof of Theorem 3.3.

Define:

$$P(s)\equiv P_0-\frac{1}{2\pi i}\int_\Gamma [R_0(\zeta)A]W(s)[AR_0(\zeta)]T(\zeta, s)d\zeta. \tag{3.12}$$

By differentiation under the integral sign, $P(s)$ is strongly $C^2$. We claim that

$$P(s)=E_s(a, b). \tag{3.13}$$

For let $\Gamma_\varepsilon$ be the polygonal path with two pieces, obtained by deleting the segments $(a-i\varepsilon, a+i\varepsilon)$ and $(b-i\varepsilon, b+i\varepsilon)$ from $\Gamma$. By standard spectral theory, since $a$ and $b$ are not eigenvalues of $H(s)$,

$$\begin{aligned} E_s(a, b) &= \operatorname*{w\text{-}lim}_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} R(\zeta, s)d\zeta \\ &= \operatorname*{w\text{-}lim}_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\{R_0(\zeta)-R_0(\zeta)AW(s)AR_0(\zeta)T(\zeta, s)\}d\zeta, \end{aligned}$$

which clearly equals (3.12). Define

$$\tilde{X}(s)=-\frac{1}{2\pi i}\int_\Gamma R(z, s)P(s)\dot{P}(s)Q(s)R(z, s)dz. \tag{3.14}$$

We must show that this exists by differentiating (3.12), and substituting into (3.14). To simplify the rather lengthy formulas, *we suppress the variables* $\zeta$ *and* $s$.

Differentiate (3.12) to obtain

$$\dot{P}=-\frac{1}{2\pi i}\int_\Gamma [R_0A\dot{W}R_0T+R_0AWAR_0\dot{T}]d\zeta, \tag{3.15}$$

and substitute into (3.14), using $R(z)P=PR(z)$ and similarly for $Q$. We obtain

$$\tilde{X} = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{2\pi i} \int_{\Gamma} \{PR(z)R_0 A[\dot{W} AR_0 T + WAR_0 \dot{T}]R(z)Q\} \, dz \, d\zeta$$

$$= P \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{2\pi i} \int_{\Gamma} \{T^*(z)(R_0(z)R_0 A)[\dot{W}(AR_0 TR_0(z)$$

$$- WW(AR_0)TA\dot{W}(ATR_0(z)] \, T(z) dz \, d\zeta\} Q, \tag{3.16}$$

where we have used $\dot{T} = -TA\dot{W}AT$. Now

$$ATR_0(z) = A[I - T(AWA)R_0]R_0(z) = AR_0(z) - ATAW(AR_0R_0(z)) \tag{3.17}$$

is continuous on $\Gamma \times \Gamma$, as is

$$AR_0TR_0(z) = ARR_0(z) = [I + AR_0AW]^{-1}(AR_0R_0(z)), \tag{3.18}$$

where we have used the factorization method. The inverse exists since $a \notin \sigma_p(H(s))$. Thus *all terms* on (3.16) *are norm continuous* on $\Gamma \times \Gamma$, and $\tilde{X}(s)$ is well-defined. Moreover, $\tilde{X}(s)$ is clearly strongly $C^2$ as well.

It remains to show that $\tilde{X}(s)$ satisfies the commutator identity (2.1),

$$[H(s), \tilde{X}(s)] = P(s)\dot{P}(s)Q(s). \tag{3.19}$$

Define

$$\tilde{X}_\varepsilon(s) = -\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} R(z,s)P(s)\dot{P}(s)Q(s)R(z,s) \, dz.$$

It is trivial to compute (see [1]) that

$$[H(s), \tilde{X}_\varepsilon(s)] = [P_\varepsilon(s), P(s)\dot{P}(s)Q(s)],$$

where $2\pi i P_\varepsilon(s) = \int_{\Gamma_\varepsilon} R(z,s) dz$. But in our case $\tilde{X}_\varepsilon \to \tilde{X}$ in norm and $P_\varepsilon \to P$ weakly, so that we obtain (3.19).

This concludes the proof of Theorem 3.3. □

## 4. Well Localized Dense Point Spectrum

In this section, we will discuss a situation where, as $s$ varies, the dense spectrum moves smoothly and a strong form of no crossing takes place. Here are the assumptions:

1. *For each $s$ in $[0,1]$, $H(s)$ has a complete set of eigenvectors $\{e_n(s)\}_{n=0}^{\infty}$ with eigenvalues $\{\lambda_n(s)\}_{n=0}^{\infty}$.*

   Let $P(s)$ be the projection onto $e_0(s)$.

2. *$P(s)$ is $C^2$ and the matrix elements of its derivatives $P^{(l)}(s)$ obey*

$$|(e_0(s), P^{(l)}(s)e_n(s))| \leq C(1+n)^{-m} \tag{4.1}$$

*for $l = 1, 2$ and some $m > 0$.*

3. *The following strong no crossing rule holds*:

$$|\lambda_n(s) - \lambda_0(s)| \geqq D_1 |n|^{-q} \tag{4.2}$$

*for some* $q > 0$.
4. *Each* $\lambda_n(s)$ *is* $C^1$ *and obeys for* $l = 0, 1$:

$$|\lambda_n^{(l)}(s)| \leqq D_2 |n|^p \tag{4.3}$$

*for some* $p$.
5. *The eigenfunctions* $e_n(s)$ *are* $C^1$ *and obey*

$$\| e_n(s) \| \leqq D_3 |n|^p \tag{4.4}$$

*for the same* $p$ *as in (4.3)*.
6. $m > 3q + 2p + \frac{1}{2}$.

There are nontrivial examples where the hypotheses hold.

**Theorem 4.1.** *Suppose that* $H(s)$ *obeys* (*i*), (ii) *of Sect. 2 and that* (1)–(6) *holds. Then* $P$ *obeys the adiabatic theorem.*

*Proof.* We need only verify the hypothesis of Theorem 2.1 (2.iii) is implied by (2), so we need only prove (2.iv). Thus we concentrate on the equation

$$[H(s), \tilde{X}(s)] = P(s)\dot{P}(s)Q(s).$$

Taking matrix elements in the basis $e_n(s)$ we see that

$$\begin{aligned}(e_i(s), \tilde{X}(s)e_j(s)) &= \delta_{i0}(1 - \delta_{j0})[(e_0(s), \dot{P}(s)e_j(s)/(\lambda_0(s) - \lambda_j(s)))] \\ &\equiv a_j(s)\delta_{i0}(1 - \delta_{j0}).\end{aligned}$$

Since $m > q + \frac{1}{2}, \Sigma |a_j(s)|^2 < \infty$ from which it easily follows that $\tilde{X}$ (normalized by $(e_0, \tilde{X}e_0) = 0$) is a well defined bounded operator which obeys the requisite equation. By just differentiating and using the hypothesis $m > 3q + 2p + \frac{1}{2}$ we prove that $\tilde{X}$ is $C^2$. □

Examples where the hypothesis (4.1–4.6) can be proven are connected with varying a coupling constant in a system which can be solved by overcoming a small divisor problem. One case is

$$h(s) = h_0 + \beta(s)\tan(\pi\alpha n)$$

with $h_0$ the discrete one dimensional Schrödinger operator and $\alpha$ an irrational with good Diophantine properties [5, 12]. In this example the dependence on $\beta$ is actually analytic.

Another example would come from the model studied by Craig [4].

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