JOURNAL OF FUNCTIONAL ANALYSIS **120**, 220–237 (1994)

The Index of a Pair of Projections

J. AVRON*

Department of Physics, Technion, Israel Institute of Technology, 32000 Haifa, Israel

R. SEILER†

Fachbereich Mathematik, Technische Universität Berlin, Strasse des 17. Juni, W-1000 Berlin 12, Germany

AND

B. SIMON‡

Division of Physics, Mathematics and Astronomy, California Institute of Technology, 253-37, Pasadena, California 91125

Communicated by L. Gross

Received March 1, 1993

We discuss pairs of self-adjoint projections with $P - Q \in \mathcal{I}_{2n+1}$, the trace ideal, and prove that for $m \geqslant n$,

$$\operatorname{tr}(P-Q)^{2m+1} = \operatorname{tr}(P-Q)^{2n+1} = \dim(\operatorname{Ker} Q \cap \operatorname{Ran} P) - \dim(\operatorname{Ker} P \cap \operatorname{Ran} Q)$$

is an integer. We also prove that there exists a unitary V interchanging P and Q if and only if this integer is 0.

1. INTRODUCTION

Pairs of projections P, Q in a Hilbert space have a fascinating algebra. If $A = P - Q$, then it was realized in the late 1940's by Dixmier, Kadison, and Mackey [7] that A^2 commutes with both P and Q.

Kato [8, 9], in analyzing pairs with $\|P - Q\| < 1$, introduced a second operator $B = 1 - P - Q$ and noted that

$$A^2 + B^2 = 1. \tag{1.1}$$

* E-mail: phr97ya@technion.technion.ac.il.

† E-mail: seiler@math.tu-berlin.de.

‡ Research partially supported by USNSF Grant DMS-9101715.

0022-1236/94 $6.00

In trying to understand some relations between pairs of projections which enter in the analysis of the quantum Hall effect [1], we discovered a further significant relation, viz.

$$AB + BA = 0. \tag{1.2}$$

The new relation is a signature of supersymmetry and implies that off the space where $A^2 = 1$, A and $-A$ are unitarily equivalent. For example, we will see it implies that if $P - Q$ is trace class, then

$$\operatorname{tr}(P-Q) = \dim(u \mid (P-Q)u = u) - \dim(u \mid (Q-P)u = u)$$

is an integer, a result only proven recently by Effros [4]. The result also shows that in that case

$$\operatorname{tr}(P-Q) = \operatorname{tr}((P-Q)^3) = \operatorname{tr}((P-Q)^5) = \cdots.$$

As mentioned, Kato found (1.1) in his study of P, Q with $\|P-Q\| < 1$. He proved then that there exists a unitary U with

$$P = UQU^{-1}.$$

We go beyond this and prove if $\|P-Q\| < 1$, then there is a unitary V with

$$VPV^{-1} = Q, \qquad VQV^{-1} = P. \tag{1.3}$$

The structure of this paper is as follows: In Section 2 we discuss the algebra of (1.1), (1.2) and show if $\|P-Q\| < 1$, there is a V with (1.3). In Section 3 we introduce the notion of index of a pair of projections relative to the index of a suitable Fredholm operator and in Section 4 relate this index to $\operatorname{tr}(P-Q)^{2n+1}$ when that trace exists. This section has relations to work of Connes [2].

In many applictions, Q and P are related by $Q = UPU^{-1}$ and we discuss this situation and relate it to formulas of Hörmander [6] and Fedosov [5] in Section 5. Finally, Section 6 discusses a single result in the non-self-adjoint case. An appendix is included, containing all the results on Fredholm operators and index theory that we need.

We will not discuss examples in detail but the following is typical and closely related to the quantum Hall case: Let $\mathscr{H} = \ell_2(-\infty, \infty)$ and let $P = \{u \mid u_n = 0;\ n \leqslant -1\}$, $Q = \{u \mid u_n = 0; n \leqslant 0\}$. Then, $\operatorname{tr}(P-Q) = 1$ and $Q = UPU^{-1}$, where U is the left shift.

2. The Kato Dual and the Equivalence of Pairs

Let P, Q be a pair of orthonormal projections. Let

$$A = P - Q.$$

We call

$$B = 1 - P - Q$$

the *Kato dual* to A. We give the name in honor of T. Kato [8, 9], who introduced B in connection with relation (1) in the next theorem. Relation (2) is new:

THEOREM 2.1. (1) $A^2 + B^2 = 1$

(2) $\{A, B\} \equiv AB + BA = 0.$

Proof. Straightforward algebra; to make it simple we use $P' = 1 - P$ so $P'P = PP' = 0$ and $P + P' = 1$. Then, $B = P' - Q$ and

$$A^2 + B^2 = P^2 + P'^2 + 2Q^2 - (P + P')Q - Q(P + P') = P + P' = 1$$

and

$$\begin{aligned} AB + BA &= Q^2 - PQ - QP' + Q^2 - P'Q - QP \\ &= 2Q^2 - (P + P')Q - Q(P + P') = 0. \quad \blacksquare \end{aligned}$$

COROLLARY 2.2. *$(P - Q)^2$ commutes with P and Q.*

Proof. $(P - Q)^2$ obviously commutes with $P - Q$ and since it equals $1 - (1 - P - Q)^2$ it commutes with $1 - P - Q$ and so with $P + Q$. $\blacksquare$

Remark. This also follows from the useful formulas

$$P(P - Q)^2 = P - PQP = (P - Q)^2 P \tag{2.1}$$

$$Q(P - Q)^2 = Q - QPQ = (P - Q)^2 Q. \tag{2.2}$$

Kato used his dual B to study pairs P, Q with $\|A\| < 1$. He found a unitary U with $UQU^{-1} = P$. Explicitly, let $P' = 1 - P$; $Q' = 1 - Q$ and

$$W = PQ + P'Q'.$$

W obeys

$$PW = WQ. \tag{2.3}$$

Note that

$$PB = P(1 - P - Q) = -PQ = BQ.$$

Thus

$$\begin{aligned} W &= 1 - P - Q + 2PQ \\ &= (1 - 2P)B = B(1 - 2Q). \end{aligned} \tag{2.4}$$

Thus, since $(1 - 2P)^2 = 1$ and $1 - 2P$ is self-adjoint

$$W^* W = B^2 = WW^*. \tag{2.5}$$

Sinced $\|A\| < 1$, $\|Bu\|^2 = \|u\|^2 - \|Au\|^2 \geqslant (1 - \|A\|^2)\,\|u\|$, so B is invertible and

$$W = U\,|B| \tag{2.6}$$

the polar decomposition has U unitary. Moreover, since $|B| = \sqrt{1 - A^2}$ commutes with P and Q, (2.3) implies that

$$PU = UQ$$

constructing the required U.

The key properties (2.3), (2.5), and definition (2.6) are Kato's but the algebra using (2.4) is somewhat simplified.

We will construct a unitary that actually interchanges P and Q.

THEOREM 2.3. *Let P and Q be orthogonal projections on $\mathscr{H}$ with $\|P - Q\| < 1$. Then there exists a unitary V with*

$$VPV^{-1} = Q, \qquad VQV^{-1} = P. \tag{2.7}$$

Proof. Let $\operatorname{sgn}(x)$ be the function on $(-\infty, \infty)$ which is 1 (resp. -1, resp. 0) for $x > 0$ (resp. $x < 0$, $x = 0$). Since $\|A\| < 1$ and $A^2 + B^2 = 1$, $\operatorname{Ker}(B) = \{0\}$ and so $V = \operatorname{sgn}(B)$, defined by the functional calculus, is unitary. Moreover, since B commutes with $\operatorname{sgn}(B)$, we have

$$VBV^{-1} = B$$

that is,

$$V(P + Q)\,V^{-1} = P + Q. \tag{2.8}$$

Now $BA = -AB$ so B^2 commutes with A. Since $|B|^{-1}$ commutes with A and $V = |B|^{-1} B$, we see $VA = -AV$, that is,

$$V(P-Q) = (Q-P)V. \tag{2.9}$$

Equations (2.8) and (2.9) imply (2.7). ∎

Remarks. (1) Kato's U has an advantage over our V for perturbation theory; namely, it is given in a way that if P is fixed and $Q_n \to P$ in norm, then $U_n \to 1$ while $V_n \to (1-2P)$.

(2) If you look at the construction, you will note that $U = V(1-2Q) = (1-2P)V$.

(3) Some insight into the difference between U and V can be seen by looking at the case $\dim(\mathscr{H}) = 2$, $\dim(P) = \dim(Q) = 1$. If $\theta < \pi/2$ is the angle between Ran Q and Ran P, U is a rotation by angle θ. V is the reflection in the line at angle $\theta/2$ to Ran P and Ran Q (angle bisector).

(4) The proof can easily be modified to work in the case $\mathrm{Ker}(B) = \{0\}$ with no restriction on the norm, that is, V exists so long as $\mathrm{Ran}\, P \cap \mathrm{Ker}\, Q = \mathrm{Ran}\, Q \cap \mathrm{Ker}\, P = \{0\}$; we will extend this in the next section.

(5) In the second edition of his book, Kato mentions our V in a supplementary note but does not remark that V interchanges P and Q.

3. The Index of a Fredholm Pair of Projections

Definition. Let P, Q be orthogonal projections on a separable Hilbert space, $\mathscr{H}$. We say the pair (P, Q) is *Fredholm* if the map, $C \equiv QP$ viewed as a map from Ran P to Ran Q is Fredholm. The index of C is called the index of the pair, written index (P, Q).

Remark. For background material and, in particular, the definition of Fredholm, see the appendix.

Proposition 3.1. *(P, Q) is a Fredholm pair if and only if*

(1) *1 and -1 are isolated points of* $\mathrm{spec}(P-Q)$;

(2) $\mathrm{Ker}(P-Q\mp 1)$ *are both finite dimensional.*

Moreover,

$$\mathrm{index}(P, Q) = \dim \mathrm{Ker}(P-Q-1) - \dim \mathrm{Ker}(P-Q+1). \tag{3.1}$$

Proof. Let $A = P - Q$ and $C = QP$, the latter viewed as a map from Ran P to Ran Q. Then

$$\operatorname{Ker}(C \upharpoonright \operatorname{Ran} P) = \{\varphi \in \operatorname{Ran} P \mid Q\varphi = 0\} = \operatorname{Ker}(A - 1) \qquad (3.2a)$$

and similarly

$$\operatorname{Ran}(C \upharpoonright \operatorname{Ran} P)^{\perp} = \{\varphi \in \operatorname{Ran} Q \mid P\varphi = 0\} = \operatorname{Ker}(A + 1) \qquad (3.2b)$$

so (3.1) holds once we prove the equivalence.

To see the right side of (3.2a) note that $1 - A = (1 - P) + Q$ is a sum of nonnegative operators so $(1 - A)\varphi$ if and only if $(1 - P)\varphi = 0$ and $Q\varphi = 0$. The right side of (3.2b) is similar.

Suppose that (P, Q) is a Fredholm pair and that φ_n is an orthonormal Weyl sequence so that $\|(A - 1)\varphi_n\| \to 0$. Then $\langle \varphi_n, (P - Q)\varphi_n \rangle \to 1$ so $\|P\varphi_n\| \to 1$, $\|Q\varphi_n\| \to 0$. It follows that $\psi_n = P\varphi_n/\|P\varphi_n\|$ has norm 1, $\psi_n \to 0$ weakly and $(PQP)\psi_n \to 0$ so $0 \in \sigma_{\text{ess}}(C^*C)$. Thus, since (P, Q) is Fredholm, such a sequence φ_n does not exist which implies $1 \notin \sigma_{\text{ess}}(A)$. Similarly, looking at C^*, $-1 \notin \sigma_{\text{ess}}(A)$. But $\pm 1 \notin \sigma_{\text{ess}}(A)$ is equivalent to (1) + (2).

Conversely, let (1), (2) hold. We can then write $A = A_0 + F$, where F is finite rank and $A_0 \leqslant (1 - \varepsilon)1$ for some $\varepsilon > 0$. Since

$$\begin{aligned} PQP &= P(1 - A)P \\ &= -PFP + P(1 - A_0)P \\ &\geqslant -PFP + \varepsilon P \end{aligned}$$

we see that $0 \notin \sigma_{\text{ess}}(C^*C)$. By (Theorem A.4) this implies that (P, Q) is a Fredholm pair. ∎

Remark. In the above, to prove (P, Q) Fredholm we only used that 1 is an isolated point of spec(A), not that -1 is isolated, which may seem surprising. But the supersymmetry (use of B) shows that $\operatorname{spec}(A) \setminus \{1, -1\}$ is invariant under λ to $-\lambda$ so the two parts of (1) are not independent.

PROPOSITION 3.2. *A necessary and sufficient condition that (P, Q) be a Fredholm pair is that $P - Q = F + D$, where F, D are self-adjoint, $\|D\| < 1$ and F is finite rank.*

Proof. If $P - Q = F + D$, then clearly (1), (2) of Proposition 3.1 hold. Conversely, if (1), (2) hold, let $P_{\pm}$ be the projections onto the eigenspace for $P - Q$ with eigenvalues ± 1 and $P_0 = 1 - P_+ - P_-$. Then

$$P - Q = F + D$$

$F = P_+ - P_-$, $D = P_0(P - Q)P_0$ as required. ∎

THEOREM 3.3. *Let (P, Q) be a Fredholm pair. Then there exists a unitary operator V with*

$$VPV^{-1} = Q, \qquad VQV^{-1} = P$$

if and only if $\mathrm{index}(P, Q) = 0$.

Proof. If such a V exists, then $V(P-Q)V^{-1} = Q - P$ and thus by (3.1), $\mathrm{index}(P, Q) = 0$.

Conversely, let P, Q be a Fredholm pair; let $P_{\pm}$ be the projection on $\mathrm{Ker}(P - Q \mp 1)$ and $P_0 = 1 - P_+ - P_-$. If $\mathrm{index}(P, Q) = 0$, $\mathrm{Ran}(P_+)$ and $\mathrm{Ran}(P_-)$ have the same dimensions, so we can find U_0 a unitary map of $\mathrm{Ran}\, P_+$ to $\mathrm{Ran}\, P_-$. Define V_0: $\mathrm{Ran}(P_+ + P_-) \to \mathrm{Ran}\, P_+ \oplus \mathrm{Ran}\, P_-$ by $V_0(\varphi \oplus \psi) = U_0^* \psi \oplus U_0 \varphi$. As in the last section, let V_1 be $\mathrm{sgn}(B)$ viewed as a map from $\mathrm{Ran}(P_0) = \mathrm{Ker}(B)^{\perp}$ to $\mathrm{Ran}(P_0)$. Let $V_0 \oplus V_1$ on $\mathrm{Ran}(P_+ + P_-) \oplus \mathrm{Ran}(P_0)$. Then $VPV^{-1} = Q$; $VQV^{-1} = P$ as required.

Remark. Since $\|P - Q\| < 1$ implies that $\mathrm{index}(P, Q) = 0$, Theorem 3.3 extends Theorem 2.3.

THEOREM 3.4. (a) *Let (P, Q) be a Fredholm pair of projections. Then so is (Q, P) and*

$$\mathrm{index}(Q, P) = -\mathrm{index}(P, Q).$$

(b) *Let (P, Q) be a Fredholm pair of projections and let U be unitary. Then (UPU^{-1}, UQU^{-1}) is Fredholm and*

$$\mathrm{index}(UPU^{-1}, UQU^{-1}) = \mathrm{index}(P, Q).$$

(c) *Let (P, Q) and (Q, R) be Fredholm pairs and suppose either $Q - R$ or $P - Q$ is compact. Then (P, R) is a Fredholm pair and*

$$\mathrm{index}(P, R) = \mathrm{index}(P, Q) + \mathrm{index}(Q, R).$$

Proof. Parts (a) and (b) are easy. To prove (c), consider the case where $P - Q$ is compact. Then

$$PR = P(P-Q)R + PQQR.$$

$PQQR$ is the product of Fredholm operators and so Fredholm (by Theorem A.7) and the first term is compact so PR is Fredholm by Corollary A.5 and (by Theorem A.7),

$$\mathrm{index}(PR) = \mathrm{index}(PQ)(QR) = \mathrm{index}(PQ) + \mathrm{index}(QR)$$

as was to be proven.

EXAMPLE. LET $\mathscr{H}_0$ be a separable Hilbert space and let $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_0$. Let $P = \mathbb{1} \oplus 0$, $R = 0 \oplus \mathbb{1}$ and

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then (P, Q) and (Q, R) are Fredholm pairs but (P, R) is not, showing that the compactness requirement in (c) of the last theorem is not superfluous.

4. THE KATO DUAL, THE TRACE, AND THE INDEX OF PAIRS OF PROJECTIONS

Our goal in this section is to prove the following fact:

THEOREM 4.1. *Suppose P and Q are self-adjoint projections and $P - Q$ is in the trace ideal $\mathscr{I}_{2n+1}$ for some n. Then*

$$\operatorname{tr}((P-Q)^{2n+1})$$

is an integer and equals $\operatorname{index}(P, Q)$. *In particular, if $P - Q \in \mathscr{I}_{2n+1}$, then*

$$\operatorname{tr}((P-Q)^{2n+1}) = \operatorname{tr}((P-Q)^{2m+1}) \tag{4.1}$$

for $m \geqslant n$.

Remarks. (1) We will give two proofs.

(2) That $\operatorname{tr}(P-Q)$ is an integer if $P - Q \in \mathscr{I}_1$ is a result of Effros [4]. His proof makes use of the invariance of $(P-Q)^2$ and does not show that the contributions of $\pm\lambda$ cancel but only that they sum up to an integer!

The first proof uses the Kato dual $B = (1 - P - Q)$ to $A = P - Q$ and "standard supersymmetry arguments":

THEOREM 4.2. *Let $m_\lambda = \dim \operatorname{Ker}(P - Q - \lambda)$. Then for $\lambda \neq \pm 1$,*

$$m_\lambda = m_{-\lambda}.$$

Proof. Fix $0 < \lambda < 1$. Let

$$\mathscr{H}_{\pm\lambda} = \{\varphi \mid A\varphi = \pm\lambda\varphi\}.$$

Since $\{A, B\} = 0$, B maps $\mathscr{H}_{+\lambda}$ to $\mathscr{H}_{-\lambda}$. Moreover on $\mathscr{H}_+$, $B^2 = 1 - A^2 = (1-\lambda)^2 \mathbb{1}$. Thus B is an invertible map of $\mathscr{H}_{+\lambda}$ to $\mathscr{H}_{-\lambda}$ and the dimensions are equal. ∎

First Proof of Theorem 4.1. By Lidskii's theorem [10],

$$\begin{aligned}\operatorname{tr}((P-Q)^{2n+1}) &= \sum \lambda^{2n+1} m_\lambda \\ &= \sum_{\lambda>0} \lambda^{2n+1}(m_\lambda - m_{-\lambda}) \\ &= m_1 - m_{-1} \qquad \text{by Theorem 4.2} \\ &= \operatorname{index}(P, Q) \qquad \text{by Proposition 3.1.} \quad \blacksquare\end{aligned}$$

Second Proof of Theorem 4.1. We can write

$$\begin{aligned}(P-Q)^3 &= (P-Q)^2 P - (P-Q)^2 Q \\ &= P - Q + QPQ - PQP \\ &= P - Q + [QP, PQ] \\ &= P - Q + [QP, [P, Q]] \\ &= P - Q + [QP, [P-Q, Q]].\end{aligned}$$

Since $(P-Q)^2$ commutes with P and Q,

$$(P-Q)^{2n+3} = (P-Q)^{2n+1} + [QP, [(P-Q)^{2n+1}, Q]]. \tag{4.2}$$

Thus, if $(P-Q)^{2n+1}$ is trace class,

$$\operatorname{tr}((P-Q)^{2n+3}) - \operatorname{tr}((P-Q)^{2n+1}) = \operatorname{tr}([X, Y]) = 0$$

with Y trace class and so (4.1) holds. Since $\|P-Q\| \leqslant 1$,

$$\lim_{m\to\infty} \operatorname{tr}((P-Q)^{2m+1}) = \operatorname{index}(P, Q)$$

by Lidskii's theorem and Proposition 3.1. ∎

5. Unitary Deformations

In many applications, the pair P, Q is special in that $Q = UPU^{-1}$, that is

Definition. We say a pair (P, U) consisting of an orthogonal projection P and a unitary U is Fredholm if the pair of projections (P, Q) with $Q = UPU^{-1}$ is Fredholm. We set

$$\operatorname{index}(P; U) = \operatorname{index}(P, Q).$$

Remark. This may seem like a special situation, but because of infinite dimensions, it is not. Given P, Q in $\mathscr{H}$, an infinite-dimensional space, let $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$, $\tilde{P} = P \oplus I \oplus 0$ and $\tilde{Q} = Q \oplus I \oplus 0$, so $\tilde{P} - \tilde{Q}$ is $(P - Q) \oplus 0 \oplus 0$. It is easy to see there is a unitary with $U\tilde{P}U^{-1} = \tilde{Q}$.

PROPOSITION 5.1. (a) *If (P, U_1) and $(U_1 P U_1^{-1}, U_2)$ are Fredholm, then so is $(P, U_2 U_1)$ and*

$$\operatorname{index}(P; U_2 U_1) = \operatorname{index}(P; U_1) + \operatorname{index}(U_1 P U_1^{-1}; U_2).$$

(b) *If (P, U) is Fredholm and V is any unitary, then (VPV^{-1}, VUV^{-1}) is Fredholm and*

$$\operatorname{index}(P; U) = \operatorname{index}(VPV^{-1}; VUV^{-1}).$$

(c) *If (P, U) is Fredholm, then for any $n \in \mathbb{Z}$, (P, U^n) is Fredholm and*

$$\operatorname{index}(P; U^n) = n \operatorname{index}(P; U).$$

Proof. Parts (a) and (b) are a direct consequence of (c) and (b) of Theorem 3.4. Part (c) follows from (a), (b).

Remark. This result suggests a way that a pair (P, U) might have a fractional index, m/n: If (P, U^n) is Fredholm with index $(P; U^n) = m$. It remains to be seen if this has anything to do with the fractional Hall effect, but the possibility is attractive.

THEOREM 5.2. *(P, U) is a Fredholm pair if and only if PUP as a map from $P\mathscr{H}$ to $P\mathscr{H}$ is Fredholm and*

$$\operatorname{index}(P; U) = -\operatorname{index}(PUP \mid P\mathscr{H} \to P\mathscr{H}) = \operatorname{index}(PU^{-1}P \mid P\mathscr{H} \to P\mathscr{H}).$$

Proof. By definition of $\operatorname{index}(P, Q)$, we have that (P, U) is Fredholm if and only if $UPU^{-1}P$ is Fredholm as a map from $P\mathscr{H}$ to $UP\mathscr{H}$ and

$$\begin{aligned} \operatorname{index}(P; U) &= \operatorname{index}(UPU^{-1}P \mid P\mathscr{H} \to UP\mathscr{H}) \\ &= \operatorname{index}(PU^{-1}P \mid P\mathscr{H} \to P\mathscr{H}) \end{aligned}$$

since U is a unitary map from $P\mathscr{H}$ into $UP\mathscr{H}$. But since $\operatorname{index}(P, Q) = -\operatorname{index}(Q, P)$

$$\begin{aligned} \operatorname{index}(P; U) &= -\operatorname{index}(PUPU^{-1} \mid UP\mathscr{H} \to P\mathscr{H}) \\ &= -\operatorname{index}(PUP \mid P\mathscr{H} \to P\mathscr{H}). \quad \blacksquare \end{aligned}$$

Finally, we want to consider the relations to trace ideals. We will let $P_U = UPU^{-1}$ and consider three expressions,

$$\alpha_n(P, U) = \mathrm{tr}((P - P_U)^{2n+1})$$

$$\beta_n(P, U) = \mathrm{tr}(([P, U]\, U^*)^{2n+1})$$

$$\gamma_n(P, U) = \mathrm{tr}((P - PUPU^{-1}P)^n) - \mathrm{tr}((P - PU^{-1}PUP)^n).$$

THEOREM 5.3. (a) *For each* $n = 0, 1, \ldots$, $P - P_U \in \mathscr{I}_{2n+1}$ *if and only if* $[P, U] \in \mathscr{I}_{2n+1}$ *and then*

$$\mathrm{index}(P; U) = \alpha_n(P, U) = \beta_n(P, U).$$

(b) *If for any* $n = 1, 2, \ldots$, *both* $P - PUPU^{-1}P$ *and* $P - PU^{-1}PUP$ *lie in* $\mathscr{I}_n$, *then* $(P - P_U) \in \mathscr{I}_{2n+1}$, *and*

$$\gamma_n(P, U) = \alpha_n(P, U) = \mathrm{index}(P; U).$$

(c) *For any* $n = 1, 2, \ldots$, *if* $(P - P_U) \in \mathscr{I}_{2n}$, *then both* $P - PUPU^{-1}P$ *and* $P - PU^{-1}PUP$ *lie in* $\mathscr{I}_n$ *and*

$$\gamma_n(P, U) = \alpha_n(P, U) = \mathrm{index}(P; U).$$

Remark. Note that $\mathscr{I}_{2n+1}$ appears in (b) but that $\mathscr{I}_{2n}$ appears in (c).

Proof. (a) $P - P_U = [P, U]\, U^*$ so $P - P_U \in \mathscr{I}_q$ if and only if $[P, U] \in \mathscr{I}_q$. The equality follows from Theorem 4.1.

(b)(c) By (2.1) and (2.2)

$$P - PP_UP = P(P - P_U)^2 = (P - P_U)^2\, P$$

so since P commutes with $(P - P_U)^2$,

$$(P - PUPU^{-1}P)^n = P(P - P_U)^{2n}. \tag{5.1}$$

Similarly

$$\begin{aligned}(P - PU^{-1}PUP)^n &= U^{-1}(P_U - P_UPP_U)^n\, U \\ &= U^{-1}P_U(P - P_U)^{2n}\, U \end{aligned} \tag{5.2}$$

and in particular

$$(P - PUPU^{-1}P)^n - U(PU^{-1}PUP)^n\, U^{-1} = (P - P_U)^{2n+1}. \tag{5.3}$$

The trace ideal implications in (b) follow from (5.3) and in (c) from (5.2). Theorems 4.1 and (5.3) imply the index equalities. ∎

Remark. The expression $\gamma_n(P, U) = \text{index}(P; U)$ is in Hörmander [6] and Fedosov [5]. Connes proved a special case of $\beta_n = \text{index}(P; U)$.

6. The Non-Self-Adjoint Case

The above analysis depended on the assumption that P and Q are self-adjoint in many ways. For example, we used (e.g., in Proposition 3.1)

$$\text{Ker}(QP \mid \text{Ran } P) = \text{Ker}(P - Q - 1) = \{u \mid (P - Q - 1)^n u = 0 \text{ some } n\}$$

and it can happen that none of these equalities occur if P, Q are non-self-adjoint. Neither of the proofs of Theorem 4.1 extend to non-self-adjoint P, Q; but, as we shall see, the theorem does hold.

We extend the definition at the start of Section 3 to include the non-self-adjoint case by dropping the term "orthogonal." Note that while

$$\text{Ker}(QP \mid P\mathscr{H} \to Q\mathscr{H}) = \text{Ran } P \cap \text{Ker } Q,$$

we have that

$$\text{Ran}(QP \mid P\mathscr{H} \to Q\mathscr{H})^{\perp} = \text{Ran } Q \cap \text{Ker } P^* \qquad \text{not} \quad \text{Ran } Q \cap \text{Ker } P.$$

We have

THEOREM 6.1. *Let P, Q be arbitrary projections on a separable Hilbert space $\mathscr{H}$. Suppose $P - Q \in \mathscr{I}_{2n+1}$, the trace ideal. Then*

$$\text{tr}((P - Q)^{2m+1}) = \text{tr}((P - Q)^{2n+1}) \tag{6.1}$$

for $m \geq n$, (P, Q) is Fredholm and this integer is index(P, Q).

Proof. The formula (4.2) holds and it implies (6.1), so we need only prove that (P, Q) is Fredholm and the value is index(P, Q). Let $X = \text{Ran } P$ and $Y = \text{Ran } Q$ and $T: X \to Y$ be QP. Let $S: Y \to X$ be PQ. Then on X

$$K_1 \equiv 1 - ST = P - PQP = P(P - Q)P$$

and on Y

$$K_2 \equiv 1 - TS = Q(Q - P)Q$$

are both compact. Hence, by Theorem A.6, (P, Q) is a Fredholm pair. Indeed, $K_1^{n+1} = P(P-Q)^{2n+1} P$ and $K_2^{n+1} = Q(Q-P)^{2n+1} Q$ are trace class; so by Theorem A.6

$$\begin{aligned}\text{index}(P, Q) \equiv \text{index}(T) &= \text{tr}(K_1^{n+1}) - \text{tr}(K_2^{n+1}) \\ &= \text{tr}(P(P-Q)^{2n+2}) - \text{tr}(Q(P-Q)^{2n+2}) \\ &= \text{tr}((P-Q)^{2n+3})\end{aligned}$$

as was required. ∎

APPENDIX. A CHILD'S GARDEN OF INDEX

For the reader's convenience, we summarize the basic properties of Fredholm and index theory for operators from one separable Hilbert space, $\mathcal{H}_1$, to another, $\mathcal{H}_2$.

LEMMA A.1. *Let T be a closed operator. Then the following are equivalent*:

(1) Ran T *is closed.*

(2) 0 *is not in or is an isolated point of* $\sigma(T^*T)$, *the spectrum of* T^*T.

(3) *There exists* $c > 0$ *so that*

$$\|T\varphi\| \geqslant c \, \|\varphi\|$$

for all $\varphi \in \text{Ker}(T)^\perp$.

Proof. (2) ⇔ (3). Part (2) is equivalent to $T^*T \geqslant c^2$ on $\text{Ker}(T^*T)^\perp = \text{Ker}(T)^\perp$ and this is equivalent to (3).

(3) ⇒ (1). Let $u_n \in \text{Ran}(T)$ converge to u. We can find $\varphi_n \in \text{Ker}(T)^*$ so $T\varphi_n = u_n$. By (3)

$$\|\varphi_n - \varphi_m\| \leqslant c^{-1} \|u_n - u_m\|$$

so φ_n is Cauchy. Let $\varphi_n \to \varphi$. Then $T\varphi = u$ since T is closed.

(1) ⇒ (3). $T\colon (\text{Ker } T)^\perp \to \text{Ran } T$ is a closed bijection. If Ran T is closed, both spaces are complete, so that (3) follows from the inverse mapping theorem. ∎

LEMMA A.2. *If* Ran T *is not closed or* Ker(T) *is infinite dimensional, then there exists an orthonormal sequence* $\{\varphi_n\}_{n=1}^{\infty}$ *so that* $\|T\varphi_n\| \to 0$ *as* $n \to \infty$.

Proof. If Ran T is not closed, 0 is not an isolated point of $\sigma(T^*T)$ so infinitely many spectral projections $P_{(1/n, 1/n-1]}(T^*T)$ are non-zero. Choose φ_n in the ranges of the non-zero projections. ▌

DEFINITION. A closed operator T from $\mathscr{H}_1$ to $\mathscr{H}_2$ is called *Fredholm* if and only if

(1) $\mathrm{Ran}(T)$ is closed.

(2) $\mathrm{Ker}(T)$ is finite dimensional.

(3) $\mathrm{Ran}(T)^\perp$ is finite dimensional.

One then defines $\mathrm{index}(T) = \dim \mathrm{Ker}(T) - \dim[\mathrm{Ran}(T)^\perp]$.

The following is well-known (see, e.g., Deift [3]):

LEMMA A.3. *Let T be a closed operator from $\mathscr{H}_1$ to $\mathscr{H}_2$. Then*

(i) $\sigma(T^*T)\backslash\{0\} = \sigma(TT^*)\backslash\{0\}$

(ii) $\dim(\mathrm{Ker}(T^*T-\lambda)) = \dim(\mathrm{Ker}(TT^*-\lambda))$ *for* $\lambda \neq 0$.

Similarly if T is bounded from $\mathscr{H}_1$ to $\mathscr{H}_2$ and S is bounded from $\mathscr{H}_2$ to $\mathscr{H}_1$, then

(iii) $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$

(iv) $\dim(\mathrm{Ker}(TS-\lambda)^n) = \dim(\mathrm{Ker}(ST-\lambda)^n)$ *for* $\lambda \neq 0$.

for any n.

THEOREM A.4. *A closed operator T is Fredholm if and only if $0 \notin \sigma_{\mathrm{ess}}(T^*T)$, the essential spectrum of T^*T and $0 \notin \sigma_{\mathrm{ess}}(TT^*)$ and then* $\mathrm{index}(T) = \dim(\mathrm{Ker}(T^*T)) - \dim(\mathrm{Ker}(TT^*))$.

Proof. By Lemma A.1, $0 \notin \sigma_{\mathrm{ess}}(T^*T)$ if and only if $\mathrm{Ran}(T)$ is closed and $\dim \mathrm{Ker}(T) = \dim \mathrm{Ker}(T^*T)$ is finite. By Lemma A.3, 0 is isolated in $\sigma(T^*T)$ if and only if it is isolated in $\sigma(TT^*)$. Since $\mathrm{Ran}(T)^\perp = \mathrm{Ker}(T^*)$, the result follows. ▌

COROLLARY A.5. *Let T be a bounded Fredholm operator. Then, there is a constant $d > 0$ so that if*

$$A = X + Y$$

with $\|X\| < d$ and Y compact, then $T + A$ is Fredholm and

$$\mathrm{index}(T+A) = \mathrm{index}(T).$$

Proof. Let $d>0$ solve $d^2+2d\,\|T\|=c^2\equiv\inf[\sigma(T^*T)\setminus\{0\}]$. We claim $T(\lambda)\equiv T+\lambda A$ is Fredholm for all $\lambda\in[0,1]$. For by the definition of d and simple perturbation theory, $0\notin\sigma_{\mathrm{ess}}((T+\lambda X)^*\,(T+\lambda X))$, $0\notin\sigma_{\mathrm{ess}}((T+\lambda X)(T+\lambda X)^*)$. Since $\sigma_{\mathrm{ess}}(H)$ is invariant under compact perturbations, $T(\lambda)$ is Fredholm for $\lambda\in[0,1]$.

Since 0 is a discrete point of $\sigma(T(\lambda))$ or a resolvent point, we can apply the Kato–Rellich eigenvalue perturbation theory. $\dim(\mathrm{Ker}(T^*(\lambda)\,T(\lambda)))$ can only change by eigenvalues $e_n(\lambda)$ of $T^*(\lambda)\,T(\lambda)$ approaching 0. By Lemma A.3, the same eigenvalues and multiplicities occur for $T(\lambda)\,T^*(\lambda)$ and for self-adjoint operators, the sum of the dimensions of the eigenvalues is invariant under perturbation, so $\mathrm{index}(T(\lambda))$ is unchanged. ∎

Our next result involves algebraic multiplicities defined by

$$k_{\mathrm{alg}}(A,\lambda)=\dim\{u\mid(A-\lambda)^n\,u=0\text{ for some }n\}.$$

Recall that for A compact and $\lambda\neq 0$, $k_{\mathrm{alg}}(A,\lambda)<\infty$ and there exists some $n_0<\infty$ so that for $n\geqslant n_0$

$$\{u\mid(A-\lambda)^n\,u=0\}=\mathrm{Ker}((A-\lambda)^{n_0}).$$

We are heading towards

THEOREM A.6. *Let T be a bounded operator from $\mathscr{H}_1$ to $\mathscr{H}_2$. Then T is Fredholm if and only if there exists an approximate inverse S from $\mathscr{H}_2$ to $\mathscr{H}_1$ so that $K_1\equiv 1-ST$ is a compact operator on $\mathscr{H}_1$ and $K_2\equiv 1-TS$ is a compact operator on $\mathscr{H}_2$ and then*

$$\mathrm{index}(T)=k_{\mathrm{alg}}(K_1,1)-k_{\mathrm{alg}}(K_2,1). \tag{A.1}$$

Moreover, if $K_1^n, K_2^n\in\mathscr{I}_1$, the trace class, then

$$\mathrm{index}(T)=\mathrm{tr}(K_1^n)-\mathrm{tr}(K_2^n). \tag{A.2}$$

EXAMPLE. Let $\mathscr{H}_1=\mathscr{H}_2=\ell_2([1,\infty))$ with natural basis $\{e_n\}_{n=1}^\infty$. Let

$$Te_n=e_{n+1},\qquad n=1,2,\ldots$$

$$Se_n=e_{n-1},\qquad n=3,4,\ldots$$

$$Se_2=0$$

$$Se_1=e_1.$$

Then $\operatorname{index}(T) = -1$. $\dim \operatorname{Ker}(ST) = k_{\text{alg}}(ST, 1) = 1$. $\dim \operatorname{Ker}(TS) = 1$ but $k_{\text{alg}}(TS, 1) = 2$. Thus (A.1) holds, but

$$\operatorname{index}(T) \neq \dim \operatorname{Ker}(ST) - \dim \operatorname{Ker}(TS)$$

so it is necessary to discuss algebraic multiplicities.

Proof of Theorem A.6. T is a closed bijection from $(\operatorname{Ker} T)^\perp$ to $\operatorname{Ran} T$, so there is an inverse S_0 from $\operatorname{Ran} T$ to $(\operatorname{Ker} T)^\perp$. Extend S_0 to $\mathscr{H}_2$ by setting $S_0 = 0$ on $(\operatorname{Ran} T)^\perp$. Then, if $P =$ Projection onto $\operatorname{Ker} T$ and $Q =$ Projection on $(\operatorname{Ran} T)^\perp$, then $S_0 T = 1 - P$ and $TS_0 = 1 - Q$ so S_0 is the required S and $k_{\text{alg}}(S_0 T, 1) = \dim(P)$ while $k_{\text{alg}}(TS_0, 1) = \dim(Q)$ so (A.1) holds.

Now, given any S with $1 - ST$ and $1 - TS$ compact, let $S(\theta) = \theta S + (1 - \theta) S_0$. By Lemma A.3, for $\lambda \neq 0$, $k_{\text{alg}}(S(\theta) T, \lambda) = k_{\text{alg}}(TS(\theta), \lambda)$ and by eigenvalue perturbation theory, the sum of the algebraic dimensions is invariant under perturbation, so (A.1) holds.

By Lidskii's theorem [10],

$$\begin{aligned}
\operatorname{tr}(K_1^n) - \operatorname{tr}(K_2^n) &= \sum_{\substack{\lambda \in \operatorname{spec}(K_1) \cup \operatorname{spec}(K_2) \\ \lambda \neq 0}} \lambda^n [k_{\text{alg}}(K_1, \lambda) - k_{\text{alg}}(K_2, \lambda)] \\
&= \sum_{\substack{\lambda \in \operatorname{spec}(K_1) \cup \operatorname{spec}(K_2) \\ \lambda \neq 0}} \lambda^n [k_{\text{alg}}(ST, 1 - \lambda) - k_{\text{alg}}(TS, 1 - \lambda)].
\end{aligned}$$

By Lemma A.3(iv), all terms with $\lambda \neq 1$ vanish, so (A.1) implies (A.2). ∎

We are heading towards a proof of

THEOREM A.7. *Let X, Y, Z be separable Hilbert spaces and $A: Y \to Z$ and $B: X \to Y$ Fredholm. Then AB is Fredholm and*

$$\operatorname{index}(AB) = \operatorname{index}(A) + \operatorname{index}(B). \tag{A.3}$$

LEMMA A.8. *Let $C: X \to Y$ be Fredholm. Then, $C = U + F$ with U invertible and F finite rank if and only if* $\operatorname{index}(C) = 0$.

Proof. If $C = U + F$, then $\operatorname{index}(C) = \operatorname{index}(U) = 0$ since $\operatorname{Ker}(U) = \operatorname{Ran}(U)^\perp = \{0\}$.

Conversely, if C is Fredholm we can write

$$X = (\operatorname{Ker} C)^\perp \oplus \operatorname{Ker} C, \qquad Y = \operatorname{Ran} C \oplus (\operatorname{Ran} C)^\perp$$

and C is invertible from $(\operatorname{Ker} C)^{\perp}$ to $\operatorname{Ran} C$. If $\operatorname{index}(C)=0$, then $\dim(\operatorname{Ker} C)=\dim((\operatorname{Ran} C)^{\perp})<\infty$, so we can find F_0 invertible from one space to the other. Let P be the orthogonal projection on $\operatorname{Ker} C$ and $F=-F_0P$ and let $U=C-F$. Then U is invertible, F is finite rank and $C=U+F$. ∎

LEMMA A.9. *Let $X=X_1\oplus X_2$, $Y=Y_1\oplus Y_2$ and let $A_1: X_1\to Y_1$ and $A_2: X_2\to Y_2$ be Fredholm. Then $A\equiv A_1\oplus A_2: X\to Y$ is Fredholm and*

$$\operatorname{index}(A)=\operatorname{index}(A_1)+\operatorname{index}(A_2).$$

Proof. Elementary. ∎

Proof of Theorem A.7. Let W be an infinite-dimensional Hilbert space. Find $A_1: W\to W$ with $\operatorname{index}(A_1)=-\operatorname{index}(A)$ and $B_2: W\to W$ with $\operatorname{index}(B_2)=-\operatorname{index}(B)$; for example, take $W=\ell_2([0,\infty))$ and let A_1, B_2 be suitable left or right coordinate shifts. Let $\tilde{X}=X\oplus W\oplus W$, $\tilde{Y}=Y\oplus W\oplus W$, and $\tilde{Z}=Z\oplus W\oplus W$. Define $\tilde{A}: \tilde{X}\to\tilde{Y}$ and $\tilde{B}: \tilde{Y}\to\tilde{Z}$ by

$$\tilde{A}(x, w_1, w_2)=(Ax, A_1w_1, w_2)$$

$$\tilde{B}(x, w_1, w_2)=(Bx, w_1, B_2w_2).$$

By Lemma A.9, $\tilde{A}$, $\tilde{B}$ are both Fredholm and both have index 0. Thus, $\tilde{A}=U_1+F_1$, $\tilde{B}=U_2+F_2$ where U_i are invertible and F_i are finite rank. Thus

$$\tilde{A}\tilde{B}=U_1U_2+U_1F_2+F_1U_2+F_1F_2=\tilde{U}+F$$

with $\tilde{F}$ finite rank and $\tilde{U}=U_1U_2$ invertible and so Fredholm. It follows that $\tilde{A}\tilde{B}$ is Fredholm and

$$\operatorname{index}(\tilde{A}\tilde{B})=0.$$

But

$$\tilde{A}\tilde{B}(x, w_1, w_2)=(ABx, A_1w_1, B_2w_2)$$

so AB is Fredholm and by Lemma A.9,

$$\operatorname{index}(AB)+\operatorname{index}(A_1)+\operatorname{index}(B_2)=0$$

that is, (A.3) holds. ∎

Acknowledgments

We thank M. Ben-Artzi for the hospitality of Hebrew University where some of this work was done and I. Kaplansky for a useful question. The research is partially supported by GIF.

Note added in proof. After the preparation of this manuscript, we learned that T. Kato discovered the key anti-commutation relation in Theorem 2.1(2) in 1971 in an unpublished manuscript. We also mention a related paper by C. Davis, [Separation of two linear subspaces, *Acta Sci. Math. Szeged* **19** (1958), 172–187]. Davis' paper and the current one illuminate each other. We thank T. Kato for telling us of his unpublished work and of the paper of Davis.

References

1. J. Avron, R. Seiler, and B. Simon, Charge deficiency, charge transport and comparison of dimensions, *Commun. Math. Phys.*, to appear.
2. A. Connes, Noncommutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* **62** (1986), 257–360; and "Géométrie Non-Commutative," InterEdition, Paris, 1990.
3. P. Deift, Applications of a commutation formula, *Duke Math J.* **45** (1978), 267–310.
4. E. Effros, Why the circle is connected, *Math. Intelligencer* **11** (1989), 27–35.
5. B. Fedosov, Direct proof of the formula for the index of an elliptic system in Euclidean space, *Functional Anal. Appl.* **4** (1970), 339–341.
6. L. Hörmander, The Weyl calculus of pseudo-differential operators, *Comm. Pure Appl. Math.* **18** (1965), 501–517.
7. R. Kadison, private communication.
8. T. Kato, "Notes on Projections and Perturbation Theory," Technical Report No. 9, University of California, 1955, unpublished.
9. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin/Heidelberg, 1966.
10. B. Simon, "Trace Ideals and their Applications," Cambridge University Press, Cambridge, 1979.

Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis