Hall Conductance and Adiabatic Charge Transport of Leaky Tori

Joseph E. Avron, Markus Klein,[a] and Ayelet Pnueli
Department of Physics, Technion–Israel Institute of Technology, Haifa, 32000, Israel

Lorenzo Sadun
Department of Mathematics, University of Texas, Austin, Texas 78712
(Received 27 February 1992)

Leaky tori are two-dimensional surfaces with g handles and r horns (punctures) that extend to infinity. They model certain features of mesoscopic systems with multiple Aharonov-Bohm-type geometries connected to infinitely long leads. For a large class of leaky tori with finite area, in the presence of a constant magnetic field and threading fluxes, we calculate exactly the persistent currents, adiabatic charge transport, and the (appropriately defined) Hall conductances.

PACS numbers: 72.10.Bg, 03.65.−w

Leaky tori, a term coined by Gutzwiller [1], are two-dimensional analytic surfaces with g handles and r punctures. The metric near the punctures is such that it describes infinitely long horns. An example with one handle and two horns is shown in Fig. 1. Leaky tori with finite area have horns (called cusps by mathematicians) that are effectively one dimensional near infinity and capture some of the features of mesoscopic systems which are multiply connected and connect to infinitely long, one-dimensional, leads. Unlike mesoscopic systems, leaky tori have no boundaries at finite distances.

Aharonov-Bohm fluxes for leaky tori fall into three distinct classes: r fluxes ϕ_j, $j=1,\ldots,r$, which thread the horns; $2g$ fluxes ϕ_j, $j=r+1,\ldots,r+2g$, through the g handles; and fluxes that pierce the surface. We shall take one piercing flux ϕ_0. We denote the fluxes collectively by $\boldsymbol{\phi}$ and with the piercing flux excluded by ϕ.

With such a surface we associate a Landau Hamiltonian (the Schrödinger operator in magnetic field B) $H(B,\boldsymbol{\phi})$. We shall describe *exact* results for (1) the low-lying eigenvalues $E_n(B,\boldsymbol{\phi})$ and their degeneracies; (2) the $2g+r$ persistent currents associated to the jth flux and nth eigenvalue defined by $-\partial_{\phi j}E_n$; (3) the charges transported from infinity along the ith horn and then back to infinity along the jth horn when some of the fluxes change adiabatically by a unit of quantum flux; and (4) the Hall conductances at fixed Fermi energy E_F, set in a gap [2].

The results hold for a large class of leaky tori in the presence of a constant magnetic field B, and parts extend to more general surfaces and Hamiltonians. For reasons of space, we shall merely outline the basic strategies involved in the more mathematical issues. Detailed proofs shall be presented elsewhere [3].

Leaky tori with finite volume and constant Gaussian curvature $K=-1$ can be represented as the quotient of the complex upper half plane $\mathbf{H}$, with the Poincaré metric $ds^2=y^{-2}(dx^2+dy^2)$, by some discrete subgroup Γ of $SL(2,\mathbb{R})$ [4]. Thus the surface is represented by a polygon in $\mathbf{H}$ (the boundary of a fundamental domain) with appropriate identifications of the sides. Recall that the geodesics are semicircles centered on the boundary of $\mathbf{H}$, $\partial\mathbf{H}=\{y=0\}\cup\{\infty\}$, and the sides of the polygon can be taken to be circular arcs. Each horn corresponds to a cusplike vertex of the polygon (with zero opening angle) on $\partial\mathbf{H}$. A fundamental polygon for the leaky torus of Fig. 1 is shown in Fig. 2. Specification of curvature, number of horns, and number of handles does not fix a unique surface, but a choice of Γ does (up to translations, scaling, and hyperbolic rotations—the isometries of $\mathbf{H}$). There is a $[6(g-1)+2r]$-dimensional family of such leaky tori, known as the moduli space. Our results turn out to be completely independent of the choice of the point in the moduli space.

By the Gauss-Bonnet theorem we have $\int K=2\pi(2-2g-r)$. Therefore, the area is $2\pi(2g+r-2)$. We set $-\hbar^2K/2m=1$, and choose the quantum flux unit to be

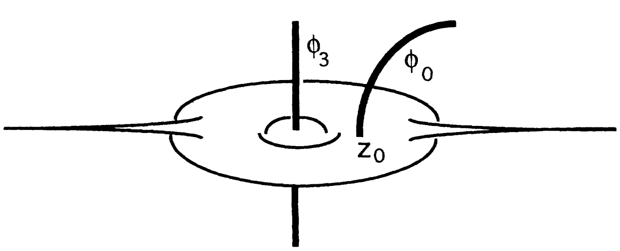

FIG. 1. Leaky torus with one handle and two horns. ϕ_3 is a flux threading the handle and ϕ_0 is a piercing flux at z_0.

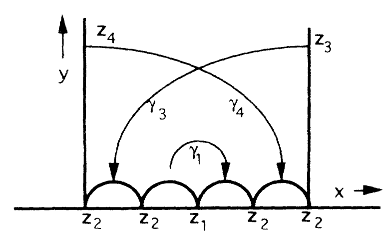

FIG. 2. The fundamental domain (polygon) for the leaky torus with one handle and two horns. The arrows show the identifications. All cusps except z_1 are identified. The points z_{3},z_4 are base points used in the text to define the handle flux; see Eq. (4).

2π.

Because of the negative curvature, the geodesic flow on such surfaces is chaotic. The Laplacian associated with compact multihandle tori has been studied in the context of chaology [1,5,6]. In classical dynamics at low energies, $[0,B^2)$, the Lorentz force dominates and one finds closed orbits, while at high energies, (B^2,∞), the negative curvature dominates and the dynamics is (presumably) chaotic [7].

We take $B\geq 1$, i.e., the magnetic field (two-form) $By^{-2}dx\wedge dy$ is a positive multiple of the area form. (Think of B as drawn outwardly normal.) The orientation of the surface fixes an orientation for the loops that encircle the piercing and horn fluxes. (Think of these fluxes as penetrating the surface.) The orientation of the surface does not pick a direction for the loops that encircle the handles: For each handle one arbitrarily chooses a direction for one loop and the direction for the second loop is then fixed by the orientation of the surface.

The fluxes and B are external parameters in the Schrödinger operator. They are related by a type of Dirac quantization condition

$$\int B-\phi_0-\sum_1^r\phi_j=0 \pmod{2\pi}. \qquad (1)$$

Note that the fluxes through the handles do not enter this relation, while the horn and piercing fluxes do.

Representing a leaky torus as a fundamental domain for a group Γ and using the fixed gauge $A_0=y^{-1}B\,dx$, the Schrödinger operator for noninteracting spinless electrons, with $\phi_0=0$, is

$$H(B,\phi)=y^2(-\partial_x^2-\partial_y^2)+2iBy\partial_x+B^2, \qquad (2)$$

acting on functions which satisfy

$$\psi(\gamma z)=u(\gamma,z)\psi(z),\quad u(\gamma,z)\equiv v_\phi(\gamma)\frac{(cz+d)^{2B}}{|cz+d|^{2B}}. \qquad (3)$$

Here $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ acts on $\mathbf{H}$ as a Möbius transformation, and $v_\phi(\gamma)$, a complex number of modulus one, is a *multiplier* system on Γ associated to the fluxes. The twisted "periodicity" condition in Eq. (3) is the analog of the familiar twisted boundary condition for flat tori [8]; in both cases the notion of periodicity takes into account gauge transformations that connect two "unit cells." The properties of a multiplier system are the consistency conditions ensuring univaluedness of the wave function on the universal covering space, namely, $v_\phi(-1)=e^{-i2\pi B}$ and

$$u(\gamma_1\gamma_2,z)=u(\gamma_1,\gamma_2z)u(\gamma_2,z),\quad \gamma_1,\gamma_2\in\Gamma.$$

We fix the relation of multipliers to fluxes by setting

$$u(\gamma_j,z_j)=e^{i\phi_j},\quad 1\leq j\leq 2g+r, \qquad (4)$$

where, for $1\leq j\leq r$, z_j is the site of the jth cusp, and γ_j generates the subgroup of Γ that leaves z_j fixed. For $r+1\leq j\leq r+2g$, γ_j is a transformation identifying sides in the fundamental polygon which dissect the g handles, and z_j is an *arbitrary* reference point on such a side. (There is no distinguished reference point on the $2g$-dimensional torus of handle fluxes which naturally corresponds to zero flux.) A piercing flux at z_0 can be added via the usual vector potential, singular at z_0.

The spectral analysis of Schrödinger operators with magnetic fields and flux tubes on leaky tori has a very long history in the theory of automorphic forms, where it is known as the spectral analysis of the Maass-Selberg Laplacian for nonclassical automorphic forms of real weight with multipliers [9]. Below we list some key facts. Towards the end of this Letter we shall indicate how they can be derived.

One distinguishes *four* energy ranges: $(-\infty,B)$, which is outside the spectrum; low energies $[B,B^2)$, where the spectrum is reminiscent of the usual *Landau levels* in the plane; intermediate energies $(B^2,B^2+\frac{1}{4})$, where except for being discrete little is known about the spectrum; and high energies $(B^2+\frac{1}{4},\infty)$, which admit scattering states if at least one horn flux is zero.

Scattering states.— Each horn which is threaded by a flux tube carrying an *integer number of flux quanta* is an open scattering channel. Each such scattering channel contributes the interval of energies $[\frac{1}{4}+B^2,\infty)$, with multiplicity 1, to the absolutely continuous spectrum. Horns that carry fluxes which are not integral are in some sense plugged, and a particle cannot leak through such horns to infinity. If all the horns are plugged, the spectrum in $[\frac{1}{4}+B^2,\infty)$ is discrete.

Maass supersymmetry.— For $B\geq 1$ and fixed multiplier system $v_\phi(\gamma)$, the spectrum of $H(B)$ coincides with the spectrum of $H(B+1)-(2B+1)$ with the ground state removed, counting multiplicity.

Spectrum of Landau levels when $\phi_0=0$.— In the interval of energies $[B,B^2]$, $B\geq 1$, for $\phi_0=0$, the spectrum has $[B-\frac{1}{2}]$ points at energies

$$E_n(B,\phi,\phi_0=0)\equiv B(2n+1)-n(n+1),$$
$$n=0,1,\ldots,[B-\tfrac{3}{2}], \qquad (5)$$

where $[x]$ denotes the integer part of x. E_n depends explicitly on B only, and implicitly on the horn fluxes through the Dirac quantization condition. It is completely *independent of the fluxes through the handles* [10].

Degeneracy of Landau levels when $\phi_0=0$.— Like the *Landau levels* on the flat torus, the energies in Eq. (5) are in general degenerate. Unlike them their degeneracy decreases with energy. Let $\lfloor x\rfloor$ be the greatest integer strictly smaller than x and set $\{x\}\equiv x-\lfloor x\rfloor\in(0,1]$. The degeneracy of the nth *Landau level* is then given by

$$D(n,\phi_0=0)=(B-n)(2g-2+r)$$
$$-\sum_{j=1}^r\left\{\frac{\phi_j}{2\pi}\right\}-(g-1)+\delta_{B,1}\delta_{v,1}. \qquad (6)$$

By Dirac quantization, the right-hand side of Eq. (6) is an integer. Equations (5) and (6) are invariant under deformations of the leaky torus within the moduli space. Since generically the spectrum and degeneracies are sensitive to deformations, this invariance is remarkable, especially since the dimension of the moduli space can be large. In particular, it will follow from this invariance that the transport properties we shall calculate are constant on the moduli space.

Adiabatic charge transport.— Suppose that initially all horn fluxes are noninteger. Thus, all horns are plugged. Now vary two of the horn fluxes, e.g., along the line $\phi_i+\phi_j=\text{const}\neq 0 \pmod{2\pi}$ by decreasing ϕ_i by 2π. The initial and final Hamiltonians are unitarily equivalent: Up to a gauge transformation, the Schrödinger operator underwent a closed cycle. In particular, the initial and final spectra, counting multiplicity, coincide. As in Laughlin's original argument, this cycle can transport net charge, and indeed it does: As ϕ_i passes through an integral flux quantum, horn i opens briefly and, according to Eq. (6), one state per Landau level is sucked in from (spatial) infinity. As ϕ_j passes through an integral flux quantum, these additional states disappear at (spatial) infinity via horn j. If N Landau levels are occupied, N charges will be transported. The cycle describes a quantum charge pump which transports integer charges. It is noteworthy that it gives integral adiabatic charge transport for systems whose area is finite. In the Hall effect and in the Niu charge pump [11] precise integers require the thermodynamic limit.

Hall conductance.— For the plane Laughlin defines the Hall conductance as the charge transported to infinity by increasing the piercing flux by 2π. In the present context the charge can be transported to infinity along any of the r horns, and moreover, by Dirac quantization, the piercing flux cannot be varied independently. We therefore define the jth Hall conductance as the charge Q_j transported to infinity along the jth horn, increasing ϕ_0 by 2π along the path $\phi_0+\phi_j=\text{const}$. All these r Hall conductances turn out to be identical. Unfortunately, since Eqs. (5) and (6) hold for $\phi_0=0$, we cannot directly follow the charge transport along the path $\phi_0+\phi_j=\text{const}$. To compute the charge transport we therefore deform the path: First, the ϕ_0 increase is compensated by an increase in B by $2\pi/\text{area}$. This changes the degeneracy of all *Landau levels* by 1. Then B decreases to its original value at the expense of the jth horn flux. This sends one particle per *Landau level* to infinity along the jth horn, while the Hamiltonian returns to its initial form up to unitary equivalence. We see that *the Hall conductance of each Landau level (for noninteracting electrons) is unity, for all leaky tori,* if the magnetic field is large enough, i.e., $B\geq 1$.

It is of interest to see where the charge is transported from. To see what happens we argue as follows: Consider first plugging all the horns by choosing the horn fluxes appropriately. None of the new states can arrive from infinity because the horns are plugged. Instead, the states must come from high energies.

These results for the Hall conductances are a sweeping generalization of what one knows for *Landau levels in* the plane where, interestingly enough, the condition on the strength of B does not enter [12].

The energy and degeneracies are independent of the fluxes through any of the handles, and so is Dirac quantization. It follows that manipulating handle fluxes only does not transport any charge from infinity even if the horns are open.

Persistent currents.— Equations (5) and (6) determine (most of) the persistent currents for the *Landau levels* in the absence of piercing fluxes. (1) The persistent currents around the handles vanish. (2) Setting $\partial_{\phi_j}=(\text{area})^{-1}\partial_B$ in view of the Dirac quantization condition, the persistent current associated to ϕ_j in the jth horn and the full nth *Landau level* is equal to

$$\partial_{\phi_j}E_n=(2n+1)D(n,\phi_0=0)/2\pi(2g-2+r)\,.$$

Having the same sign for different energy levels, these currents add up coherently (while in a flux driven ring they have alternating sign). (3) The persistent currents around the piercing flux involve variation in ϕ_0 and cannot be calculated from Eqs. (5) and (6) alone.

We now briefly outline a derivation of Eqs. (1), (5), and (6). Equation (5) can be found, e.g., in Rölcke [9]. Equations (1) and (6) do not appear in the literature in the form given here, because the notion of fluxes is foreign to the tradition of the theory of automorphic forms, which instead has been dominated by the notion of *multipliers* described above. Multipliers are related to fluxes by Eq. (4), but, unfortunately, are not gauge invariant. Once one establishes Eq. (4) and uses the correspondence between classical automorphic (cusp) forms and the L^2 eigenfunctions of $H(B,\phi)$ (see, e.g., Rölcke), Eq. (6) can be found in Hejhal or Petersson [9]. Both the Dirac quantization condition and the formula for the degeneracy are due to Petersson, and go back to 1938. His proof of the degeneracy formula is based on the Riemann-Roch theorem. In [3] we plan to give derivations of Eqs. (1), (5), and (6) motivated by quantum mechanics rather than automorphic forms and analytic number theory. The basic tool is Maass supersymmetry which identifies the ground-state degeneracy with an index which can be computed from the small time asymptotics of the heat kernel. Because of the punctures the manifold is noncompact, and one needs to compute corrections to the heat kernel from the boundaries. Like most higher-order corrections to the heat kernel, this is a computational effort. The boundary terms lead to the flux-dependent terms in Eq. (6) (the handle fluxes drop because they are not boundary terms). The fluxes may be thought of as a one-dimensional version of the three-dimensional η invariant for the signature [13].

We close with remarks on open problems. (1) The notion of Hall conductance used here can naturally be generalized to leaky tori with infinite area where $\mathrm{Deg}(n) = \infty$, along the lines of [14]. Since the Hall conductance for leaky tori is area independent, it is natural to conjecture that the Hall conductance for each Landau level is unity also for leaky tori with infinite area, but we have no proof which holds in any generality. For the entire upper half plane, however, this holds by explicit computation. (2) There are additional $\binom{2g}{2}$ interesting adiabatic transport coefficients associated with transport of charges around the handles due to the fluxes through the handles. Such transport coefficients are related to first Chern classes [15]. From the spectral results in this paper it follows that, since variation of the handle fluxes never gives level crossing, all these first Chern classes as well as higher Chern classes are constant on the moduli space. By analogy with other adiabatic transport coefficients, we expect some of them to be nontrivial in general.

We thank D. Freed for pointing out the relation with η invariants. The research is supported by BSF, the Fund for the Promotion of Research at the Technion, Elron-Elbit Electronics Research grant, Minerva, and the SFB Geometrie und Physik at TU-Berlin.

[a]On leave of absence from Technische Universität Berlin, Fachbereich Mathematik, D-1000 Berlin 12, Germany.

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