I. REVIEW OF LINEAR ALGEBRA- EXERCISES

1) Prove that the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is not diagonalizable.

2) Let $\mathbf{v} \in \mathbb{R}^{\mathbf{3}}$ be a vector with $\|\mathbf{v}\| = \mathbf{1}$ and $\theta \in \mathbb{R}$ any real number. Prove that

$$\exp(i\theta\mathbf{v} \cdot \sigma) = \cos\theta\,\mathbf{I} + \mathbf{i}\sin\theta\,(\mathbf{v} \cdot \sigma),$$

where I is the identity operator on $\mathbb{C}^2$ and σ is the Pauli vector operator of components $\sigma_x, \sigma_y, \sigma_z$.

3) Find the *left polar* and *right polar* decompositions of the matrix A defined in Ex. 1.

4) Consider two spin 1/2 particles defined on the Hilbert spaces V, W and the corresponding sets of spin operators

$$s_x = 1/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ s_y = 1/2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ s_z = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

on V, and analogously,

$$S_x = 1/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_y = 1/2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ S_z = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

on W.

Write explicitly, on the tensor product space $V \otimes W$, the matrices $s_j \otimes I$, $I \otimes S_k$, $s_j \otimes S_k$, with $j, k = x, y, z$.

5) With the same definitions as in Ex. 4, consider in $V \otimes W$ the *singlet state*

$$|\Psi\rangle = 1/\sqrt{2}\{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\},$$

where $|0\rangle, |1\rangle$ are the eigenvectors of σ_z corresponding to the eigenvalues $1, -1$ respectively.

Show that the mean values over the state $|\Psi\rangle$ satisfy

$$\langle s_j \rangle = \langle S_k \rangle = 0;\ \langle s_j \otimes S_k \rangle = -1/4\delta_{jk}.$$