EXERCISES FOR LECTURE II- 26/10/2004

1) **Hilbert–Schmidt inner product**

Let L_V be the set of linear operators on a Hilbert space V. It is easily seen that L_V is a linear space. Show that the function $\langle\, ,\, \rangle$ on $L_V \times L_V$ defined by

$$\langle A, B\rangle =: tr[A^\dagger B],\ A, B \in L_V$$

is an inner product. Show that if the dimension of V is d, the one of L_V is d^2. Find a basis of hermitian matrices for L_V, orthonormal with respect to the above inner product.

2)Consider the two–qubit *singlet state*

$$|\Psi\rangle = 1/\sqrt{2}\{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\},$$

where $|0\rangle$, $|1\rangle$ represent the vectors of the computational basis, and let

$$\rho = |\Psi\rangle\langle\Psi|$$

be the corresponding *density operator*.
Show that

$$\rho = 1/4(I \otimes I - \sum_m \sigma_m \otimes \sigma_m),$$

where I is the identity and σ_m, $m = x, y, z$ are the Pauli operators.

3) Using only CNOTs and Toffoli gates, construct a quantum circuit that performs the transformation

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

4) Consider the operator $\sigma \cdot \mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^{\mathbf{3}}$, $\|\mathbf{v}\| = \mathbf{1}$. Show that $\sigma \cdot \mathbf{v}$ has eigenvalues ± 1 and that the projectors onto the corresponding eigenspaces are $P_\pm = \frac{1}{2}(I \pm \sigma \cdot \mathbf{v})$. Compute the probabilities of obtaining $+1$ for a measurement of $\sigma \cdot \mathbf{v}$ when the initial states of the system are $|0\rangle, |1\rangle$.