EXERCISES FOR LECTURE III- 02/11/2004

1) **B-C-H Identity**

For A, B hermitian operators on some Hilbert space H, with $[A, B] \neq 0$, show that if

$$[A, [A, B]] = [B, [A, B]] = 0,$$

then

$$\exp(A)\exp(B) = \exp(A + B)\ \exp([A, B]/2).$$

2) Show that any 2×2 *real projector* with unit trace can be written as

$$P_\theta = 1/2\,(I + \sigma_x \sin\theta + \sigma_z \cos\theta).$$

3) Consider the problem of perfectly distinguishing between the two states

$$|u\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix},\ |v\rangle = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \alpha \in (0, \pi/4),$$

assuming that they are given with equal prior probabilities $p_u = p_v = 1/2$. Construct a suitable $POVM$ with three elements, E_u, E_v, E_0, where 0 labels the *inconclusive answer*, to identify these states.

4) Consider the action of the $CNOT$ gate on the two-qubit basis defined by the vectors $\{|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}^{\otimes 2}$. Show that, with respect to this new basis, the state of the target is not changed, while the state of the control is flipped if the target starts as $|-\rangle$, otherwise it is unchanged. Draw this circuit using a $CNOT$ gate and two Hadamard gates at the input and at the output, and its equivalent circuit represented by a swapped-$CNOT$ gate (as proven by the above calculations).

5) Let C be a $CNOT$ with qubit 1 as control and qubit 2 as target. Show that:

i) $CX_1C = X_1X_2$

ii) $CZ_1C = Z_1$

iii) $CY_2C = Z_1Y_2$,

where 1, 2 refer to the first, second qubit respectively.