EXERCISE 4, 09/11–16/11/2004

1) Superdense coding

Consider the quantum superdense coding protocol and let E be any *positive operator* acting on Alice's qubit.
Show that $\langle\psi|E \otimes I|\psi\rangle$ takes *the same value* when $|\psi\rangle$ is any of the four vectors of the Bell basis. Suppose that a third party, Eve, intercepts Alice's qubit on its way to Bob.
Can Eve infer which of the four possible pairs of bits, $00, 01, 10, 11$ is Alice sending to Bob? If yes, how? If no, why not?

2)Projector operators

Let $|u\rangle, |v\rangle$ be normalized vectors. Show that $|u\rangle\langle u|$ and $|v\rangle\langle v|$ are projectors. Moreover, show that $|u\rangle\langle u| + |v\rangle\langle v|$ is a projector if and only if $\langle u|v\rangle = 0$. Generalize this result to an arbitrary number of vectors.

3)Composite systems

3.1) Suppose that a composite system of A and B is in the state

$$|\psi\rangle = |a\rangle \otimes |b\rangle,$$

where $|a\rangle$, $|b\rangle$ are pure states on H_A, H_B respectively. Show that the reduced density operator of the system A alone is also a pure state.
3.2) Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number one.

4) Schmidt decompositions

Consider in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the two-qubit state $|\psi\rangle$ of components (0.1; 0.3+0.4 i; 0.5 i; -0.7).
Find the Schmidt decomposition of $|\psi\rangle$.