EXERCISE 5 (DUE BY 16/12/2004)

1) **Quantum state distance**

Consider two density operators ρ_1, ρ_2 defined on an N-dimensional Hilbert space $\mathcal{H}$ and a measurement defined by the complete set of one dimensional orthogonal projectors

$$\{E_\alpha; \alpha = 1, ..., N\}$$

with

$$\sum_\alpha E_\alpha = I.$$

Recall that the probability of detecting outcome α by measuring ρ_1, ρ_2 is respectively

$$P_{1(2)}(\alpha) = Tr[\rho_{1(2)} E_\alpha].$$

Let

$$d(p_1, p_2) \doteq 1/2 \sum_{\alpha=1}^{N} |P_1(\alpha) - P_2(\alpha)|$$

define the *distance* between these two probability distributions. (Notice that the above functions defines a metric on the set of probability distributions. The distance is zero if the two distributions are equal and maximal if they have support on disjoint sets.)

Show that

$$d(p_1, p_2) \leq 1/2 \sum_i |\lambda_i|,$$

where λ_i are the eigenvalues of the Hermitian operator $(\rho_1 - \rho_2)$.

2) **Third order Schmidt decompositions**

Consider a three–component quantum system ABC described by a pure state $|\psi\rangle$ defined on a $n \times n \times n \equiv n^3$ dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Show that the three–dimensional equivalent of the Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |i\rangle^A |i\rangle^B |i\rangle^C$$

is in general not possible.
(Hint: partition first the total Hilbert space into two components and apply then standard Schmidt theorem twice.)

3) **Hadamard gate**

Express the Hadamard gate H as a product of R_x, R_z rotations and a phase factor $e^{i\phi}$, for some $\phi \in \mathbb{R}$.

4) **Controlled–rotation gate**

Consider the *controlled-R* gate defined by replacing the X operator of the CNOT by the single qubit rotation $R_x(\theta)$. In matrix form, this gate can be written as

$$controlled - R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ 0 & 0 & i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Show that this gate can be implemented using two CNOT gates and two single qubit rotations.