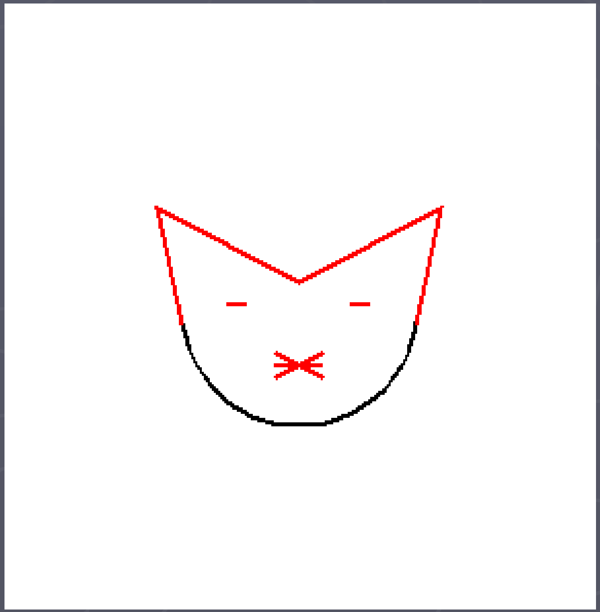
The Baron and the cat



Movie: O Gat

With Oded Kenneth
inspired by J. Wisdom

Baron and cat

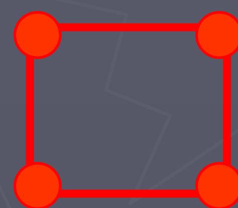
- Translations and rotations as Berry's phases
- A cat rotates without angular momentum
- Can Baron von Munchausen move without momentum?
- Moving on a slippery surfaces

Rotations without angular momentum: Physics

Rotations without angular momentum: Mathematics

Generators of deformations

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[X, Z] = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2R$$

The commutator of deformations= a rotation

Newton’s law

Newton:

Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

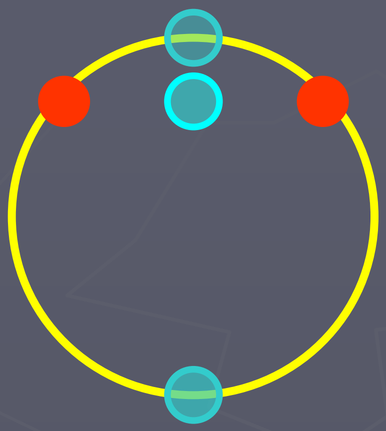
Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed

The version we teach students:

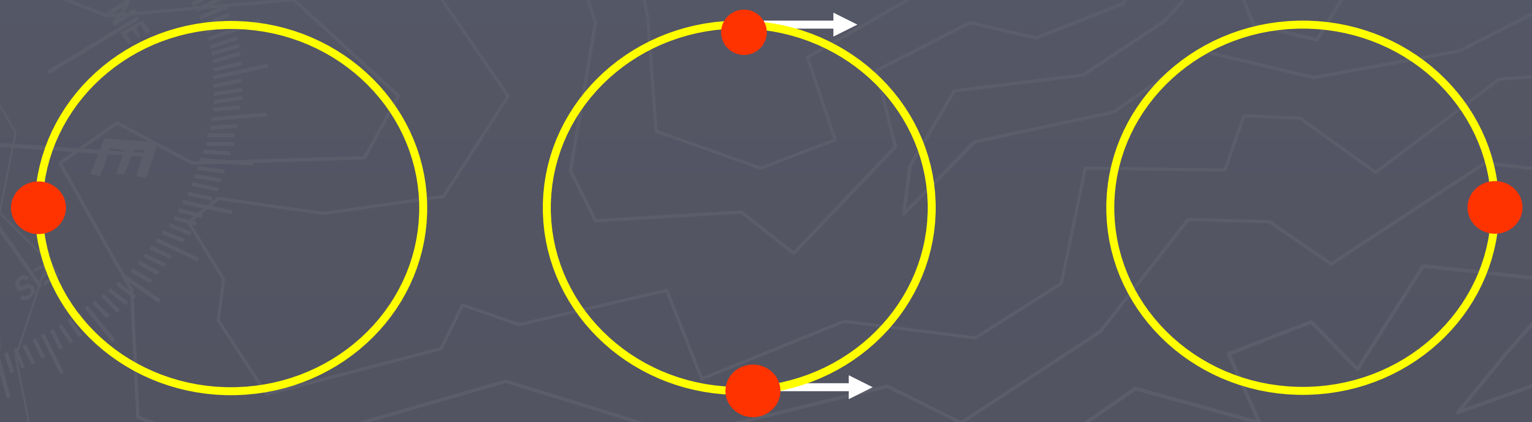
The center-of-mass of the system will remain at rest in the absence of external forces.

CM is Euclidean notion

Ambiguous center of mass

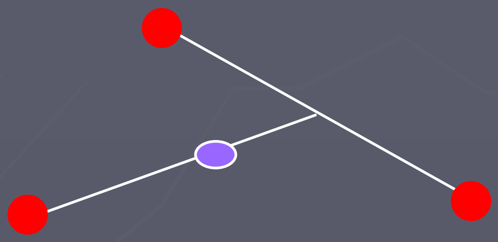


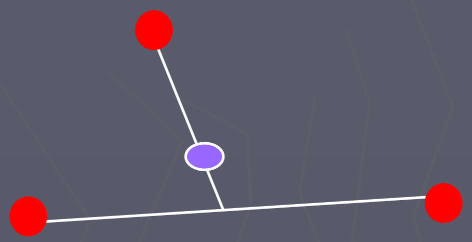
Ambiguous center of mass allows for translation with internal forces



Ambiguous center of mass: Geometry

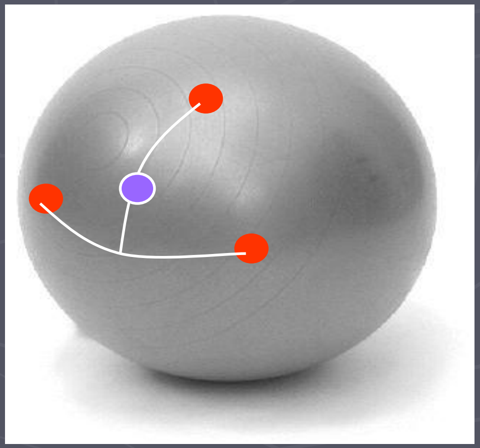
Euclidean plane





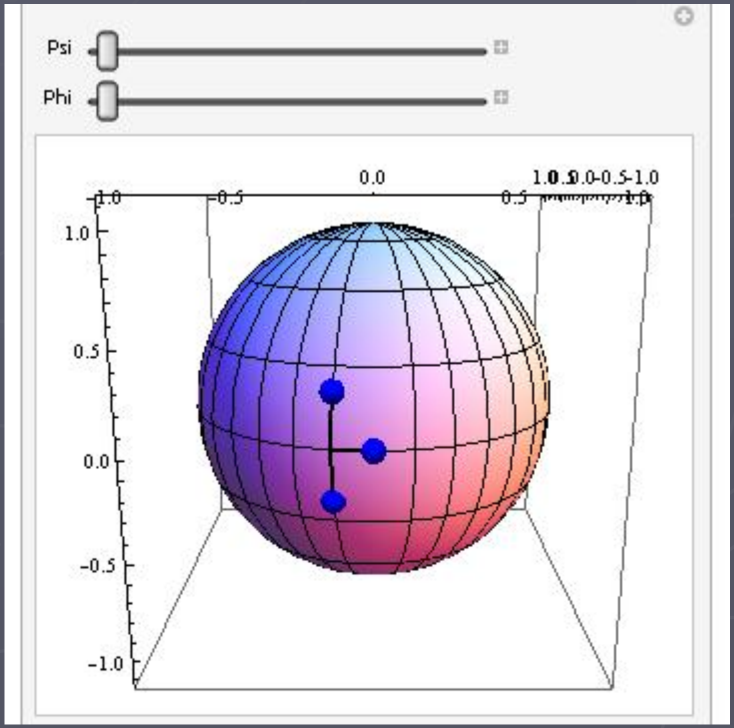
The three medians of a (Euclidean) triangle intersect at one point

On a sphere



Swimming triangles

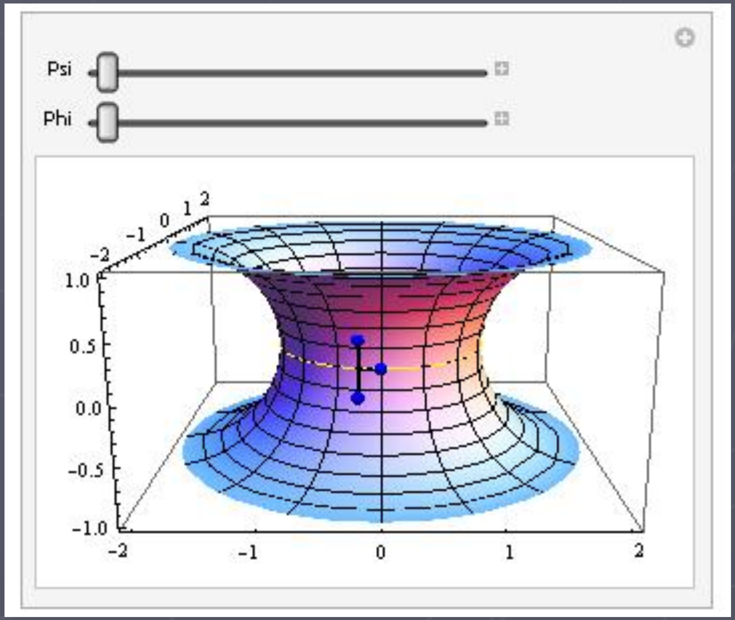
On sphere



Movies: Oren Raz

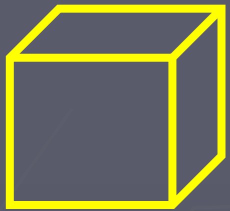
Swimming triangles

Inside a torus



Movies: Oren Raz

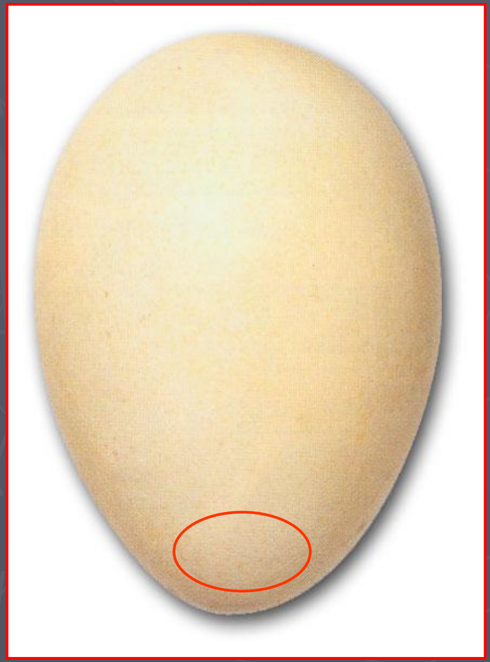
Misfits



Shape: mutual distances between pt masses $\binom{n}{2}$

Freedoms: nd coordinates

$n > 2d + 1$ overconstrained.



Rigid bodies (generically) fit nowhere

Freedom to move needs:

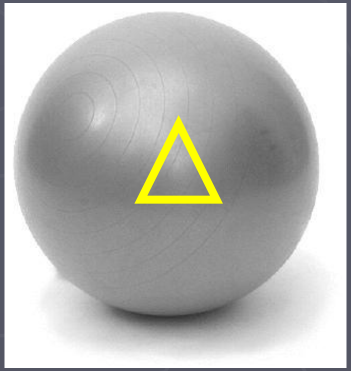
- Flexibility--of shape
- Accommodation--of space (homogeneous)

Symmetric spaces

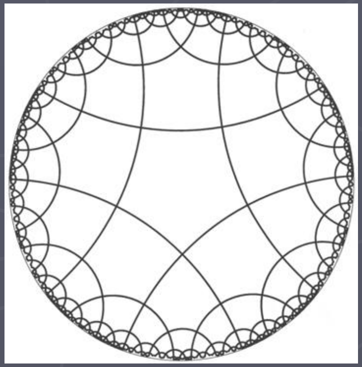
Symmetric space: Homogeneous and isotropic.

Symmetric space: if a rigid body fits somewhere, it fits any where.

Examples:



Sphere

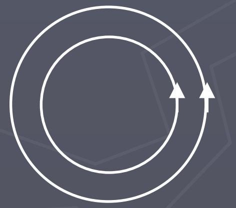


Lobachevski plane

Killing forms

Killing for Euclidean translation $\xi_t = dx$

Killing for Euclidean rotation $\xi_R = xdy - ydx$



No objective notion of translation and rotation (Euclidean)

$$\xi_R \to xdy - (y-a)dx = \xi_R + a\xi_t$$

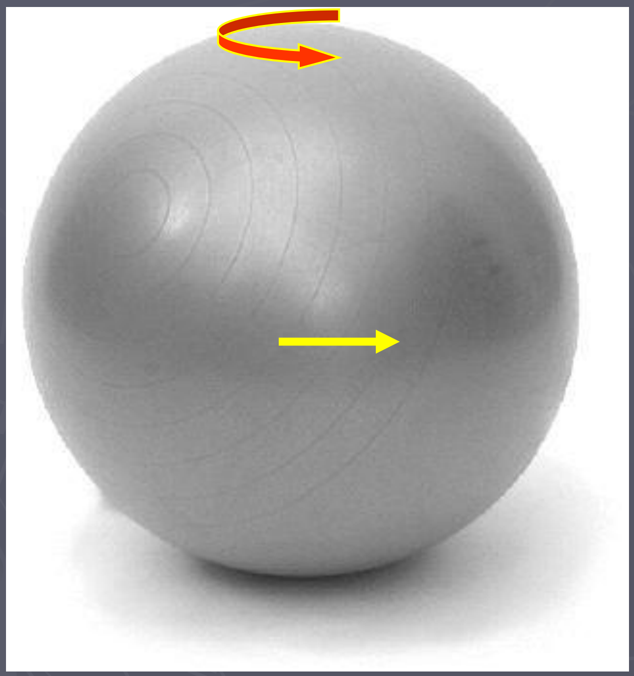
Killing fields: Translation & rotations

Defining property: No strain

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

Relation to Riemann

$$-\xi_{\ell;i;j} = R_{\ell jik}\xi^k$$



Determined by initial data (value and gradients) at a point

$$\xi_\mu(0), \qquad \xi_{[\mu,\nu]}(0)$$

Rotation about z axis:
Looks like translation near equator
And like rotation at poles

Translation:
value, no grad

Rotation: rot grad,
Vanishing value

Controls: Deformations

Example: Euclidean case

$$\eta_1 = x\partial_x - y\partial_y$$

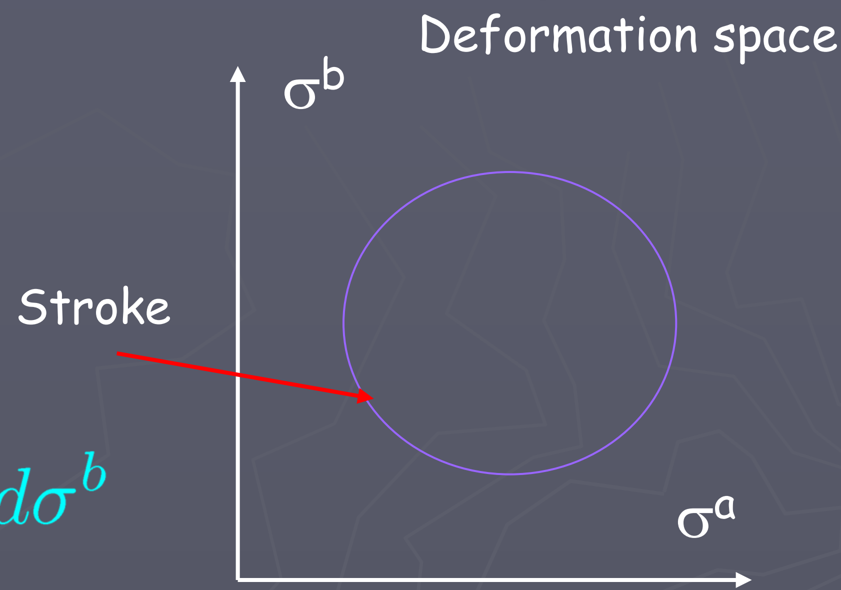
$$\eta_2 = x\partial_y + y\partial_x$$

Vector field for deformations depend on choice of origin: Ambiguity in Killing

$$\eta_1 \rightarrow \eta_1 + a\xi_x - b\xi_y$$

Ambiguities in killing and deformations are gauge freedoms

Swimming as Holonomy



Swimming distance

$$\bar{d}\tau = \langle d\xi | \eta_a, \eta_b \rangle \; d\sigma^a \wedge d\sigma^b$$

Killing 2-form

Deformation fields

$$\langle \xi | \eta \rangle = \sum m_n \xi(x_n) \cdot \eta(x_n)$$

Space allows for swimming if the Killing 1-form is not closed

Cats spin and Barons lie

Translations $\xi_t = dx \longrightarrow d\xi_t = 0$

The Baron lies

Rotations

$$\xi_R = xdy - ydx \longrightarrow d\xi_R = 2dx \wedge dy$$

Cats spin

Small swimmers on symmetric surface

Property of space

Size of stroke

$$\delta x \approx 8R \left(\frac{\sum m_n x_n y_n^2}{M} \right) dA$$

Geometry of aswimmer

Wisdom
Swim away from Black hole
Relativistic motions

AK:
Symmetric spaces
Non relativistic bodies

Conservation laws

A system of particles

$$P_\xi = \sum_n p_n \cdot \xi(x_n), \quad p_n = \frac{\partial L}{\partial \dot{x}_n} = m\dot{x}_n$$

Killing vector field

Remarkably: Conservation laws are swimming equations

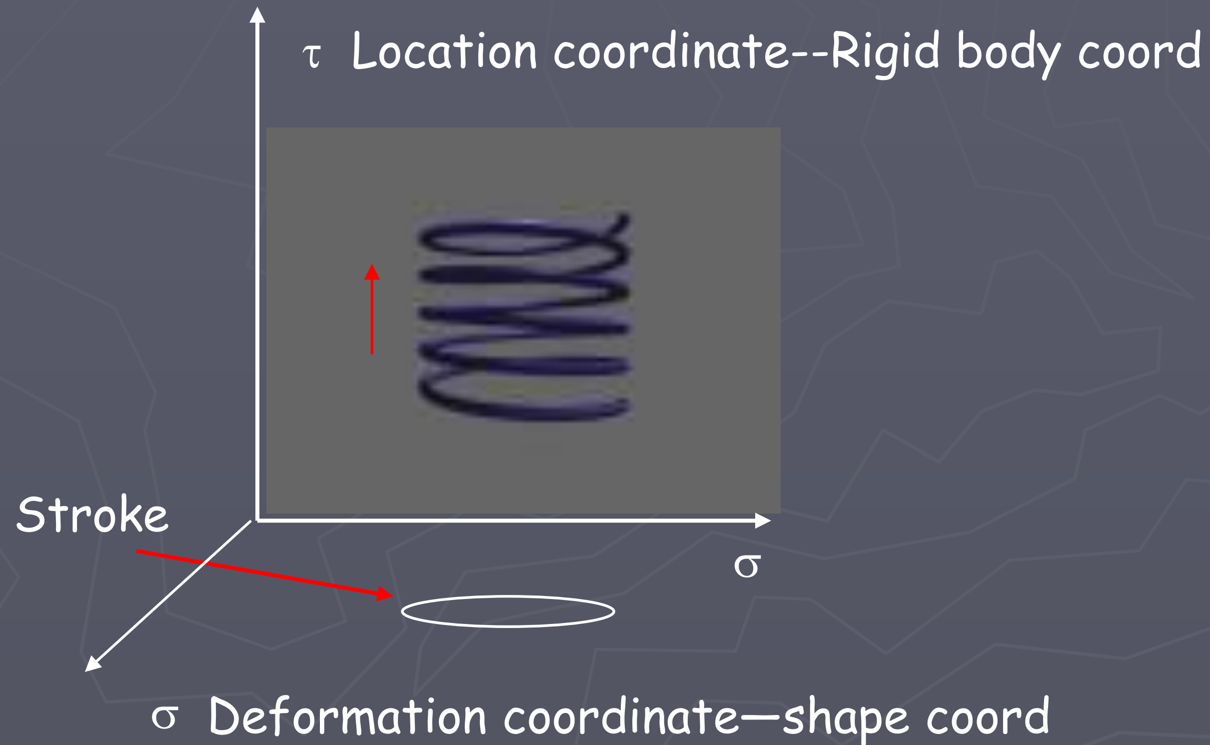
$$P_\xi = 0 \longrightarrow \sum m_n \xi(x_n) \cdot dx_n = 0$$

As many equations as freedoms (Killing fields)

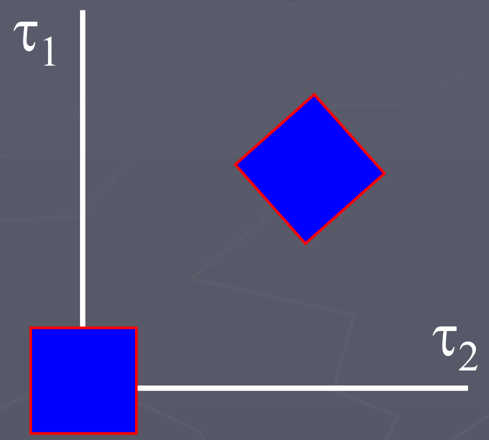
Eq. independent of time parameterization: Geometric

Coordinates: Choosing gauge

Swimming with periodic strokes:



Coordinates in location (rot+trans) space

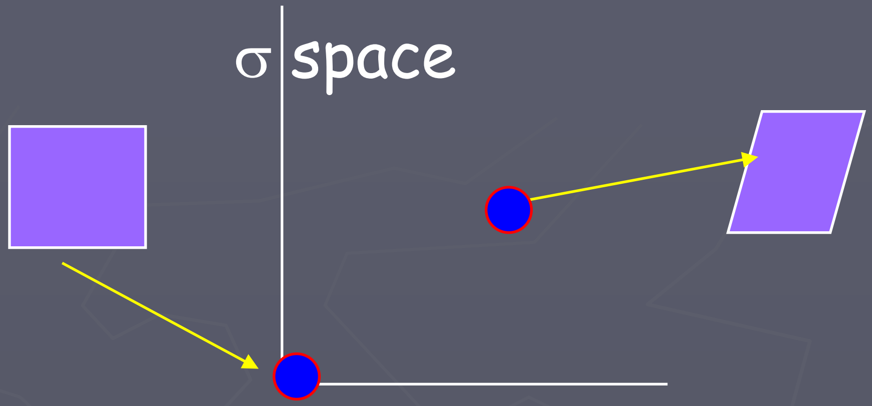


Shifted and rotated

$$\dot{x}(t) = \tau \cdot \xi\big(x(t)\big), \quad x(0) \to x_\tau(1)$$

Decomposition of rigid body motion to translation+rotation
- Depends on order
- Depends on choice of Killing (Origin in Euclidean)

Deformation σ



$$\dot{x}(t) = \sigma \cdot \eta\big(x(t)\big), \quad x(0) \rightarrow x_\sigma(1)$$

- Depends on order and choice of deformations

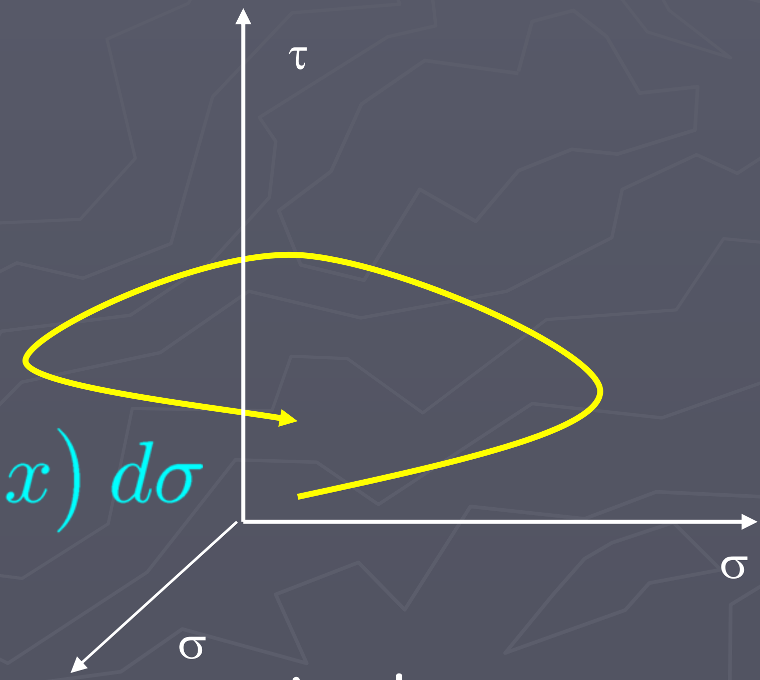
Coordinates: shape+location

Coordinates for deformations + Euclidean motions

$$S(\tau, \sigma) = e^{\tau \xi} e^{\sigma \eta} S$$

$$x_0 \longrightarrow x(\tau, \sigma; x_0)$$

$$dx(\tau, \sigma; x_0) = (\partial_\tau x)\, d\tau + (\partial_\sigma x)\, d\sigma$$



Motions legit if it consistent with conservation laws

Equation of Motion

Infinitesimal motion parameterized by deformations and rigid body motion

$$dx = \xi \text{đ}\tau + \eta d\sigma,$$

Substitute in conservation law

$$\sum m_n \xi(x_n) \cdot dx_n = 0$$

Gives a linear (system) of equations for dτ

$$\langle \xi_\alpha | \xi_\beta \rangle \text{đ}\tau^\beta + \langle \eta_b | \xi_\beta \rangle d\sigma^b = 0$$

Gauge choice: CM & frames

Choice of gauge (at initial time)

$$\langle \xi_\alpha | \eta_a \rangle = 0 \qquad \langle \xi_\alpha | \xi_\beta \rangle = \delta_{\alpha\beta}$$

Fixes non-uniqueness in Killing and deformations

Example:

$$\xi_1 = dx \qquad \eta_1 = x\partial_x - y\partial_y$$

$$\langle \xi_1 | \eta_1 \rangle = \sum m_n \, x_n$$

Vanishing gives cm

Leading order easy

Equation of motion

$$\langle \xi_\alpha | \xi_\beta \rangle đ\tau^\beta = \langle \xi_\alpha | \eta_b \rangle d\sigma^b$$

Gauge condition

$$\langle \xi_\alpha | \eta_b \rangle (\sigma = \tau = 0) = 0, \quad \langle \xi_\alpha | \xi_\beta \rangle (\sigma = \tau = 0) = \delta_{\alpha\beta}$$

Hence

$$đ\tau = O\big((d\sigma)^2\big)$$

Swimming comes from the next order harder