Topological quantum numbers

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Technion

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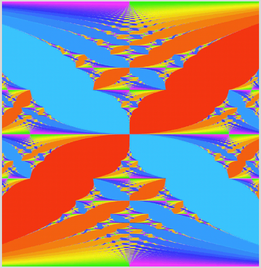
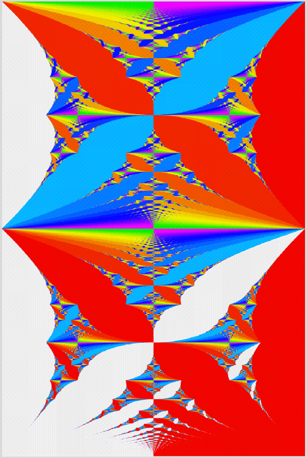


Outline

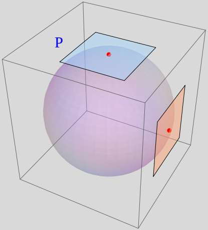
1. Curvature and topology of surfaces
   - Gauss Bonnet
2. Geometry of QM
   - Adiabatic curvature
3. Geometric transport
   - Chern numbers
4. Homotopy of simple matrices
   - The mother of parity:
   - Integer and parity band labels
5. Fredholm index
   - QHE
6. Hofstadter Butterfly
   - Duality



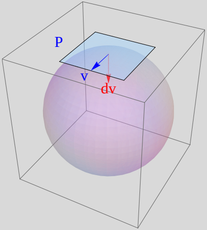
Landau-TKNN butterfly

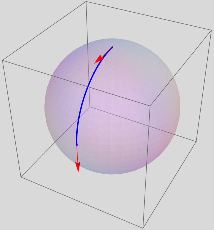
Tangent, Geodesics, Parallel transport

- Tangent plane



- Parallel transport: No motion in $P \Longleftrightarrow P \cdot dv = 0$



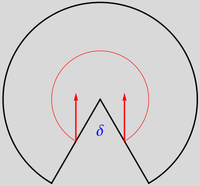


Spagetti and Levi Civita

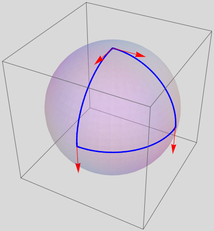
- Curvature is Holonomy density
- $\Omega(point) = \frac{\delta angle \ \ mismatch}{\delta \ Area}$

  $= \frac{\pi/2}{4\pi R^2/8} = \frac{1}{R^2}$
- Local; Intrinsic; Signed

Tullio Levi-Civita (1873-1941)



Parallel transport on a cone: Curvature at vertex



Parallel transport: Levi-Civita connection

Gauss-Bonnet

- Euclidean triangle

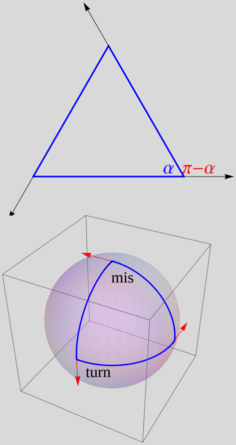
$$\sum turns = \sum(\pi - \alpha_j) = 2\pi$$

- Spherical triangle

$$holonomy + \sum turns = \frac{\pi}{2} + 3\frac{\pi}{2} = 2\pi$$

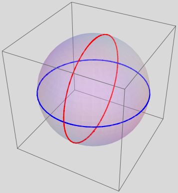
- Gauss-Bonnet:

$$\int \Omega + \sum(\pi - \alpha_j) = 2\pi$$



$holonomy + turns = 2\pi$

Gauss Bonnet

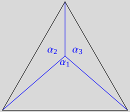


- Euler characteristic

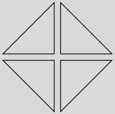
$$F - E + V = 2(1-g)$$

- Apply Gauss-Bonnet to triangulation

$$\int \Omega + \sum (\pi - \alpha_j) = 2\pi F$$



- $\sum \alpha_j = 2\pi V$
- $\frac{1}{2\pi} \int \mathit{Curvature} = 2(1-g)$
- Relate geometry to topology

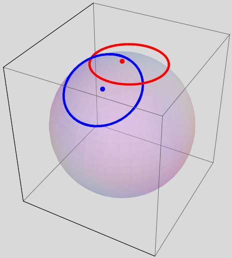


Projections

- Projections:

  One dimensional: $P = |\psi\rangle\langle\psi|$
  Multidimensional $P = \sum_j \underbrace{|\psi_j\rangle\langle\psi_j|}_{\langle\psi_j|\psi_k\rangle=\delta_{jk}}$

- Defining property: $P^2 = P$
- Nice under tensoring: $(P_1 \otimes P_2)^2 = P_1 \otimes P_2$
- Mixing is bad: $\alpha_1 P_1 + \alpha_2 P_2$ not projection
- Orthogonals add:
  $(P_1 + P_2)^2 = P_1 + P_2 \Leftrightarrow P_1 P_2 = P_1 P_2 = 0$

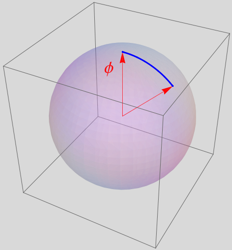


Qubit: Phase circles link

Controlled projections

- Controls: ϕ; Determine : $P(\phi)$
- Example: Qubit $P, \hat{\phi} \in \mathbb{S}^2$

$$P = \frac{1+\hat{\phi}\cdot\sigma}{2} = \begin{pmatrix} 1+\phi_3 & \phi_1 + i\phi_2 \\ \phi_1 - i\phi_2 & 1-\phi_3 \end{pmatrix}$$



Sphere as control space

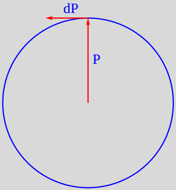
Theorem (Kato)

$$P(dP)P = 0$$

Pf: $P\,dP + (dP)\,P = dP$

$$\underbrace{P\,dP}_{cancel} + P(dP)\,P = \underbrace{P\,dP}_{cancel}$$

Analog of $\hat{\phi}\cdot\hat{\phi} = 1 \Rightarrow \hat{\phi}\cdot d\hat{\phi} = 0$



Controlled Hamiltonians; Spectral Projections

- Controls: ϕ parameters in $H(\phi)$
- Example: Berry's model

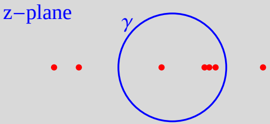
$$H(\phi) = \phi \cdot \sigma, \phi \in \mathbb{R}^3$$

- Spectral projection:

$$P = \sum |\psi_j\rangle \langle\psi_j| = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{H-z}$$

- H matrix valued, analytic in ϕ
- $\frac{1}{H(\phi)-z}$, $z \notin Spec(H)$; matrix, analytic in ϕ
- Smooth (in H) if γ not pinched

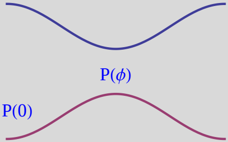
$$dP = -\frac{1}{2\pi i} \oint_\gamma \frac{1}{H-z} (dH) \frac{1}{H-z}\, dz$$



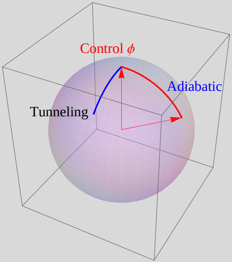
Projection on encircled eigenvalues

Spectral evolution

- U_k: evolution in spectral subspaces
- Generator: $\quad idU_k = AU_k$
- Differentiate $P(\phi) = U_k(\phi)P(0)U_k^\dagger(\phi)$
- Gives $0 = i[A, P] + dP$
- Kato's choice $A = i[dP, P]$
- A parallel transports
  $$P\, d\,|\psi\rangle = P\,[dP, P]\,|\psi\rangle = 0$$
- A analog of Levi-Civita connection (aka Christoffel symbo) Γ
- Aiabatic thm: $A = H + \underbrace{i[\dot{P}, P]}_{small}$



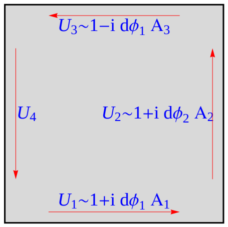
Two spectral bands



Tunneling: transverse to control

Adiabatic (Berry) curvature: Projections

- Controls: ϕ_μ
- $A = i[dP, P] = A_\mu \, d\phi_\mu, \quad A_\mu = i[\partial_\mu P, P]$
  $U_1 \approx 1 + iA_1 d\phi_1$
- Holonomy of small loop $d\phi_1 \wedge d\phi_2$
- $U_4 U_3 U_2 U_1 \approx 1 + i \, \Omega_{12} d\phi_1 \wedge d\phi_2$
- Curvature $\Omega_{12} = -i \, P \, [\partial_1 P, \partial_2 P] \, P$



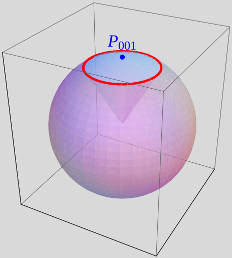
Ω as holonomy

Adiabatic curvature: Wave functions

- $$P = |\psi\rangle\langle\psi|$$
  $$\partial P = |\partial\psi\rangle\langle\psi| + |\psi\rangle\langle\partial\psi|$$
- Mindless computation: $\Omega_{12} = -i\,P\,[\partial_1 P, \partial_2 P]\,P$:
  $$\Omega_{12} = 2\,Im\,\langle\partial_1\psi|\partial_2\psi\rangle P$$
- Acts in P as multiplication by function (not matrix)
- Canonical form of Berry curvature:
  $$\mathrm{Tr}\,\Omega_{12} = 2\,Im\,\langle\partial_1\psi|\partial_2\psi\rangle$$

Berry's model

- $H = \vec{\phi} \cdot \vec{\sigma}$
- $P(0,0,1) = \frac{1+\sigma_3}{2}$
- $\partial_1 P = \frac{\sigma_1}{2}, \ \partial_2 P = \frac{\sigma_2}{2}$
- $\Omega_{12} = -\frac{i}{4} \frac{1+\sigma_3}{2} [\sigma_1, \sigma_2] = \frac{P}{2}$
- Berry's curvature $\frac{1}{2}$ area
- Holonomy= half spherical angle



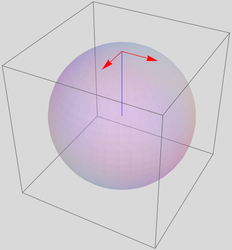
Berry's holonomy:1/2 spherical angle

Curvature: Properties

$$\Omega = \sum \Omega_{\mu\nu} d\phi_\mu \wedge d\phi_\nu, \quad \Omega_{12}(P) = -i\, P\, [\partial_1 P, \partial_2 P]\, P$$

Curvature turns controls space to phase space:

$$\Omega_{12}(P) = -\Omega_{21}(P); \quad d\,\mathrm{Tr}\,\Omega = 0$$



Sphere as Phase space

- Hilbert is flat: $\Omega_{12}(\mathbb{I}) = 0$
- Tensor product—additive:

  $$\Omega(P \otimes Q) = \Omega(P) \otimes Q + P \otimes \Omega(Q)$$

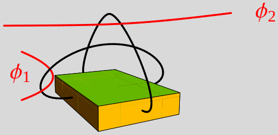
- Direct sum— trace additive

  $$\Omega(P \oplus Q) = \Omega(P) + \Omega(Q) \underbrace{-iP(dP)(dQ)Q - iQ(dQ)(dP)P}_{0 \ \mathit{trace}}$$

- Electron-hole anti-symmetric

QHE: Loop currents

- $H(\phi)$ controlled by AB fluxes
- EMF in loop 1 : $\dot{\phi}_1$
- Current in loop 2: $\partial_2 H = \frac{\partial H}{\partial \phi_2}$



QHE controlled by two AB fluxes

- $|\psi\rangle$ evolved by (time dependent) Schrödinger equation:

$$i\frac{d|\psi\rangle}{dt} = H(\phi)\,|\psi\rangle$$

Theorem

Looping charge is boundary value: $Q_2 = \int_0^1 \langle\psi|\,\partial_2 H\,|\psi\rangle\, dt = i\langle\psi|\partial_2\psi\rangle\Big|_0^1$

Pf: $\langle\psi|\,\partial_2 H\,|\psi\rangle = i\partial\langle\psi|\dot\psi\rangle - i\langle\partial\psi|\dot\psi\rangle + i\langle\dot\psi|\partial\psi\rangle = i\,\frac{d\langle\psi|\partial\psi\rangle}{dt}$

Hall conductance is geometric

- Weak emf $\dot{\phi}_1$ small; $H(\phi)$ evolves adiabatically
- Kato evolution: $|\psi(t)\rangle \approx U_k(t)\,|\psi(0)\rangle$
- Kato generator $H \to i[\dot{P}, P]$
- Substitute Kato in *Looping charge is boundary value* identity

$$Q_2 = i\langle\psi|\partial_2\psi\rangle\Big|_0^1 = \int_0^1 \langle\psi|\,\partial_2 H\,|\psi\rangle\,dt$$

Theorem

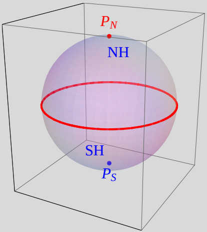
Looping charge= line integral of curvature

$$Q_2 = i\int \mathrm{Tr}([\partial_2 P, \dot{P}]P)dt = \int \mathrm{Tr}(\Omega_{21})d\phi_1$$

in adiabatic limit.

First Chern number

- Holonomy of equator: an angle
- Equator: $\pm$ bdry of both North & South hemispheres
- $$\int_{NH} \mathrm{Tr}\ \Omega_{12}\ dS = -\int_{SH} \mathrm{Tr}\ \Omega_{12}\ dS \quad mod(2\pi)$$



Theorem (Chern)

For closed (boundary-less) surface S total curvature quantized:

$$\frac{1}{2\pi}\int_S \mathrm{Tr}(\Omega_{jk})\, dS \in \mathbb{Z}$$

Wigner von Neumann

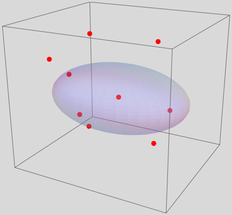
Theorem (Wigner, von Neumann)

Degenerate hermitian matrices are codimension 3

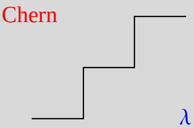
- Example: Tracelss, Hermitian 2×2 matrices

$$H^2(\vec{k}) = (\vec{k}\cdot\vec{\sigma})^2 = \sum_{i,j=1}^{3} k_i k_j \sigma_i \sigma_j = |\vec{k}|^2$$

- Degeneracy: $|\vec{k}| = 0$; 3 conditions



Topology of Control space: crossing pts removed



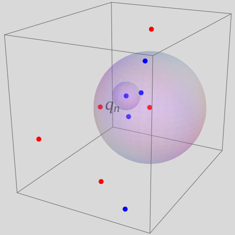
Typical interpolation: Step function, unit steps

What is counted?

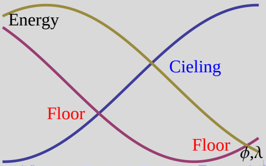
- n-th conic crossing point locally

$$H_n(\delta\vec{k}) = \sum_{i,j=1}^{3} g_{ij}(n)\,\delta k_i\,\sigma_j, \quad det\ g \neq 0$$

- $q_n\ \delta\vec{k}\cdot\vec{\sigma}, \quad q_n = sign\,det\,g(n)$
- Chern # is Gauss law for q_n



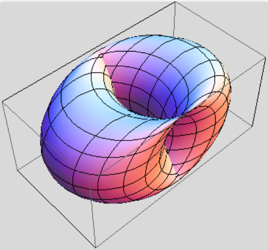
Charged crossing points



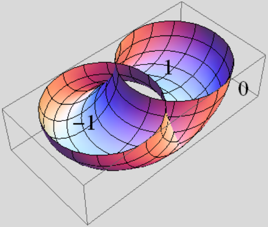
2-level system with $\mathbb{T}^2$ controls

- Control—two angles: $\phi \in \mathbb{T}^2$
- 2-level systems $H = \vec{k}(\phi) \cdot \vec{\sigma}, \quad \vec{k} \in \mathbb{R}^3$
- $\mathbb{T}^2$ controlled 2-level: Torus in $\mathbb{R}^3$
- Example: $\vec{k}(\phi) = \sum_{j=1}^{2} (\vec{a}_j \cos\phi_j + \vec{b}_j \sin\phi_j)$
- 2-level system; phase changing knob y

$$H(\phi, y) = (\vec{k}(\phi) - \vec{y}) \cdot \vec{\sigma}$$



Immersed torus in $\mathbb{R}^3$



Chern Phase diagram

Topology of space of Non-degenrate hermitian matrices

Theorem

Simple (non-degenrate) $n \times n$ *hermitian matrices homotopic to* $U(n)/U^n(1) = SU(n)/U^{n-1}(1)$

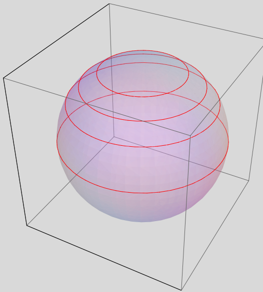
- Hermitian matrices: brought to diagonal form by $U(n)$
- If simple, order $\begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & \dots & \dots & \dots \end{pmatrix}, \quad \lambda_1 < \lambda_2 \dots$
- Deform $\begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & \dots & \dots & \dots \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \dots \\ 0 & \dots & \dots & \dots \end{pmatrix}$
- Diagonal hermitian commute with diagonal unitary: $U^n(1)$,

Exact sequence of fibration

Exact sequence of fibration:
Machine to compute $\pi_k(G/H)$ if $\pi_k(G)$ and $\pi_k(H)$ known (and simple)

$$\dots \pi_k(H) \to \pi_k(G) \to \pi_k(G/H) \to \pi_{k-1}(H) \to \pi_{k-1}(G) \dots$$

- Fact: $0 \to A \to B \to 0$ implies $A = B$
- Fact about $SU(2)$: $\pi_2 = \pi_1 = 0$
- Evidently $\pi_1(U^m(1)) = \mathbb{Z}^m$



$\pi_1 = 0 \Leftrightarrow$Simply connected

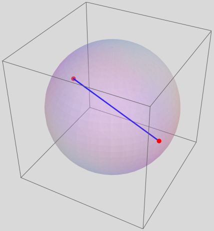
Theorem

Hermitian $n \times n$ simple matrices have integer band invariant: $\pi_2 = \mathbb{Z}^{n-1}$

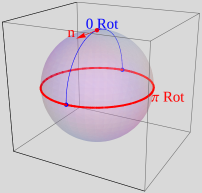
- $n-1$ means: zero sum rule
- $\pi_k, \ k \geq 2$ give global, **not band**, invariants.

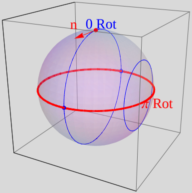
The mother of $\mathbb{Z}_2$

- Rotations: $SO(3)$; (3 Euler angles.)
- $Rot(\hat{n},\theta) = Rot(-\hat{n}, 2\pi - \theta)$
- $SO(3) =$ Ball of radius π antipodal identified
- $\pi_1(SO(3)) = \mathbb{Z}_2$



Ball with antipodals identified





Path $(0, 2\pi)$ rotation — non-contractible; Path $(0, 4\pi)$ — contractible

Homotopy groups

- Time reversal with spin $1/2$ = quaternionic hermitian
- Fact: *Unit quaternions* $= S^3$
- Fact $\pi_3(S^3) = \mathbb{Z}$; easy
- Fact: $\pi_4(S^3) = \mathbb{Z}_2$; fancy

Theorem (Sadun, Segert, Simon)

Quaternionic hermitian $n \times n$ matrices, have and integer and parity band invariant $\pi_4 = \mathbb{Z}^{n-1}, \ \pi_5 = \mathbb{Z}_2^{n-1}$

- $n-1$: zero sum rule
- $\pi_k, \ k \neq 4, 5$; global invariants

Theorem

Real symmetric $n \times n$ simple matrices have parity band invariant with sum rule and a global invariant: $\pi_1/\mathbb{Z}_2 = \mathbb{Z}^{n-1}$

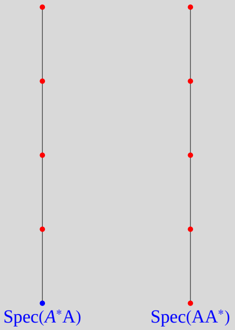
Topological quantum numbers for matrices

- $\mathrm{Tr}(P_1 - P_2) \in \mathbb{Z}$, P_{12} projections
- Fredholm Index:
  $Ind(A) = dim\ Ker(A^*A) - dim\ Ker(AA^*) \in \mathbb{Z}$
- $dim\ Ker\ (A^*A)$ quantized, but unstable
- Topological i.e. stable under perturbations :
  $$Ind(A + \epsilon B) = Ind(A)$$

Theorem (Archimedes)

$Spec(A^*A)/\{0\} = Spec(AA^*)/\{0\}$

Proof:
$A^*A\,|\psi\rangle = \lambda\,|\psi\rangle \longrightarrow (AA^*)A\,|\psi\rangle = \lambda A\,|\psi\rangle$//



Hall conductance as Fredholm Index

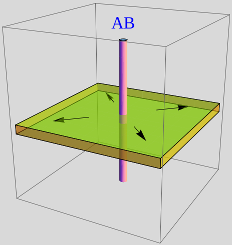
- Example: Laughlin charge pump:
  $AB - flux : |m\rangle \to |m+1\rangle$
- $dim\ Ker\ A^\dagger = 1; dim\ Ker\ A = 0$

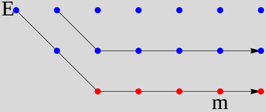
Theorem (Bellissard, ...)

Hall conductance is a Fredholm index:
$Ind(PUP) = \mathrm{Tr}(P - UPU^\dagger)^{2k+1};$
P projects below gap ; $U = e^{i\theta}$ AB-flux tube.

- Non-interacting electrons
- Thermodynamic systems; ∞ of electrons
- Does not use BZ; Works with disorder

Laughlin Hall pump



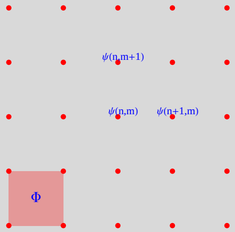


AB pump flow: occupied, empty

Hofstadter model

$$H_{tb}\psi(n) = \underbrace{\psi(n+1) + \psi(n-1)}_{2\cos k_1} + \underbrace{2\cos(k_2 - \Phi n)}_{2\cos(k_2 - A)}\psi(n)$$

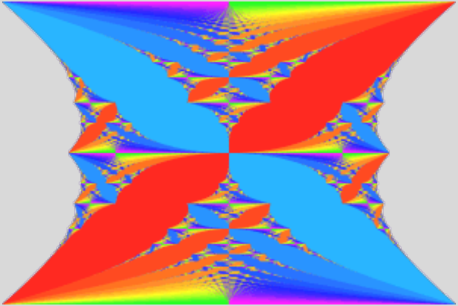
- Φ flux through unit cell (small)
- AB periodic in Φ
- Controls k: Bloch momenta.
- If $\Phi = 2\pi p/q \rightarrow q \times q$ periodic matrix
- q bands with Chern numbers



Hopping on $\mathbb{Z}^2$ with constant Φ per plaquette

Hofstadter-TKNN butterfly

$\uparrow \Phi$



Hofstater-Azbel-TKNN butterfly $\longrightarrow \mu$

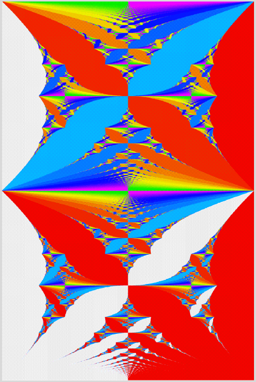
- Colors: Chern(Fermi projections)
- Gaps colored; Spectrum set of measure 0
- *Negative*; *Positive*
- Electron-Hole (Right-left) anti-symmetric
- $Chern(0) = Chern(\mathbb{I}) = 0$; Empty & Full band are inert

To draw fractal pictures: need TKNN Diophantine equation

Landau Butterfly

$$(H_L\psi)(n) = \underbrace{\psi(n+1) + \psi(n-1)}_{2P\cos(x)P} + \underbrace{2\cos\left(k_2 - \frac{n}{\Phi}\right)}_{2P\cos(y)P}\psi(n)$$

- Magnetic field Φ
- μ Horizontal
- Φ^{-1} Vertical
- Empty Chern=0
- Full Chern =1
- No AB periodicity



Landau-TKNN butterfly

Duality; Diophantine

- Density, non-interacting fermions $\rho = \frac{1}{N}\,\mathrm{Tr}\left(\frac{1}{1+e^{\beta(H-\mu)}}\right)$
- $H_l(\Phi) = H_{tb}(1/\Phi)$
- To fix N:

$$\text{Density of state} = \begin{cases} \Phi & \text{Full Landau} \\ 1 & \text{Full band} \end{cases}$$

- Hall conductance (Streda): $n = \partial_\Phi \rho$

Theorem (TKNN)

$\rho_l(\Phi) = \Phi\,\rho_{tb}(1/\Phi) \longrightarrow n_l + \Phi n_{tb} = \rho \quad n_{l,tb} \in \mathbb{Z}$,

Mother of diophantine equations: $n_1^m + n_2^m = n_3^m$;
4 integer unknowns, discrete solutions

Takehome message

- Controlled quantum states have interesting geometry
- Topological quantum numbers characterize gap preserving deformations
- Certain ∞ dimensional projections have topological characterization

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