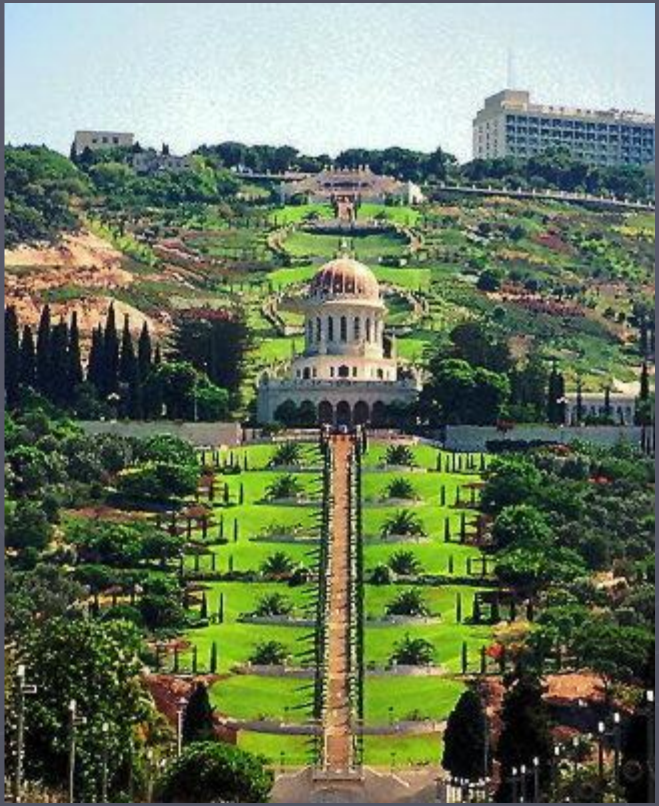
Classical dynamics in magnetic fields

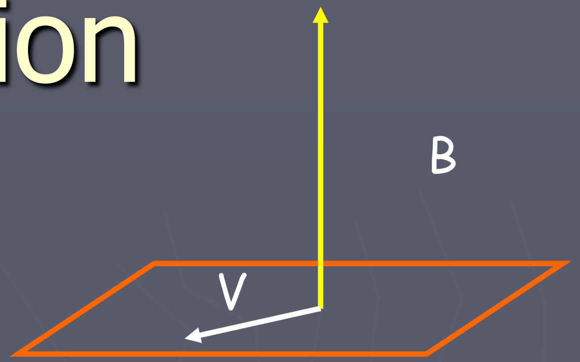


Bahai temple, Haifa

Outline

- Dynamics in the plane
- Drifts
- Manifolds

Equation of motion



Newton-Lorentz

$$m\dot{v} = e\ (E + v \times B)$$

B fixes a time scale. Larmor or cyclotron frequency: $\omega = \dfrac{eB}{m}$

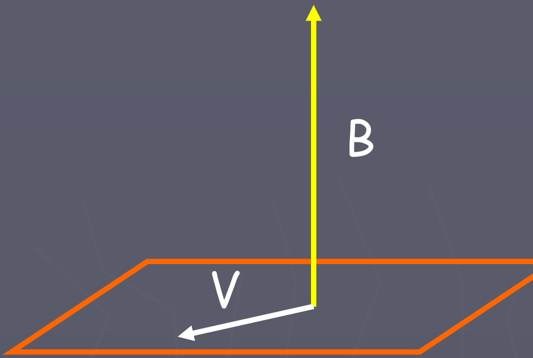
$$\omega(1\ Gauss) \approx 1\ Mhz$$

The velocity scale relating E and B, in MKS, is the velocity of light

B source free $\nabla \cdot B = 0$

E and B not independent $\partial_t B = -\nabla \times E$

Larmor orbit



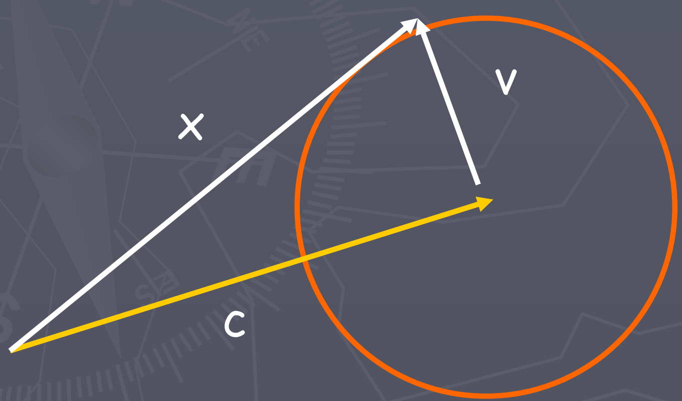
$$m\dot{v} = e\ v \times B$$

B,e,m=consts

$$\frac{d\,(mv)}{dt} = \frac{d(e\,x \times B)}{dt}$$

Elementary integration (plane) $mv = e\ (x - c) \times B$

Center of landau orbit



$(\vec{c}, E_k)$ 3 Const of motion

B does no work

Kinetic energy: $E_k = \frac{m}{2}\, v^2$

$$\dot{E}_k = m\dot{v}\cdot v = e\,(E + v\times B)\cdot v = eE\cdot v$$

Conserved when E=0

Lagrangian mechanics

$$L = \frac{m}{2} v^2 + e(-V + v \cdot A)$$

$$B = \nabla \times A, \qquad E = -\nabla V - \partial_t A$$

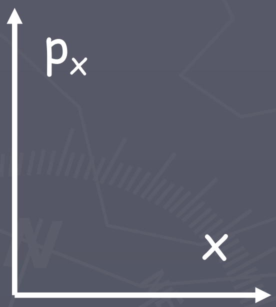
$$p = \frac{\partial L}{\partial v} = mv + eA,$$

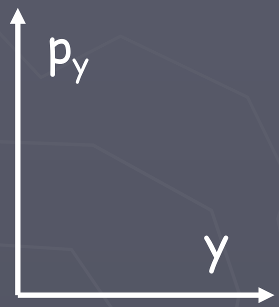
Poisson brackets

Definition: $$\{A, B\} = (\nabla_x A) \cdot (\nabla_p B) - (\nabla_p A) \cdot (\nabla_x B)$$

Canonical pair

$$\{x_j, p_k\} = \delta_{jk}, \ \{x_j, x_k\} = \{p_j, p_k\} = 0$$





$$\{mv_x, mv_y\} = \{p_x - eA_x, p_y - eA_y\} =$$

$$\{p_x, -eA_y\} + \{-eA_x, p_y\} = e\nabla \times A = eB$$

An interesting canonicl pair

From now on m=1

$$\{v_x, v_y\} = \omega$$

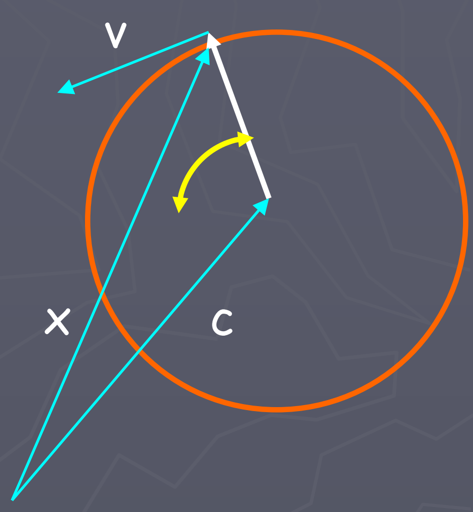
$$\{c_y, c_x\} = \frac{1}{\omega}$$

Self canonical, independent pair

$$\{v_j, c_j\} = 0$$

Larmor & Averaging

$$\dot{v}_x = \frac{1}{2}\{v_x, v^2\} = v_y\{v_x, v_y\} = \omega v_y,$$



$$\frac{d(v_x + iv_y)}{dt} = i\omega(v_x + iv_y) \Rightarrow v_x + iv_y = v_0 e^{i\omega t}$$

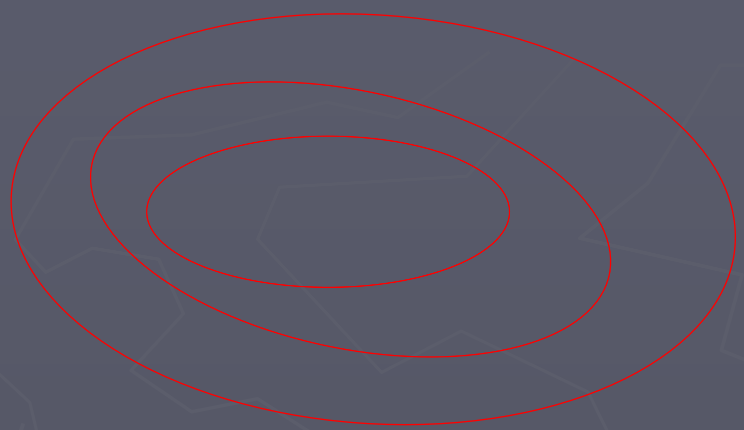
Geometry +
Dim analysis

$$c_x = x + \frac{v_y}{\omega}, \;\; c_y = y - \frac{v_x}{\omega}$$

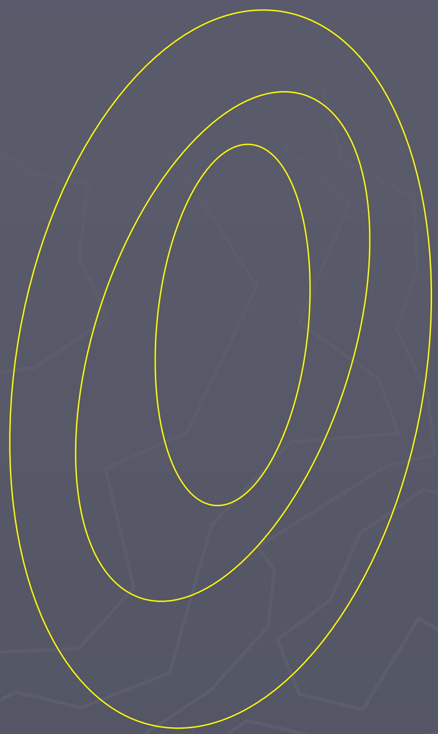
$$\langle x \rangle = c, \quad \langle v \rangle = 0, \quad \left\langle v^2 \right\rangle = 2E_k$$

$$\langle c \rangle = c$$

General motion in the plane



Contours of constant B



Contours of constant V

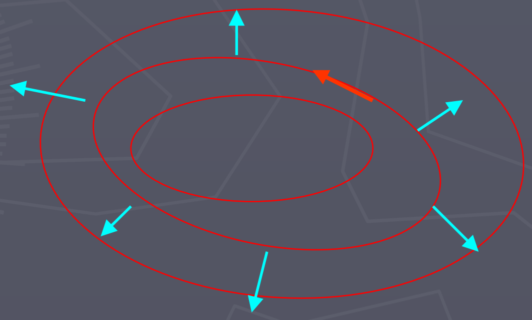
In general motion can be arbitrarily complicated
Simplifies when B large

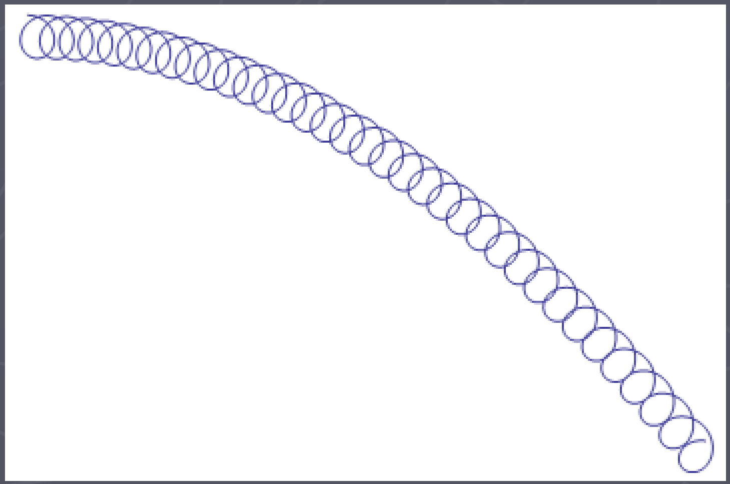
Large B and fixed E

$$c_x = x + \frac{v_y}{\omega}, \quad c_y = y - \frac{v_x}{\omega}$$

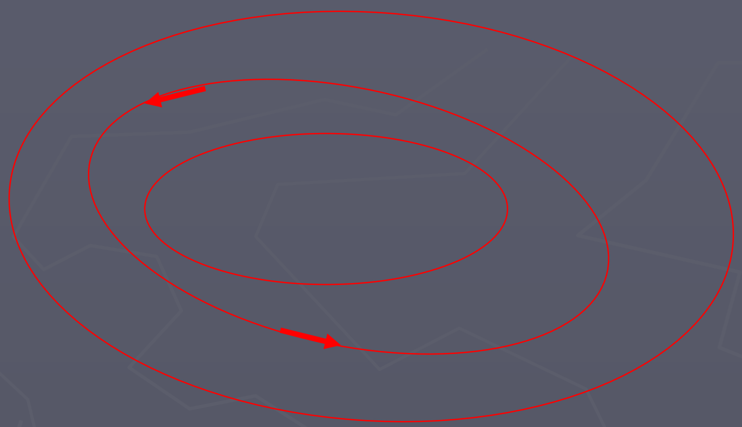
small

The vector fields: $\nabla\omega,$ ∇V drive Chiral drift currents

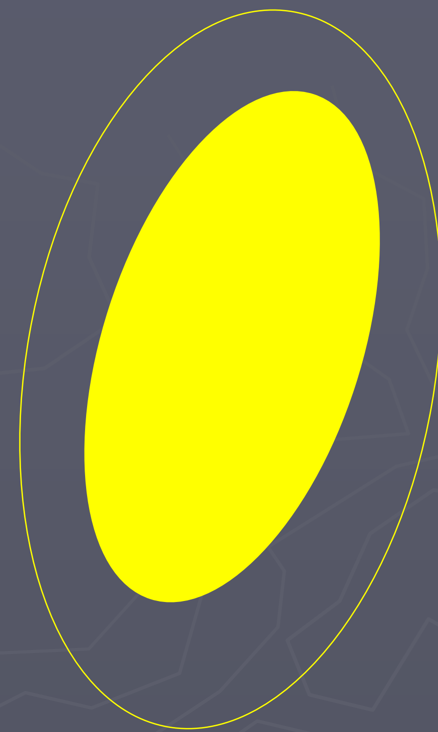




B large vs B=0



B large: Chiral drift along equipotential



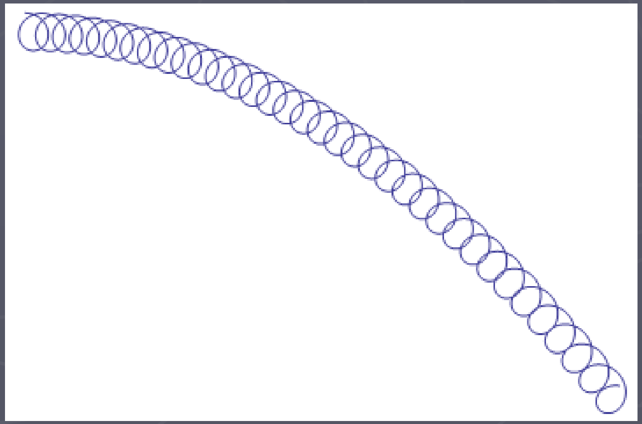
B=0: motion confined by equipotential

Drift in a potential

Weak external potential

$$H = \frac{1}{2} v^2 + V(x)$$

Average over fast dynamics



$$\langle V \rangle = V(c)$$

For constant B

$$\{v_j, c_k\} = 0$$

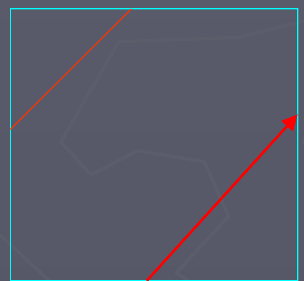
One gets

$$\dot{c} = \{c, H\} \approx \{c, V(c)\} = \left(\frac{1}{\omega} \times \nabla\right) V$$

Slow chiral drift along equi-potentials

Manifolds

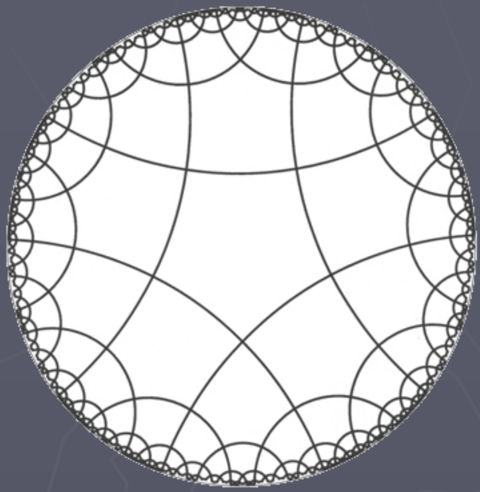
Torus



R=0

Sphere

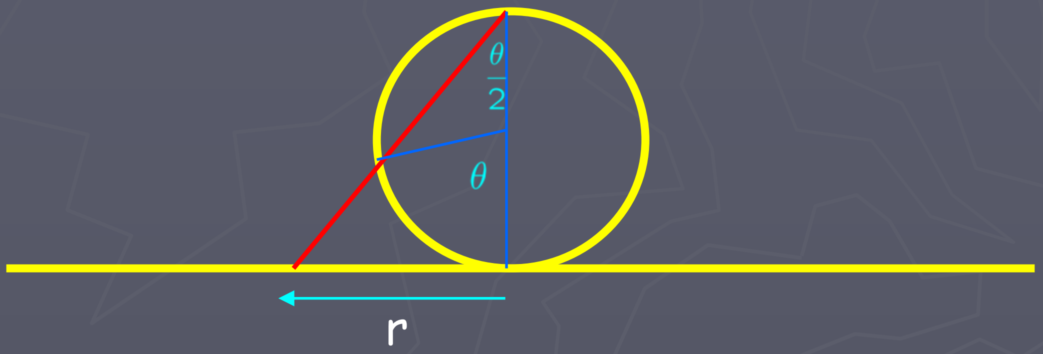
R=1



R=-1

Metric: Spherical projection

Exercise: Show that the metric on the
Sphere induces the metric on
The plane



$$(d\ell)^2 \;=\; \frac{(dr)^2 + r^2\,(d\phi)^2}{(1+r^2)^2} = \frac{|dz|^2}{(1+|z|^2)^2}, \qquad z = x + iy = re^{i\phi}$$

$$(d\ell)^2 \;=\; \frac{|dz|^2}{(1+R|z|^2)^2}, \qquad R = 0, \mp 1$$

Fields on manifolds

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & B \\ -E_2 & -B & 0 \end{pmatrix}$$

$$F^{*\alpha} = \frac{1}{2}\varepsilon^{\alpha\mu\nu} F_{\mu\nu} = \begin{pmatrix} B \\ -E_2 \\ E_1 \end{pmatrix}$$

Homogeneous Maxwel equation (Bianchi identity):

$$\partial_\alpha F^{*\alpha} = \partial_t B + \nabla \times E = 0$$

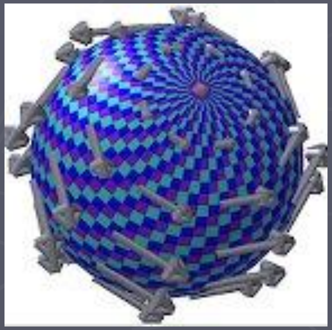
Lenz rule

No analog of $\nabla \cdot B = 0$ Any B is OK

Forbidden fields

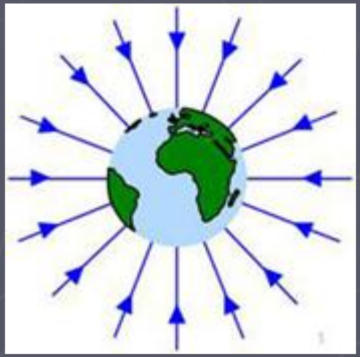
Constant electric fields on manifolds may not exist

There is no constant tangent vector filed on a sphere



No constant E

Constant normal vector field: A notion of area



Consant B

Lagrangian mechanics

$$L = \frac{m}{2} g_{ij} \ v^i v^j + e \ v^j A_j$$

Example: On the Poincare half plane

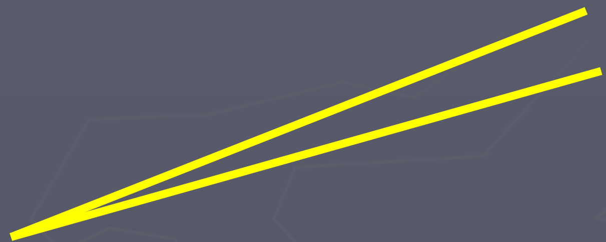
$$L = \frac{\dot{x}^2 + \dot{y}^2 + eBy\dot{x}}{y^2}$$

Lagrangian mechanics needs potentials.

Potential may not exist

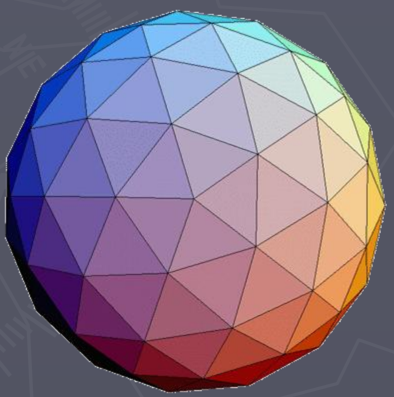
Classical mechanics is local. Enough to construct A on patches.
QM is global and there will be consistency conditions to worry about.

Focusing and divergence

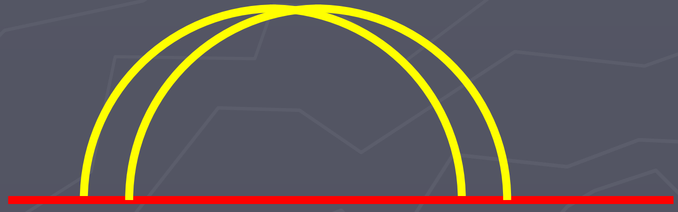


No magnetic field: Linear divergence

Magnetic field: Focusing



Positive curvature is focusing

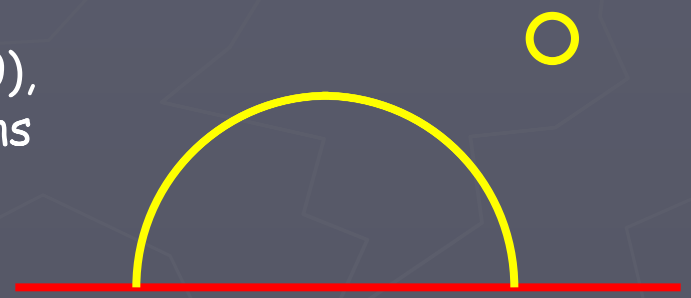


Negative curvature is diverging

Competition on the Half plane

$$L = \frac{\dot{x}^2 + \dot{y}^2 + eBy\dot{x}}{y^2}$$

Large energy, (y near 0),
Negative curvature wins



Small energy (y near 1):
B wins

Competition between curvature and magnetic field

Chaos + energy liberate:
Allow for propagation to infinity