Quantum dynamics in magnetic fields

Technion

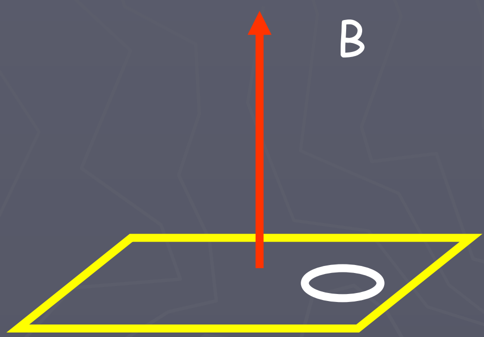
Outline

- Landau Hamiltonian
- Magnetic translations
- Gauge transformations
- Aharonov Bohm
- Dirac quantization

Landau Hamiltonian

$$v = p - eA, \quad m = 1$$

$$p = -i\hbar\nabla, \quad \nabla \times A = B$$



Non-relativistic, spinless Landau Hamiltonian

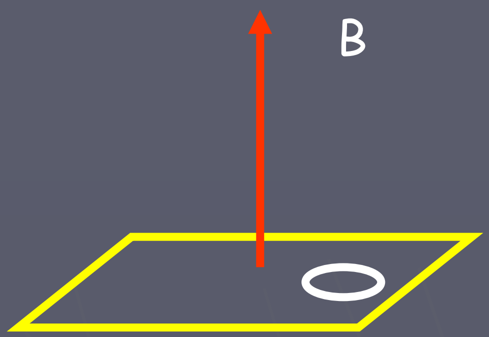
$$H_L(v^2) \;=\; \frac{1}{2}v^2$$

$$\psi \in \mathcal{H} = L^2(R^2)$$

Only kinetic energy

Spectral analysis without special functions

Commutations



$$[v_x, v_y] \;=\; -i\omega$$

When B is constants velocities satisfy Heisenberg

Uncertainty relation with B the effective Plank constant

The velocities are canonically conjugate:

By Von Neumann uniqueness

$$[v_x, v_y] = -i\omega, \Leftrightarrow v_y = -i\omega \; 1 \otimes \partial_{v_x}$$

Ancillary space

$$L^2(R^2) = L^2(R) \otimes L^2(v_x \in R)$$

Need to identify the ancillary space

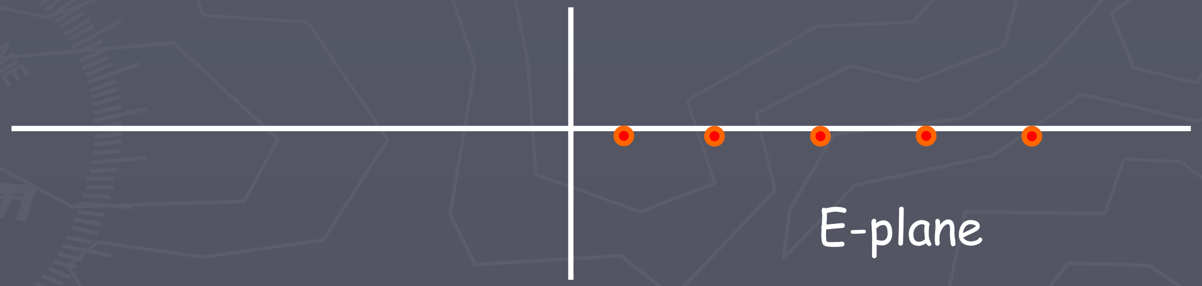
Spectrum without Eigenfunctions

Elementary facts about harmonic oscillator

$$Spec\left(\frac{p^2+x^2}{2}\right)=\left\{|\hbar|\left(n+\tfrac{1}{2}\right),n\in 0,1,\ldots\right\},\qquad [p,x]=-i\hbar$$

implies

$$Spec\left(\frac{v^2}{2}\right)=|\omega|\{n+1/2,\ n=0,1,\ldots\},\qquad [v_x,v_y]=-i\omega$$



The ancillary space is hidden in the (infinite) Multiplicity of each eigenvalue

Landau degeneracy

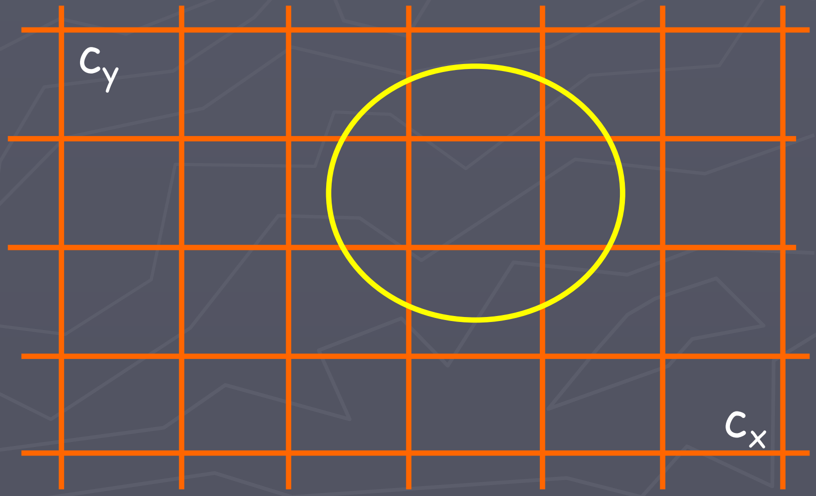
$$H = \frac{1}{2}v^2, \quad [v_j, c_k] = 0 \rightarrow [H, c_k] = 0,$$

A second canonical pair $$[c_y, c_x] = \frac{i}{\omega}$$

$$L^2(R^2) = L^2(c_x \in R) \otimes L^2(v_x \in R)$$

By uncertainty principle,
There is a state per area $\frac{1}{2\pi\omega}$

Density of states in each Landau level is $\frac{eB}{h}$



Magnetic translations

Exercse: Verify that $\Pi = p + \frac{e}{2} B \times x$ Generates shifts:

$$\left(T_a \psi\right)(x) = \left(e^{i\Pi a}\psi\right)(x) = e^{iB\cdot x \times a/2}\psi(x-a)$$

Gauge correction

Shift states without changing
their kinetic energy

$$T_a H(v) = H(v) T_a$$

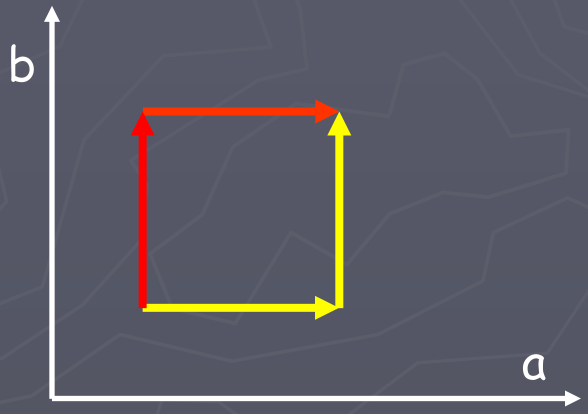
Non-commuting shifts

Exercise: show that the shifts

$$\left(T_a\psi\right)(x) = e^{iB\cdot x\times a/2}\psi(x-a)$$

Are non-commuting

$$T_aT_b = e^{ia\times b\cdot B}T_bT_a$$



Anti-symmetric under
a b interchange

Accumulated phase is the enclosed flux

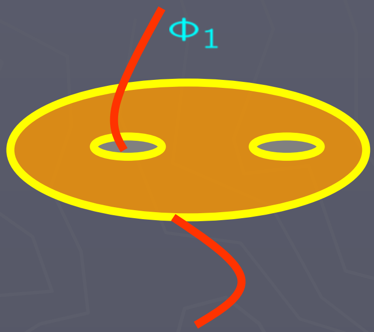
Flux and inaccessible fields

Gauge invariants:

$$B = \nabla \times A$$

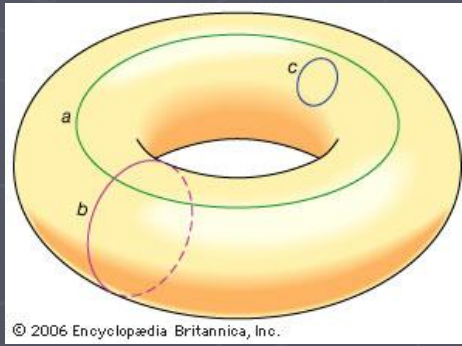
$$\Phi = \oint A \cdot d\ell$$

Flux tubes



In a contractible space no extra information in the flux

$$\oint_{\partial\Omega} A \cdot d\ell = \int_{\Omega} B \cdot dS$$



In a ring, punctured plane or torus
There are inaccessible fields
with accessible fluxes

Quantum Flux

$$dim[p] = dim[eA] \to dim[-i\hbar\nabla_x] = \mathrm{dim}[eA]$$

$$\to \frac{\hbar}{e} = \mathrm{dim}[A \cdot x] = dim[flux]$$

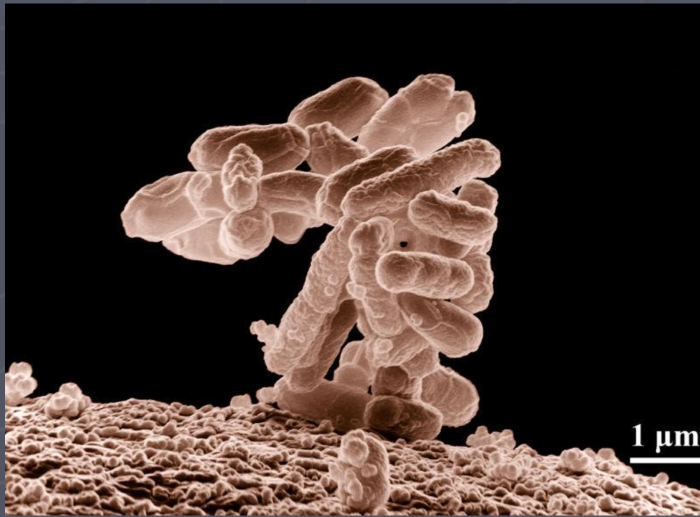
Quantum flux from fundamental constants only

$$\Phi_0 = \frac{h}{e} = 2 \times 2.06783363610^{-7}\ Gauss \times cm^2$$

The earth magnetic field (1G) through microorganism

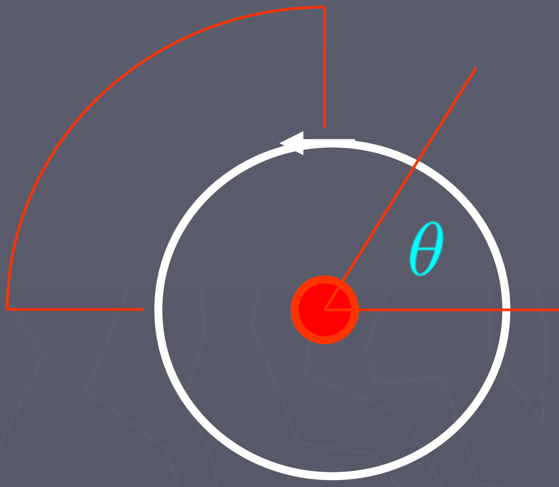
A quantum unit of magnetic field requires area scale : the Planck area

$$\frac{\hbar G}{c^3}$$



A flux tube

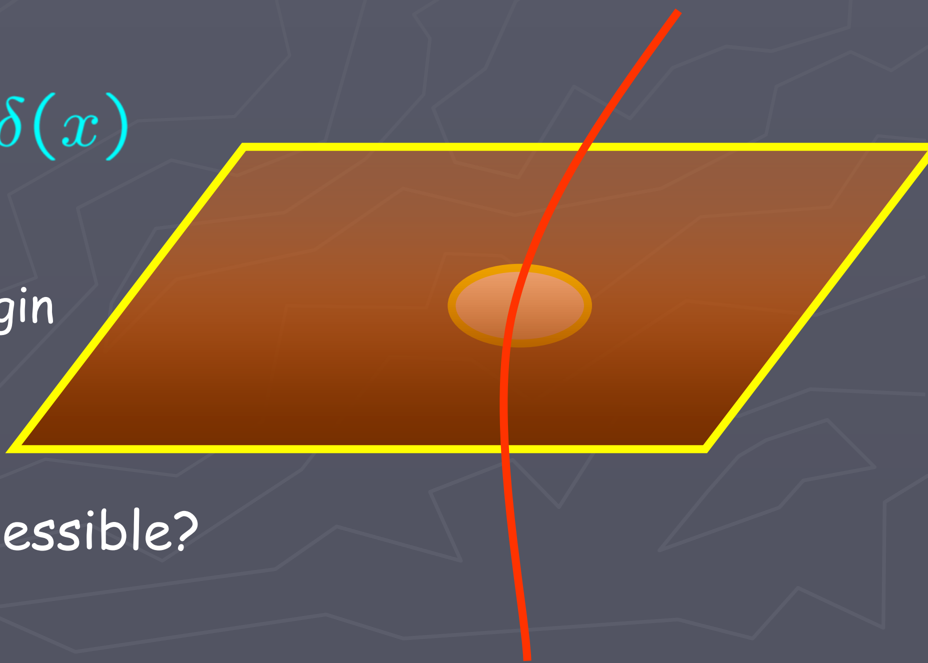
$$A = \frac{\Phi}{2\pi r}\hat{\theta}$$



$$\oint A \cdot d\ell = \begin{cases} \Phi, & \text{loop surrouns origin;} \\ 0, & \text{otherwise.} \end{cases}$$

This says that

$$B = \nabla \times A = \Phi\delta(x)$$



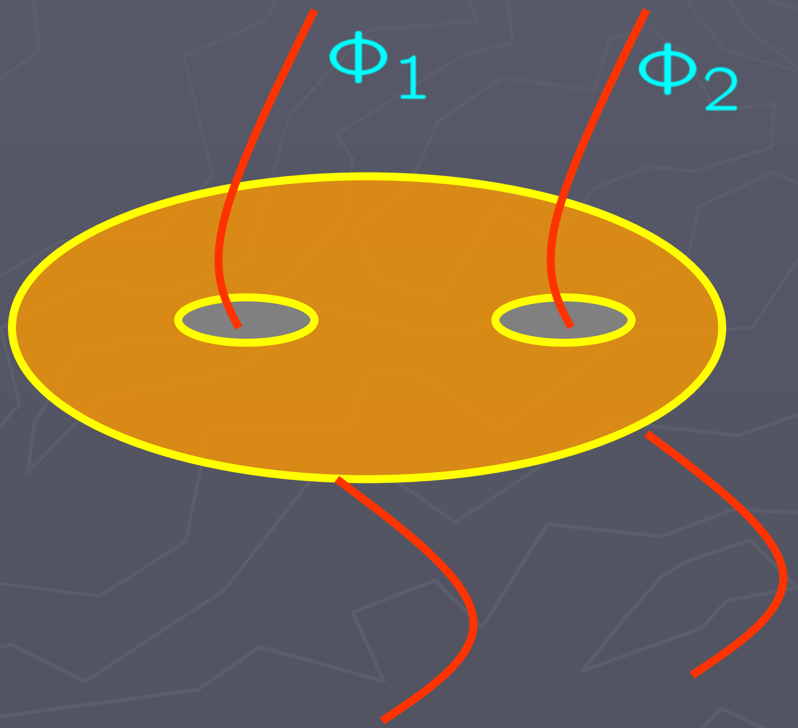
No field except at origin

What if the origin is inaccessible?

Aharonov-Bohm

Classical mechanics can be formulated with fields
Quantum mechanics is formulated with potentials.
Hamiltonians and Lagrangians rely on potentials

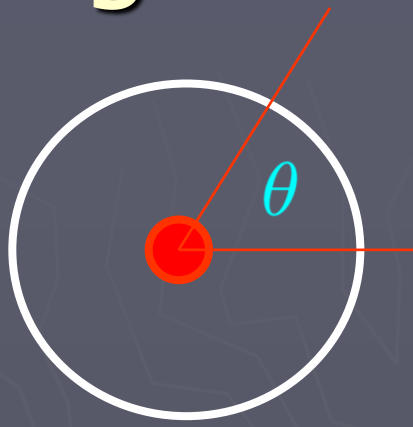
$$H(B, \Phi_1, \Phi_2, \ldots)$$



Physical observables
Can be sensitive to
Fluxes of inaccessible fields

Example: Particle on a ring

$$A = \frac{\Phi}{2\pi r}\hat{\theta}$$



Quantum particle on a ring of unit radius

The velocity is observable

$$v(\Phi) = -i\hbar\partial_\theta + e\Phi/2\pi$$

Eigenfunctions: $e^{in\theta}$

$$Spec(v) = \{n + \Phi/\Phi_0 \,|\, n \in Z\}$$

Measurement of velocity will detect the fractional part of the flux

Gauge transformations

Gauge=multiplicative unitary.
A change of basis in Hilbert space.

$$A \to A + \nabla\Lambda(x), \quad \psi \to e^{ie\Lambda/\hbar}\psi$$

Does not affect position and velocity

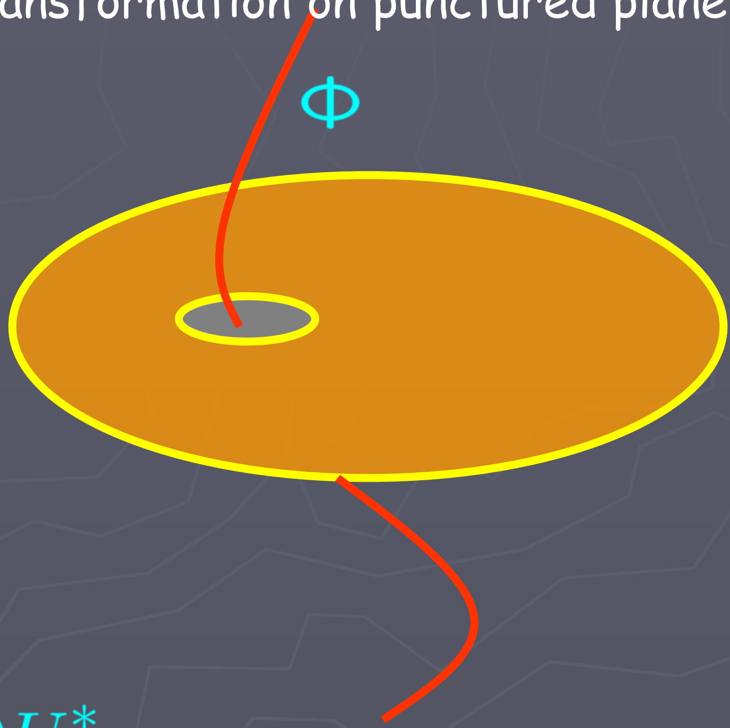
$$x \to UxU^\dagger = x, \quad U = e^{-ie\Lambda/\hbar}$$

$$v(A) = (-i\hbar\nabla - eA) \to Uv(A)U^\dagger = v(A + \nabla\Lambda)$$

Quantized flux tubes

$$U = \frac{x+iy}{|x+iy|} = e^{i\theta}$$ is a smooth gauge transformation on punctured plane

$$R^2/\{0\}$$



A pure gauge $$\nabla\Lambda = \frac{\Phi_0}{2\pi r}\hat{\theta}$$

Preserves B but sends

$$\Phi \to \Phi + \Phi_0$$

This implies

$$H(B,\Phi+\Phi_0) = U H(B,\Phi) U^*$$

Quantized fluxes are invisible

Dirac monopoles

Why is the charge of proton the same (up to sign) as the charge of the Electron?

Coulomb law for charges and magnetic charges

$$E = \frac{e}{4\pi}\frac{\hat{r}}{r^2}, \qquad B = \frac{g}{4\pi c}\frac{\hat{r}}{r^2},$$

Quantum mechanics is consistent with monopoles of charge g only if all electric charges e are such that

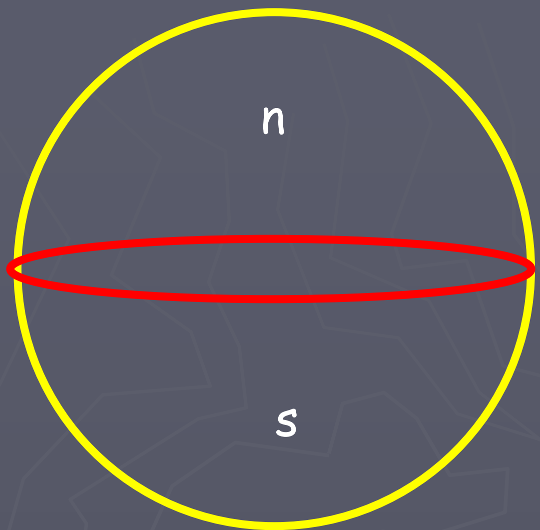
$$eg = 0 \ Mod \ hc$$

All charges e must be multiple of a single unit

A single monopole will quantize all charges

Vector potentials on manifolds

B=const, A is may not be globally defined



$$\Phi = \int B \cdot dS = \int_n B \cdot dS + \int_s B \cdot dS$$

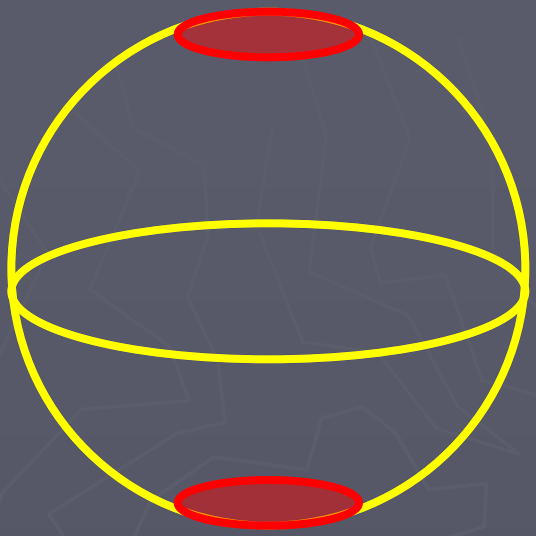
$$\int_n B \cdot dS = \oint A_n \cdot d\ell$$

$$\int_s B \cdot dS = -\oint A_s \cdot d\ell$$

$$\Phi \neq 0 \Rightarrow A_n \neq A_s$$

Dirac quantization a-la BA

Restrict a quantum particle to the equator



• The flux, modulo the unit of quantum flux, is observable and thus physical

$$\oint A_n \cdot d\ell = \oint A_s \cdot d\ell \;\; mod(\Phi_0)$$

By Stokes, total magnetic flux through manifold

$$\Phi = \oint (A_n - A_s) \cdot d\ell = 0 \;\; mod\left(\Phi_0 = \frac{h}{e}\right)$$

Translation: The total curvature (flux) of a connection A that defines a vector bundle (a gauge covariant Hilbert space) is quantized