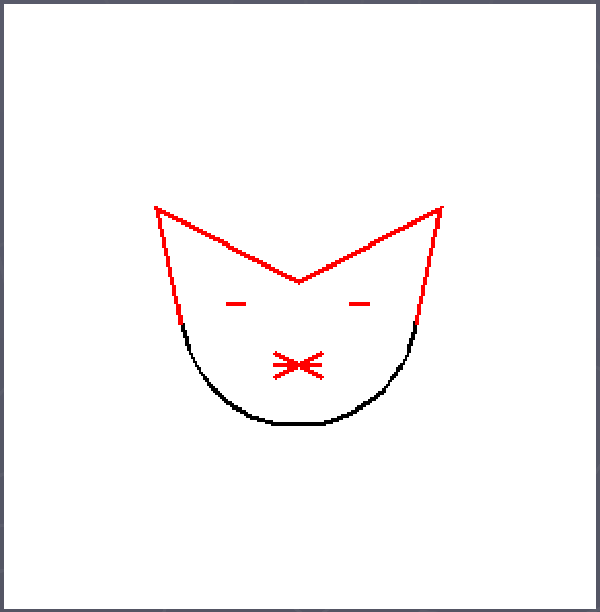
The Baron and the cat

Joint work with Oded Kenneth
Original idea: J. Wisdom

Baron and cat

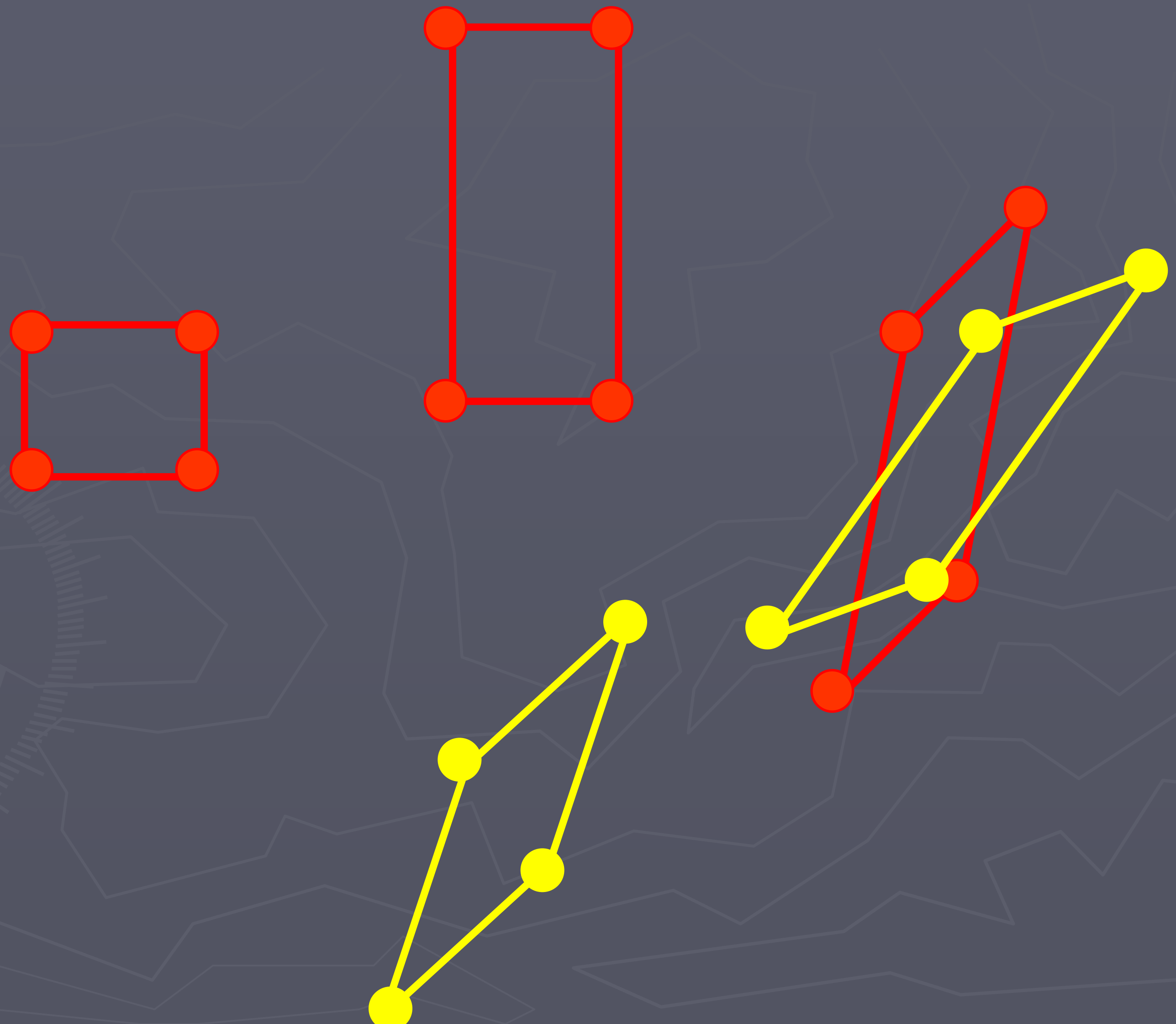
- A cat can rotate with zero angular momentum
- Can Baron von Munchausen lift himself?
- Translate with zero momentum?



Movie: omri Gat

H. Knoerer: Cats always fall on their feet

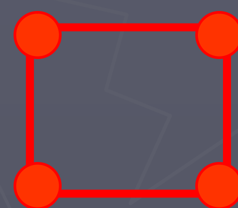
Rotations without angular momentum: Physics



Rotations without angular momentum: Mathematics

Generators of deformations

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[X, Z] = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2R$$

The commutator of deformations= a rotation

Newton's law

If a system experiences no external force,
the center-of-mass of the system will remain at rest.

http://webphysics.davidson.edu/physlet_resources/bu_semester1/c12_cofm_motion.html

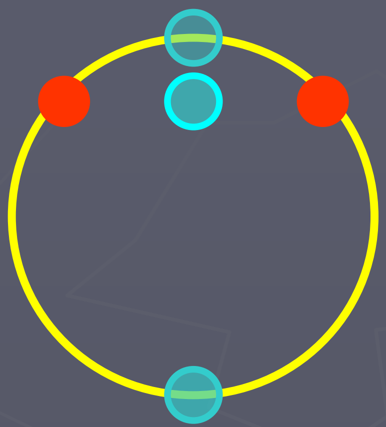
Newton:

Lex I: Corpus omne perseverare in statu suo quiescendi vel Movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

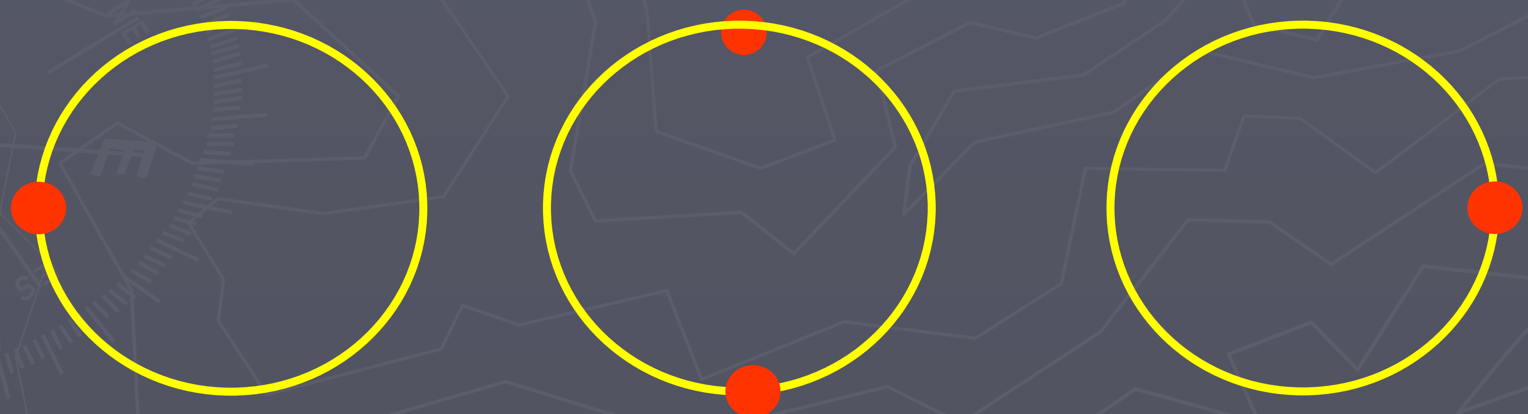
Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed

Wikipedia

Ambiguous center of mass: Topology

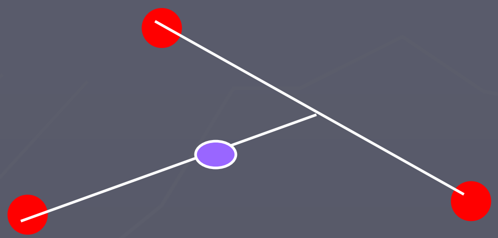


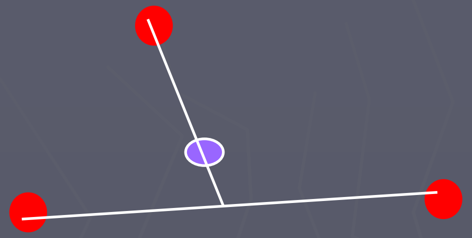
Ambiguous center of mass allows the Baron to move itself



Ambiguous center of mass: Geometry

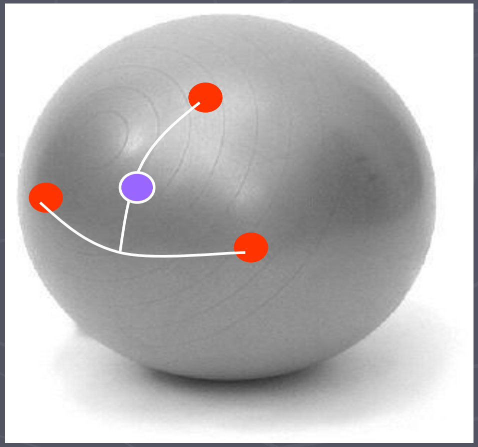
Euclidean plane





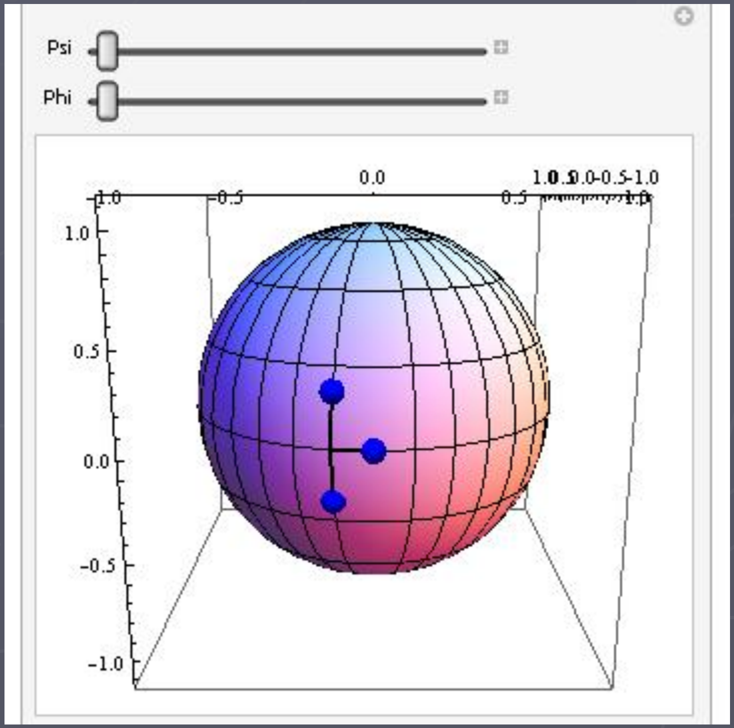
The three medians of a (Euclidean) triangle intersect at one point

On a sphere



Swimming triangles

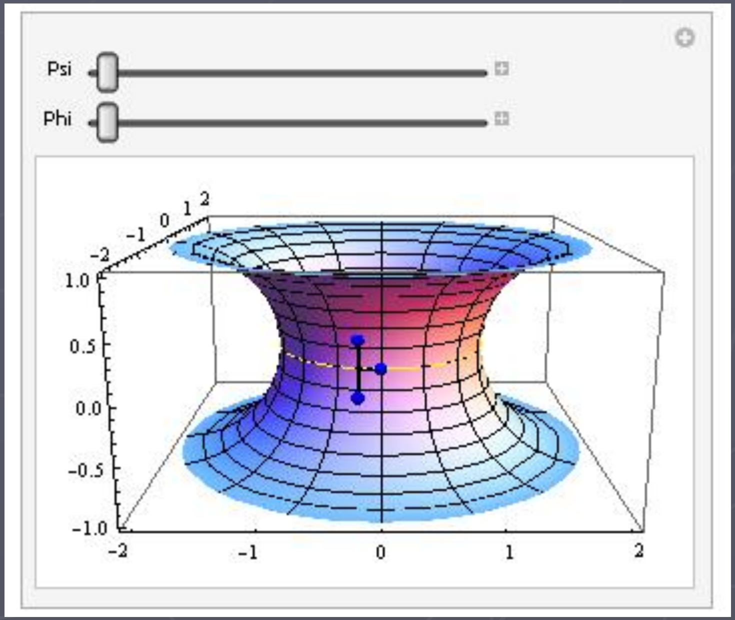
On sphere



Movies: Oren Raz

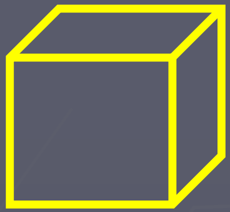
Swimming triangles

Inside a torus



Movies: Oren Raz

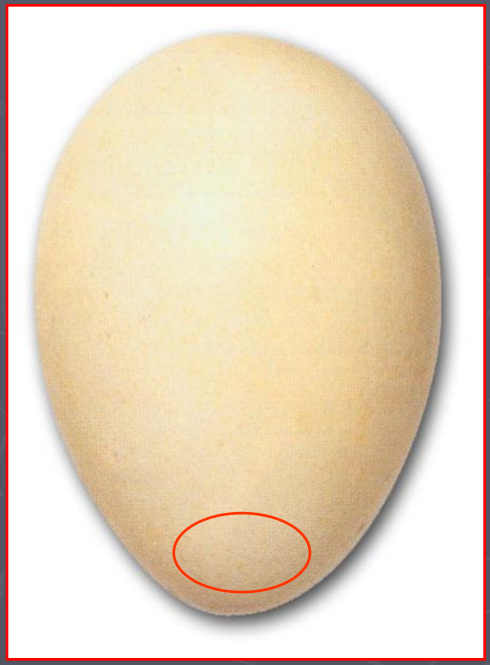
Shape fixes location



Shape: All mutual distances $\binom{n}{2}$

Location: nd coordinates

$n > 2d + 1$ overconstrained.



Large rigid bodies generically fits nowhere

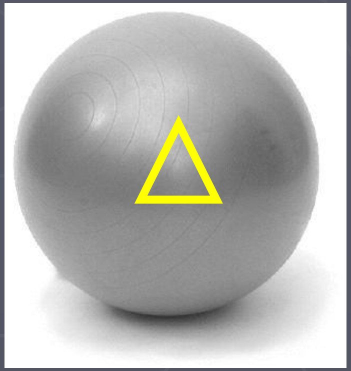
Generically, shape fixes a location

When can rigid bodies move?

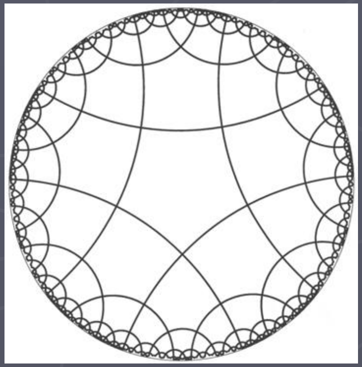
Euclidean space is a symmetric space: Homogeneous and isotropic.

In a symmetric space if a rigid body fits somewhere, it fits any where.

Examples:



Sphere



Lobachevski plane

In a symmetric space, shape does not determine location. You may try to swim.

Killing forms

Killing for Euclidean translation $\xi_t = dx$

Killing for Euclidean rotation $\xi_R = xdy - ydx$

Controls: Deformations

Example: Euclidean case

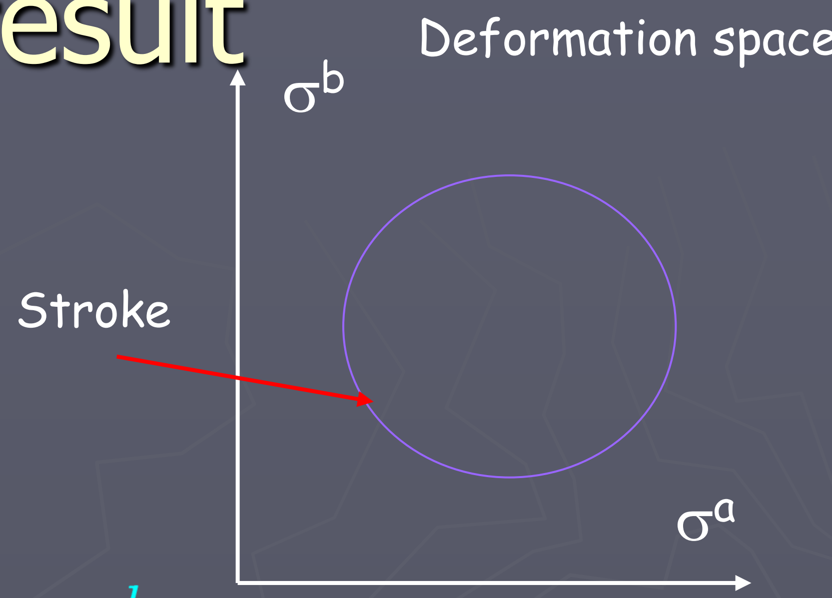
$$\eta_1 = x\partial_x - y\partial_y$$

$$\eta_2 = x\partial_y + y\partial_x$$

Vector field for deformations depend on choice of origin: Ambiguity in Killing

$$\eta_1 \to \eta_1 + a\xi_x - b\xi_y$$

Main result



Swimming distance

$$\text{đ}\tau = \langle d\xi | \eta_a, \eta_b \rangle \; d\sigma^a \wedge d\sigma^b$$

Killing 2-form

Deformation fields

Space allows for swimming if the Killing 1-form is not closed

Cats spin, the Baron lies

Translations $$\xi_t = dx \longrightarrow d\xi_t = 0$$

The Baron lies

Rotations

$$\xi_R = x\,dy - y\,dx \longrightarrow d\xi_R = 2\,dx \wedge dy$$

The cat falls on its feet

Small swimmers on symmetric surface

Property of space

Size of stroke

$$\delta x \approx 8R \left(\frac{\sum m_n x_n y_n^2}{M} \right) dA$$

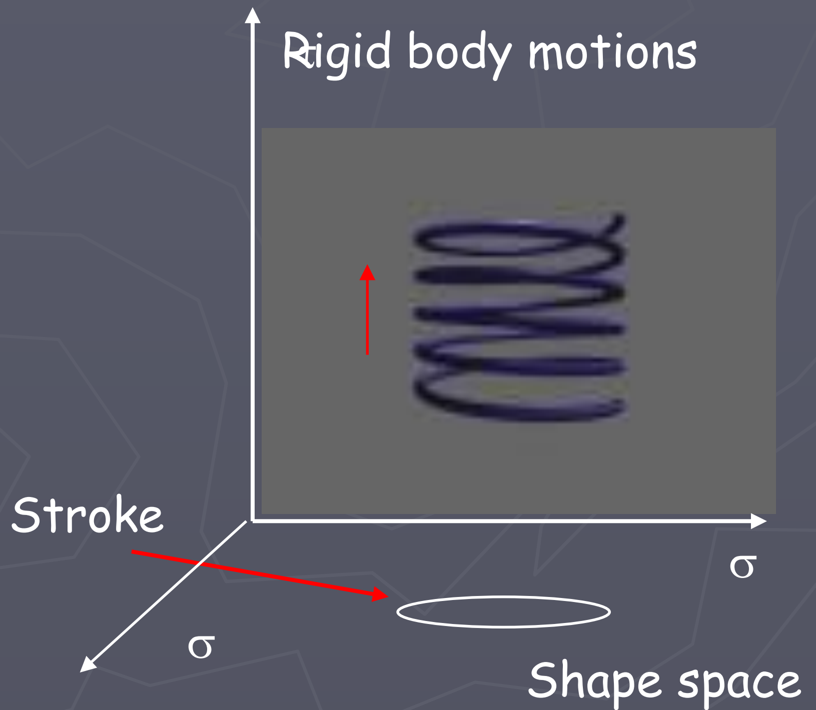
Geometry of aswimmer

Wisdom
Swim away from Black hole
Relativistic motions

AK:
Symmetric spaces
Non relativistic bodies

The swimming problem

Swimming with periodic strokes:



Conservation of momentum

A system of particles

$$P_\xi = \sum_n p_n \cdot \xi(x_n), \quad p_n = \frac{\partial L}{\partial \dot{x}_n} = m\dot{x}_n$$

Killing vector field

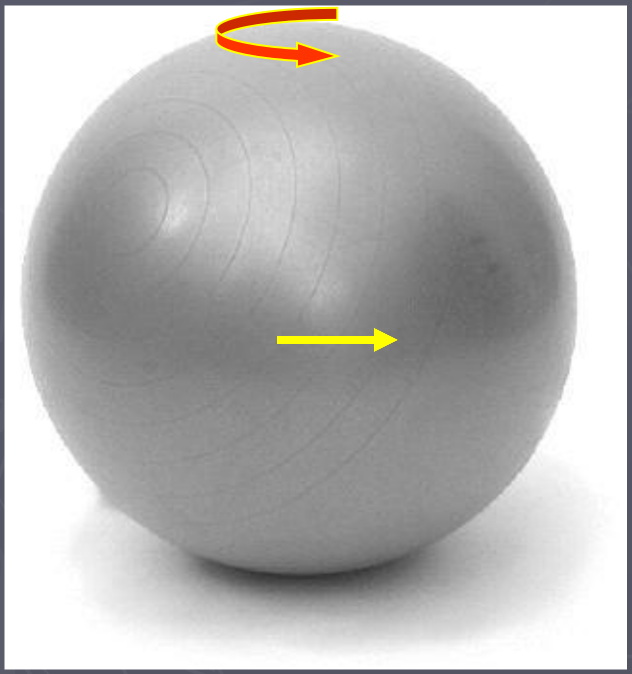
Swimming equations

$$P_\xi = 0 \longrightarrow \sum m_n \xi(x_n) \cdot dx_n = 0$$

As many equations as Killing fields:
Can't move in other directions

Eq. independent of time parameterization: Geometric

Killing fields: Translation & rotations



Rotation about z axis:
Looks like translation near equator
And like rotation at poles

Defining property: No strain

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

Relation to Riemann

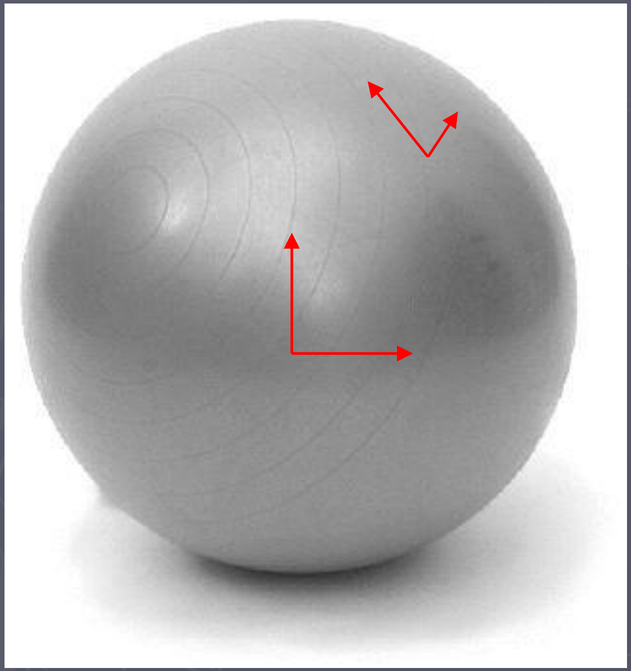
$$\xi_{\ell;i;j} = -R_{\ell jik}\xi^k$$

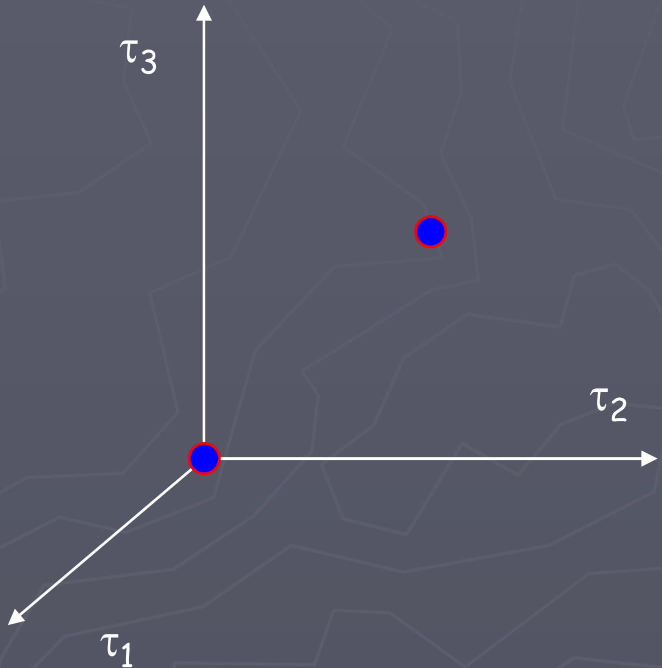
Determined by initial data at a point

$$\xi_\mu(0), \quad \xi_{[\mu,\nu]}(0)$$

translation — Rotation

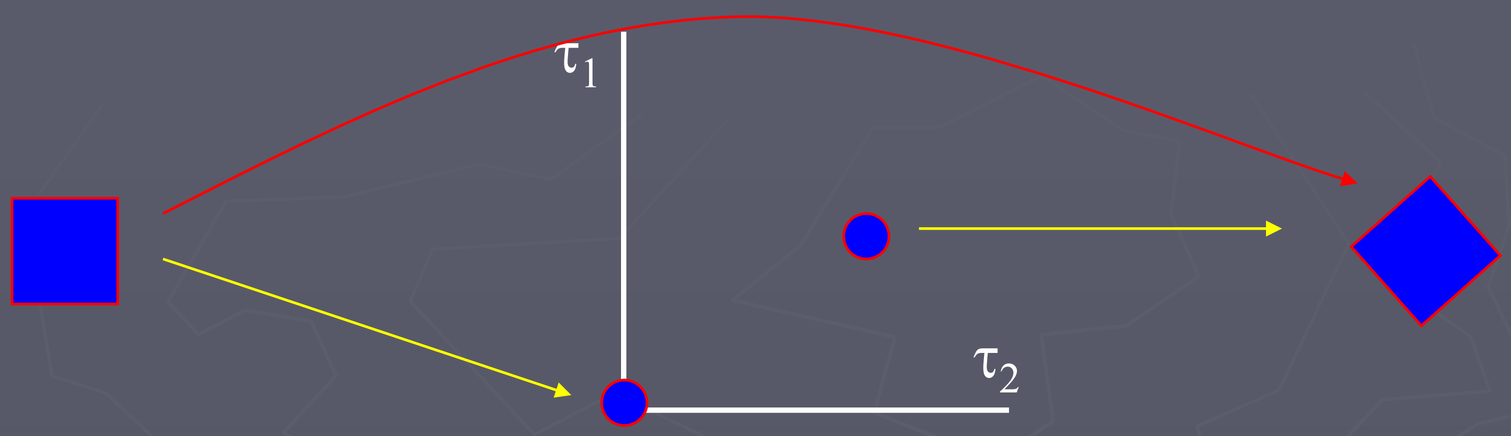
Rigid body motion: Coordinates





Putting coordinates for
The analog of Euclidean
motions

Coordinates for rigid body motion



$$\dot{x}(t) = \tau \cdot \xi\big(x(t)\big), \quad x(0) \to x_\tau(1)$$

$$S(\tau) = e^{\tau\xi} S$$

Symbolic notation for transported shape

Gauge choice instead of CM

$$\langle \xi | \eta \rangle = \frac{1}{M} \sum m_n \xi(x_n) \cdot \eta(x_n)$$

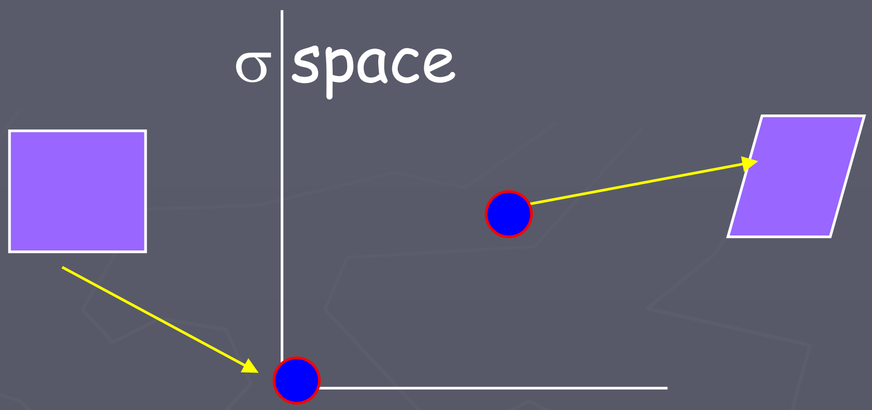
Euclidean example:

$$\langle \xi_1 | \eta_1 \rangle = \frac{1}{M} \sum m_n \, x_n$$

Gauge choice: Analog to choosing cm as fiducial pt for translations

$$\langle \xi_j | \eta_a \rangle = 0$$

Coordinates for deformed shapes



$$\dot{x}(t) = \sigma \cdot \eta\big(x(t)\big), \quad x(0) \to x_\sigma(1)$$

Symbolically $$S(\sigma) = e^{\sigma\eta} S$$

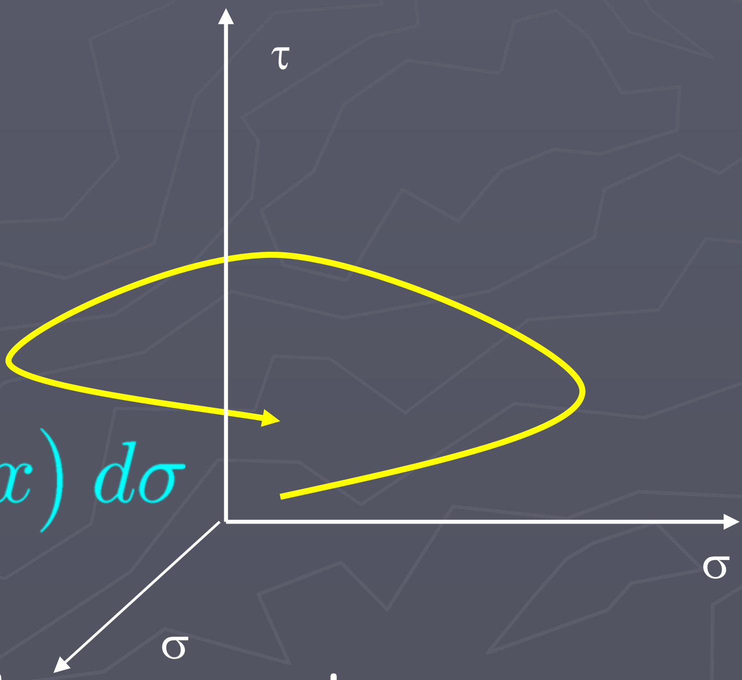
Total space

Coordinates for deformations + Euclidean motions

$$S(\tau,\sigma) = e^{\tau\xi} e^{\sigma\eta} S$$

$$x_0 \longrightarrow x(\tau,\sigma;x_0)$$

$$dx(\tau,\sigma;x_0) = (\partial_\tau x)\, \text{đ}\tau + (\partial_\sigma x)\, d\sigma$$



Motions is legit if it consistent with zero total momentum

Equation of Motion

To leading order with σ and τ small

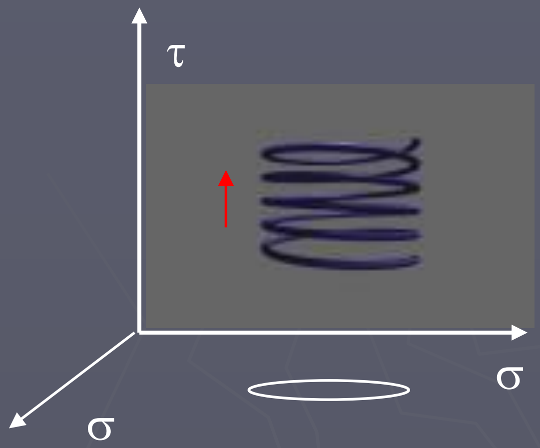
$$dx = \xi d\tau + \eta d\sigma, \quad \xi(x(\sigma,\tau)), \quad \eta(x(\sigma,\tau)),$$

Substitute in conservation law

$$\sum m_n \xi(x_n) \cdot dx_n = 0$$

Gives a linear (system) of equations for $d\tau$

Main result



Swimming distance

$$\bar{d}\tau = \langle d\xi | \eta_a, \eta_b \rangle \; d\sigma^a \wedge d\sigma^b$$

Killing fields

Deformation fields

Shape space

$$\bar{d}\tau^\alpha = \langle \partial_{[k}\xi^\alpha_{j]} | \eta^k_a \eta^j_b \rangle \; d\sigma^a \wedge d\sigma^b$$

Role of Curvature

Killing field for symmetric spaces can be
Explicitly expressed in terms of the curvature

$$\xi_{\ell;i;j} = -R_{\ell jik}\xi^k$$

In a symmetric space

$$R_{ijk\ell} = R\left(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\right)$$

Swimming in the direction of the k-th Killing field

$$\delta x^k = 2R\Big(Q^k d\sigma^a \wedge d\sigma^b\Big)$$

Stroke and swimmer data

Geometry of embedding space

$$Q^k = \frac{1}{M}\sum m_n \; x_n^i \; \eta_a^i(x_n)\eta_b^k(x_n)$$

Application to astrophysics?

How does swimming affect the motion of galaxies of size x?

$$\frac{\Delta x}{x} \propto R\,(\delta x^2) \approx R\,v^2\,(dt)^2$$

Hubble constant 75 km/sec/Mpc gives a measure of the curvature

$$R \sim H_0^2 \approx 10^{-20}\ [year]^{-2}$$

Perhaps not surprisingly, it takes the age
of the universe for a significant swimming

The hero: Oded Kenneth

Thanks to Amos Ori