Adiabatic evolutions

Temple mount

Outline

- Adiabatic invariants
- The adiabatic theorem
- Berry's phase and curvature

Levitron (Berry)

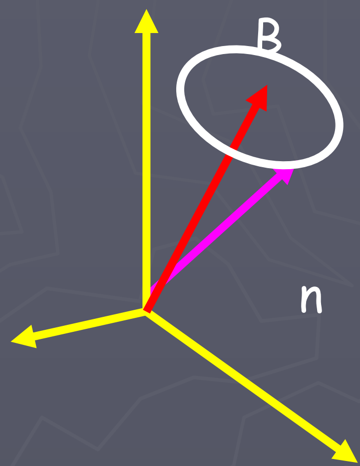
$B \cdot J = 0$ is harmonic if J is fixed.

$|B| = B \cdot \hat{B}$ is not.

Spin precession

$$H(B) = B \cdot \sigma$$

$$\rho = \frac{1 + x \cdot \sigma}{2}$$



Heisenberg equation of motion

$$-i\dot{\rho} = [H, \rho] \Rightarrow \rho \to e^{iHt} \rho e^{-iHt}$$

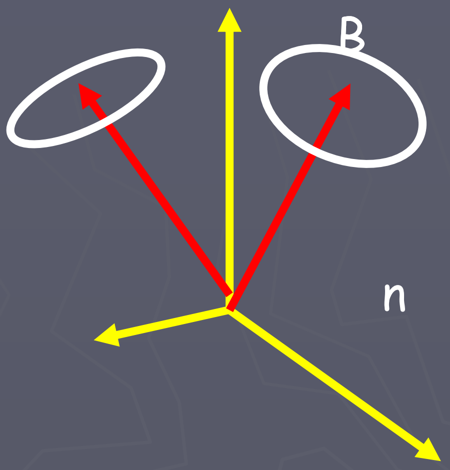
Unitary evolution preserves the length of |x|

The direction n precesses $\dot{n} = B \times n, \quad n = \hat{x}$

In the (weak) magnetic field of the earth (~1G), nuclear (weak) spins precess
At audio frequencies, ~10Khz

Adiabatic invariants

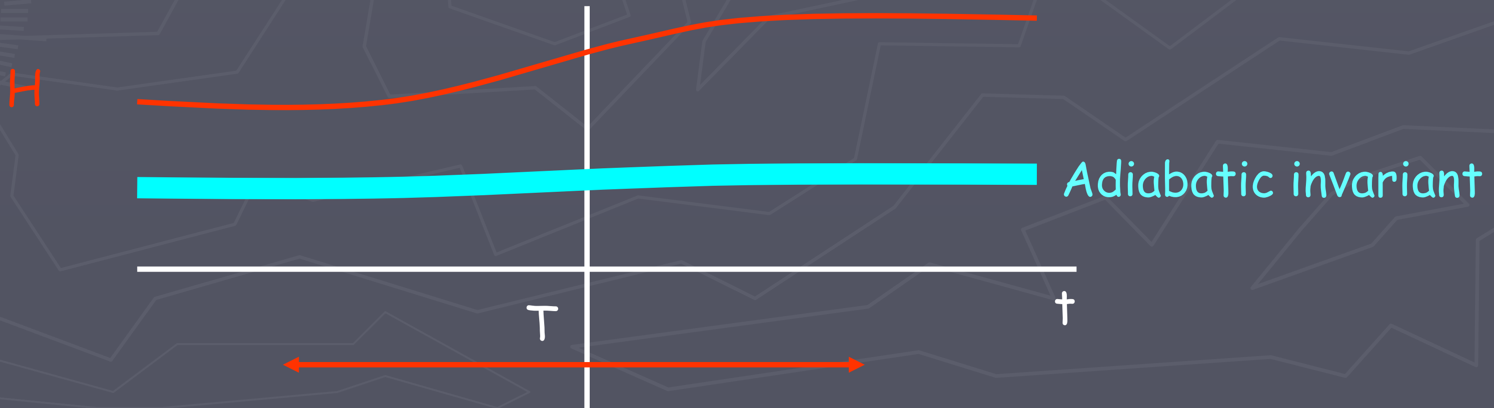
$H(s)\,, \qquad s = \frac{t}{T} = \text{scaled time, } T \gg 1$



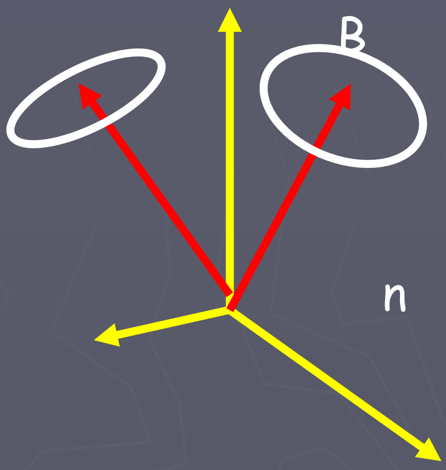
Adiabatic: Slow time dependence = T large
= fast intrinsic time scale i.e. B large

$$\langle X \rangle = \int_{period} X(t)dt$$

Adiabatic invariants:
<X>, fast time average that survives O(1) change in H
when rate is slow:
No accumulation of errors



Example: spin drag



From eq. of motion $\dot{n} = B \times n$

Conclude first $\langle n \rangle \,\|\, B$

Good, but not enough

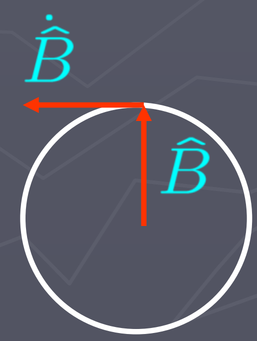
$$\dot{\left(n \cdot \hat{B}\right)} = \overbrace{n \times B \cdot \hat{B}}^{=0} + n \cdot \dot{\hat{B}} = O\left(\frac{1}{T}\right)$$

Errors do not accumulate for long times:

$$\left\langle n \cdot \dot{\hat{B}} \right\rangle = \langle n \rangle \cdot \dot{\hat{B}} + O\left(\frac{1}{T^2}\right)$$

but

$$\langle n \rangle \cdot \dot{\hat{B}} \propto B \cdot \dot{\hat{B}} = 0$$

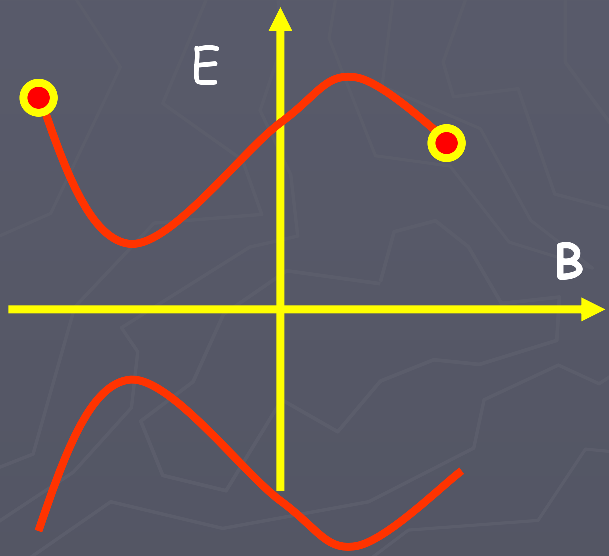


The adiabatic theorem (ref:Teufel)

Replaces solving an evolution equation by solving a family of spectral problems

In ordinary times H is slowly changing

$$-i\partial_t \left|\psi\right\rangle = H\left(s\right)\left|\psi\right\rangle, \quad s = \frac{t}{T}$$



In adiabatic times H is large

$$-i\partial_s \left|\psi\right\rangle = TH(s)\left|\psi\right\rangle, \quad T >> 1$$

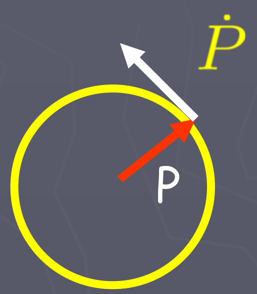
Spectral projections are adiabatic invariants

$$P_{s=0} \to P_s + o(1), \quad s = O(1)$$

Kato geometric evolution

Properties of projections

$$P^2 = P \to P\dot{P} + \dot{P}P = \dot{P}$$
$$P\,\dot{P}\,P = 0$$



The evolution of spectral projections is generated by

$$f(H) - i[\dot{P}, P]$$

Proof:

Heisenberg equation of motion for P:

$$\begin{aligned} i\Big[f(H) - i[\dot{P}, P], P\Big] &= \Big[[\dot{P}, P], P\Big] \\ &= \Big[\dot{P}P - P\dot{P}, P\Big] \\ &= \dot{P}P^2 - P\dot{P}P - P\dot{P}P + P^2\dot{P} \\ &= \dot{P}P + P\dot{P} = \dot{P} \end{aligned}$$

Comparison of dynamics

Generators of spectral flow:

$$f(H) - i\left[\frac{dP}{dt}, P\right] = f(H) - \frac{i}{T}[\dot{P}, P], \qquad \dot{P} = \frac{dP}{ds}$$

P

Comparison with physical evolution

$$H_A = H - \frac{i}{T}[\dot{P}, P]$$

Generates spectral evolution:
Satisfies the adiabatic theorem without error
Good starting point for comparison

Miracle: H approximates the spectral evolution for large intrinsic time scales: of order T.
A Riemann-Lebesgue statement: E is large implying rapid oscillations

Outline of proof

Compare dynamics

$$U_A^\dagger U - 1 = -iT\int ds\, U_A^\dagger (H_A - H) U = -\int ds\, U_A^\dagger [\dot P, P] U$$

A-priori O(1)

Hypothesis: $[\dot P, P] = [X, H]$ has bounded solutions X

$$U_A^\dagger U - 1 = -\int ds\, U_A^\dagger [X, H] U \approx \frac{i}{T}\int ds\, \left(\dot U_A^\dagger X U + U_A X \dot U\right) =$$

$$= \frac{i}{T}\int ds\, \left(\partial_s (U_A X U) - U_A \dot X U\right) = O\left(\frac{1}{T}\right)$$

Bdry term O(1)

QED

Commutator equation

Want to solve: $$[\dot{P}, P] = [X, H]$$

The solution is non-unique since

$$X \to X + g(H)$$

Solution always exists if P is protected b gaps

The case of H with discrete spectrum

Represent the commutator equation in the instantaneous basis of H

$$\langle n|\,[\dot{P}, P]\,|m\rangle = \langle n|\,X\,|m\rangle\,(E_m - E_n)$$

$$\langle n|\,X\,|m\rangle \text{ if } E_m \neq E_n$$

Determines the off-diagonals of X if levels do not cross
"Smallest" X has vanishing diagonal

Parallel transport

- Tulio Levi Civita
- 1873 Padua –1941 Rome
- Curvature=failure of parallel transport

Iexplore.exe

Adiabatic connection

We can use the notion of adiabatic generator to evolve wave-functions

$$i\left|\dot{\psi}\right\rangle = H_A\left|\psi\right\rangle, \qquad H_A = -i[\dot{P}, P]$$

Independent of time re-parameterization

$$id\left|\psi\right\rangle = \mathcal{A}\left|\psi\right\rangle, \qquad \mathcal{A} = -i[dP, P]$$

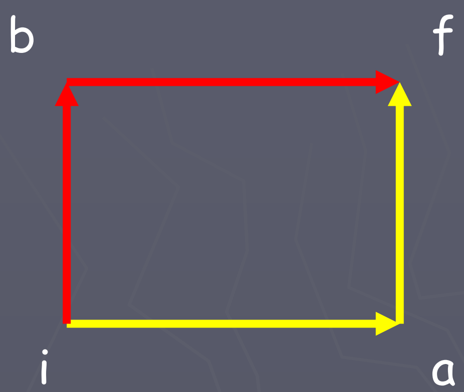
Exercise: Show that for $$P_\pm = \frac{1 \pm \hat{B}\cdot\sigma}{2}$$

One has $$\mathcal{A} = \frac{\hat{B}\times d\hat{B}\cdot\sigma}{2}$$

Matrix valued differential (gauge field).

Holonomy

Adiabatic connection



$$id\,|\psi\rangle = \mathcal{A}\,|\psi\rangle\,, \qquad \mathcal{A} = -i[dP, P]$$

The solution of the differential equation is path dependent

$$\mathcal{A} = \sum \mathcal{A}_j db_j, \qquad \mathcal{A}_j = -i[\partial_j P, P]$$

$$idP \wedge dP = i[\partial_1 P, \partial_2 P]\ db_1 \wedge db_2$$

Prop. to area

Manifestly gauge invariant

Stokes and commuting failure

$$\mathcal{A} = \sum \mathcal{A}_j db_j, \qquad \mathcal{A}_j = -i[\partial_j P, P]$$

Stokes failure

$$\begin{aligned} \partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 &= -i\partial_1[\partial_2 P, P] + i\partial_2[\partial_1 P, P] \\ &= 2i[\partial_1 P, \partial_2 P] \end{aligned}$$

Failure to commute on Range P

$$\begin{aligned} -[\mathcal{A}_1, \mathcal{A}_2]P &= \Big[[\partial_1 P, P], [\partial_2 P, P]\Big]P \\ &= \Big([\partial_1 P, P](\partial_2 P)P - [\partial_2 P, P](\partial_1 P)P\Big) \\ &= -P[\partial_1 P, \partial_2 P]P \end{aligned}$$

Holonomy

$$i dP \wedge dP = i[\partial_1 P, \partial_2 P] \; db_1 \wedge db_2$$

Berry's model

Instantaneous projections

$$P_{\pm} = \frac{1 \pm \hat{B} \cdot \sigma}{2} \qquad holonomy = \frac{d\hat{B} \times d\hat{B} \cdot \sigma}{2}$$

At the north pole:

$$\frac{d\hat{B}_x d\hat{B}_y \sigma_z}{2} = \frac{d\Omega}{2}\sigma_z \to \pm\frac{d\Omega}{2}$$

On the equator you get the -1