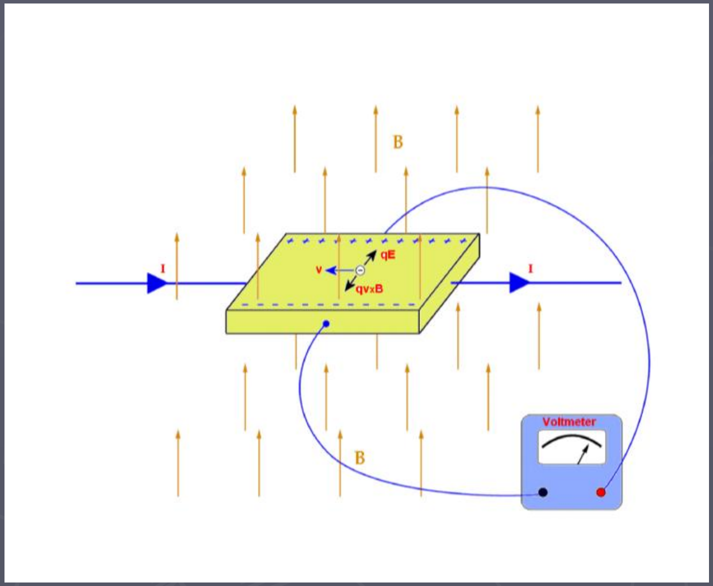
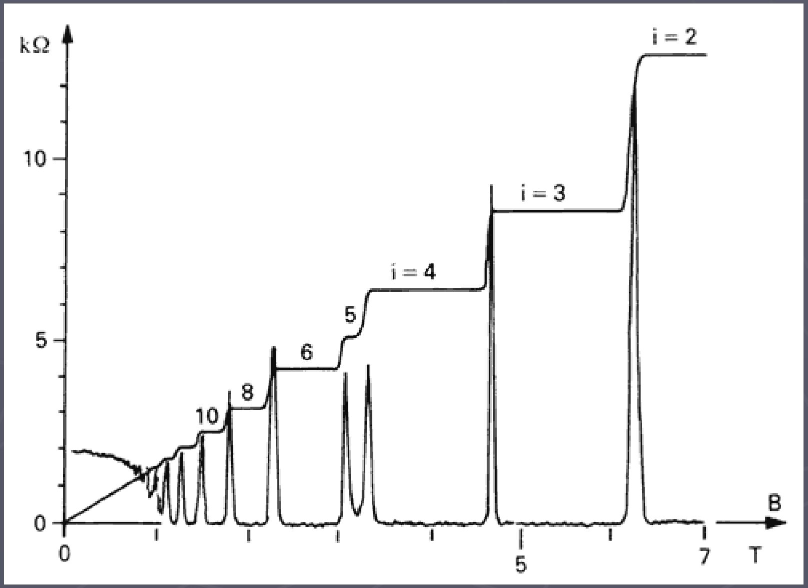
Geometry of q-transport

- Facts
- Skew Conductance=curvature
- From geometry to topology
- Chern numbers
- Comparing infinite projections
- Bellissard formula

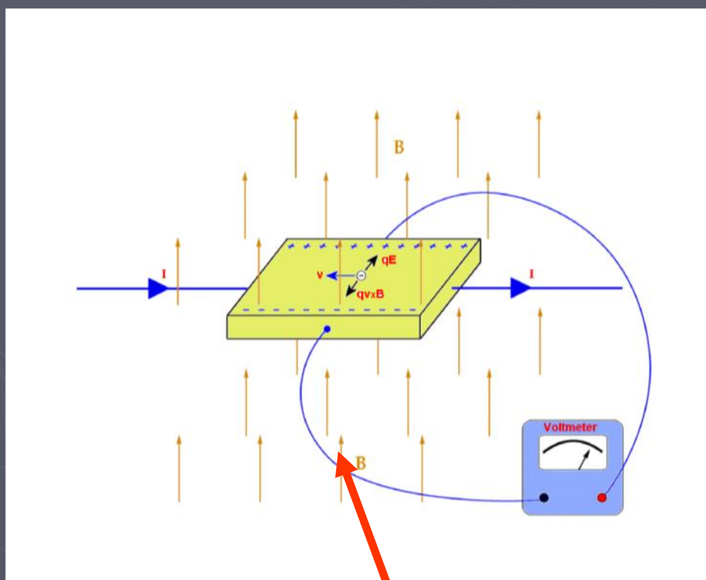
Facts

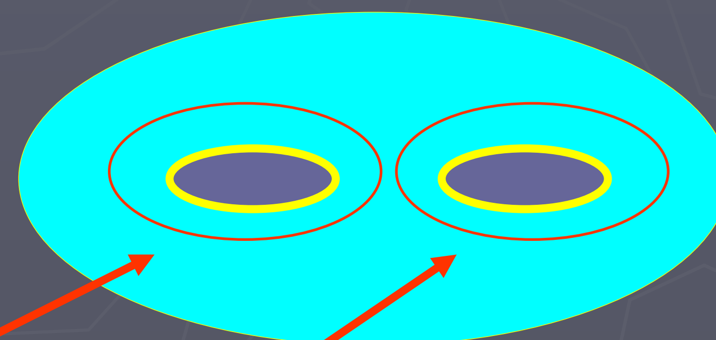




$$\sigma_H = \frac{I}{V} = n\frac{e^2}{h}$$

Current and emf loop





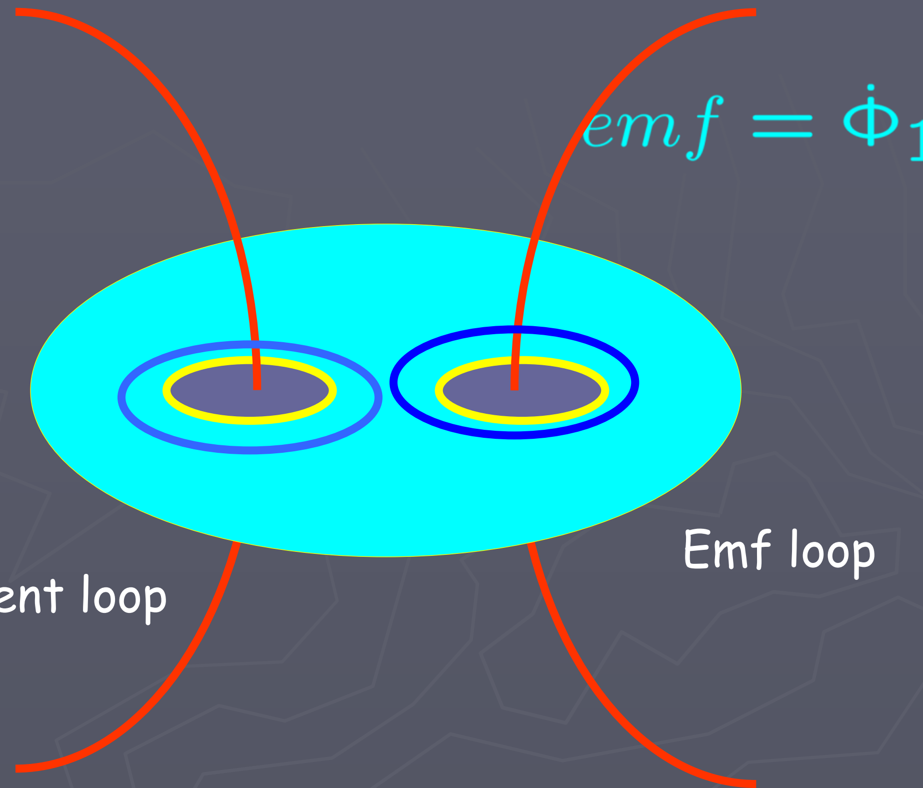
Current loop

Voltage loop

Skew Conductance=Curvature

$$H(\Phi_1, \Phi_2)$$

$$I_h = \frac{\partial H}{\partial \Phi_2}$$



In the adiabatic limit, the Hall conductance is the adiabatic curvature

$$\sigma_H = 2\, Im \, \langle \partial_2 \psi_A | \partial_1 \psi_A \rangle$$

Charge transport

Thm: If H generates the evolution of ψ then

$$\langle\psi|\,\partial_2 H\,|\psi\rangle = i\partial_t\langle\psi|\partial_2\psi\rangle$$

Charge transport is bdry term

$$\langle Q\rangle = \int_0^T dt\ \langle\psi|\,\partial_2 H\,|\psi\rangle = i\langle\psi|\partial_2\psi\rangle\Big|_0^T$$

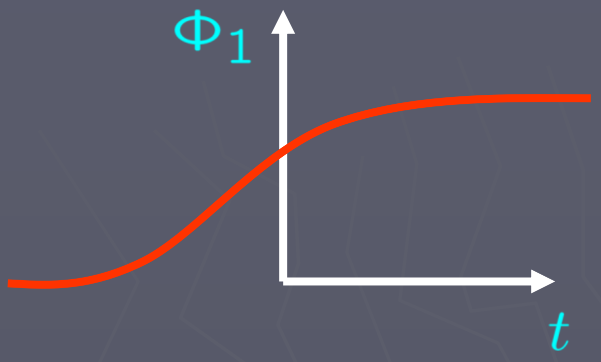
Proof:

$$\begin{aligned}\langle\psi|\,\partial_2 H\,|\psi\rangle &= \partial_2\,\langle\psi|\,H\,|\psi\rangle \\ &- \langle\partial_2\psi|\,H\,|\psi\rangle - \langle\psi|\,H\,|\partial_2\psi\rangle\end{aligned}$$

Use Schrodinger $\quad H\,|\psi\rangle = i\partial_t\,|\psi\rangle$

$$\begin{aligned}\langle\psi|\,\partial_2 H\,|\psi\rangle &= i\partial_2\langle\psi|\partial_t\psi\rangle \\ &- i\langle\partial_2\psi|\partial_t\psi\rangle + i\langle\partial_t\psi|\partial_2\psi\rangle \\ &= i\partial_t\langle\psi|\partial_2\psi\rangle\end{aligned}$$

Linear response=Adiabatic limit



In adiabatic limit

$$|\psi\rangle \longrightarrow |\psi_A\rangle$$

$$\psi(t,\Phi_2) \longrightarrow \psi_A(\Phi_1(t),\Phi_2)$$

$$\langle Q\rangle = i\langle\psi|\partial_2\psi\rangle\Big|_0^T \longrightarrow i\langle\psi_A|\partial_2\psi_A\rangle\Big|_0^T$$

$$H \longrightarrow H_A = H - i[\dot P, P] = H - i[\partial_1 P, P]\dot\Phi_1$$

Unwind the previous computation

$$i\partial_1 \langle\psi_A|\,\partial_2\,|\psi_A\rangle = \langle\psi_A|\,\partial_2 H_A\,|\psi_A\rangle$$

involves only
Adiabatic data

Technicalities

$$\langle\psi_A|\,\partial_2 H_A\,|\psi_A\rangle \;=\; \partial_2\,\langle\psi_A|\,H_A\,|\psi_A\rangle$$
$$-\;\langle\partial_2\psi_A|\,H_A\,|\psi_A\rangle - \langle\psi_A|\,H_A\,|\partial_2\psi_A\rangle$$

The first term is Feynmann-Hellman term

$$\langle\psi_A|\,H_A\,|\psi_A\rangle \;=\; \langle\psi_A|\,H - i[\dot{P},P]\,|\psi_A\rangle$$
$$=\langle\psi_A|\,H\,|\psi_A\rangle \;=\; E$$

The second term is

$$\langle\partial_2\psi_A|\,H_A\,|\psi_A\rangle = i\langle\partial_2\psi_A|\partial_1\psi_A\rangle\,\dot{\Phi}_1$$

In conclusion

$$\langle\psi_A|\,\partial_2 H_A\,|\psi_A\rangle \;=\; \partial_2 E - i\Big(\langle\partial_1\psi_A|\partial_2\psi_A\rangle - \langle\partial_2\psi_A|\partial_1\psi_A\rangle\Big)\dot{\Phi}_1$$

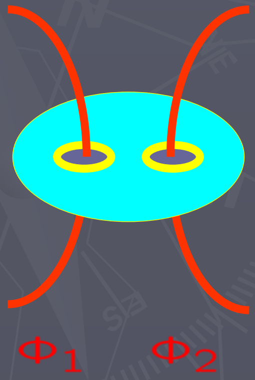
Persistent currents conductance

From geometry to topology

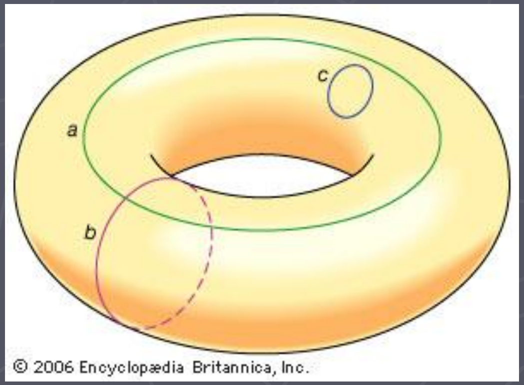
For any closed manifold,

$$\frac{1}{2\pi i}\int_M Tr(PdPdPP) = integer$$

Application to Hall effect



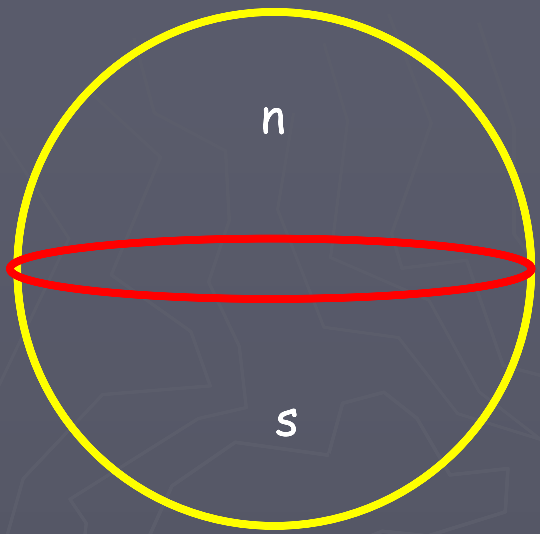
Aharonov-Bohm periodicity
Implies that flux space
Is a torus.



Corollary: The average Hall conductance over the flux torus is quantize

Total adiabatic curvature quantized

Use Dirac quantization argument:



$$\int B = \int_n B + \int_s B, \qquad B = i\langle d\psi | d\psi \rangle$$

$$\int_n B = \oint_{equator} A_n, \qquad A_n = i\langle \psi_n | d\psi_n \rangle$$

But, Berry's phase is a physical observable so

$$\oint_{equator} i\langle \psi_s | d\psi_s \rangle = \oint_{equator} i\langle \psi_n | d\psi_n \rangle \ Mod\ 2\pi$$

It follows that

$$\int B = 0 \ Mod\ 2\pi$$

A theory of Hall effect?

- Yes: Sufficiently general (allows e-e interact)
- No: Too general (disorder, 2 dim irrelevant)
- Worry: Where are the fractions? (non-deg)
- Too weak: Only the average ransport quantized

Orthogonal projections

Orthogonal projections:

$$P^2 = P, \qquad P = P^\dagger$$

Pair of projections

$$P \qquad Q, \qquad Q_\perp = 1 - Q$$

Non-commuting trigonometry

Let:

$$S = P - Q, \qquad C = P - Q_\perp$$

Then

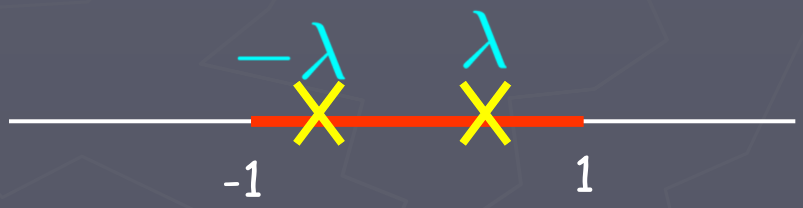
$$S^2 + C^2 = 1$$

$$CS + SC = 0$$

Corrolaries

The spectrum of P-Q is

1. Contained in [-1,1]
2. Balanced in (-1,1)



Theorem: Suppose S is compact. Then

$$Tr(P-Q)^{2n+1} = dimKer(P-Q-1) - dimKer(P-Q+1)$$

An integer independent of n

Allows for comparing infinite dimensions

Non-commutative trigo

$$S = P - Q, \qquad C = P - Q_\perp$$

Compute:

$$\begin{aligned} SC &= (P - Q)(P - Q_\perp) \\ &= P - PQ_\perp - QP \\ &= PQ - QP \end{aligned}$$

Then also

$$CS = PQ_\perp - Q_\perp P$$

It follows that

$$CS + SC = P - P = 0$$

Non-commutative Pythagoras

$$S = P - Q, \quad C = P - Q_\perp$$

Compute:

$$\begin{aligned} S^2 &= (P-Q)(P-Q) \\ &= P - PQ - QP + Q \\ &= PQ_\perp + QP_\perp \end{aligned}$$

Then also

$$C^2 = PQ + Q_\perp P_\perp$$

It follows that

$$S^2 + C^2 = P + P_\perp = 1$$

Spectrum of P-Q

Recall $S = P - Q$

From $S^2 + C^2 = 1$

Follows $Spec(S) \subseteq [-1, 1]$

Spectrum of P-Q: Symmetry

Suppose S compact. If

$$S\left|\psi\right\rangle = \lambda\left|\psi\right\rangle, \quad \left|\lambda\right| < 1$$

From the non-commutatively of C and S

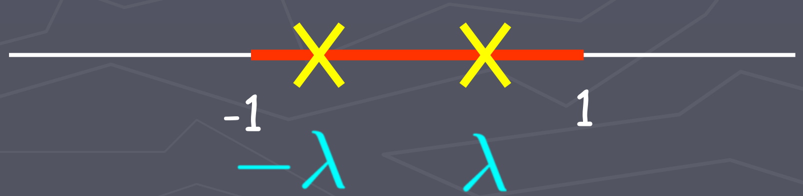
$$-CS\left|\psi\right\rangle = SC\left|\psi\right\rangle = -\lambda C\left|\psi\right\rangle$$

$$S\left|\phi\right\rangle = -\lambda\left|\phi\right\rangle, \quad \left|\phi\right\rangle = C\left|\psi\right\rangle$$

Where

$$\left\langle\phi|\phi\right\rangle = \left\langle\psi\right|C^2\left|\psi\right\rangle = 1-\lambda^2 \neq 0$$

Spectrum is balanced



Comparing infinite dimensions

When P and Q are finite dimensional then

$$Tr(P-Q) = Tr(P) - Tr(Q) \in Z$$

How would you compare two infinite dimensional projections?

Example: Full Landau level

$TrP = \infty$ # electrons in Landau level

$TrQ = \infty$ # electrons punctured Landau level

Expect: # electrons in puncture = number of flux quanta

Comparing infinite dimensions

When P and Q are finite dimensional then

$$Tr(P-Q)^{2n+1} = \sum \lambda_j^{2n+1} =$$
$$deg(P-Q-1) - deg(P-Q+1)$$

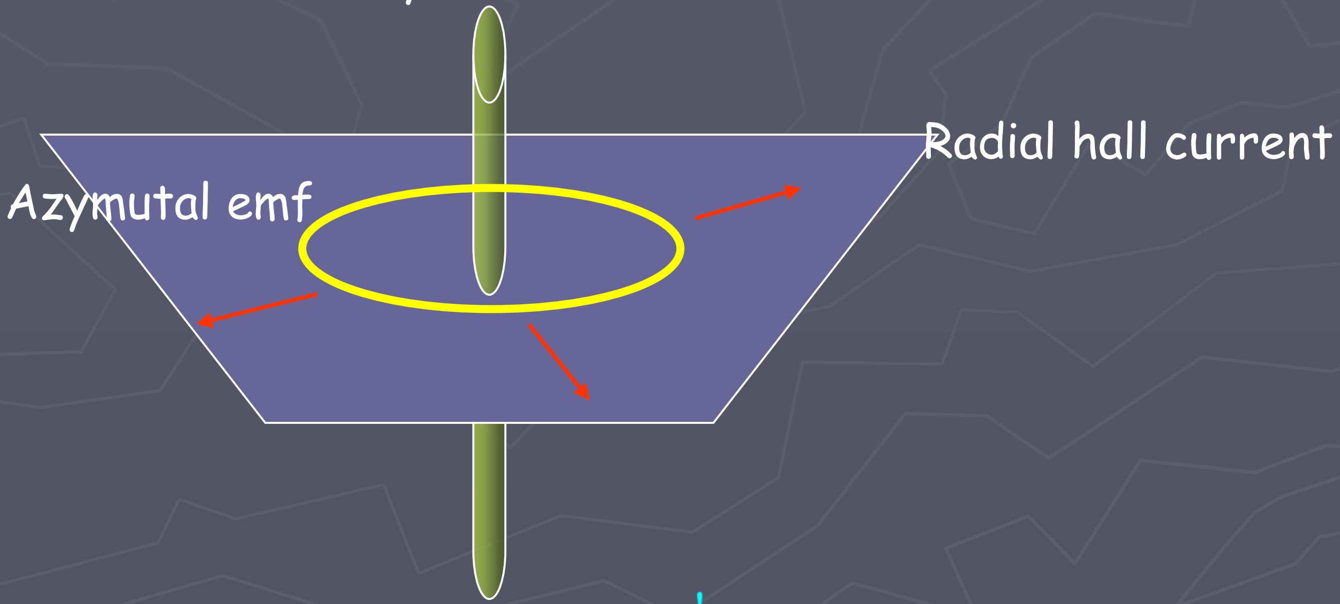
Can work also when P and Q are infinite dimensional

Application to QHE

Laughlin flux tube: singular gauge transformation

$$U = \frac{z}{|z|}$$

Crank up the flux adiabatically



Expect $Tr(P - U^\dagger P U) \in Z$

Bellissard formula

- P a spectral projection in two dimensions.
- $\langle x| P |y\rangle$ decays sufficiently fast in $x-y$
- U AB flux tube
- Mild conditions about translation invariance

The Hall conductance is

$$Index(PUP) = Tr(P - U^{\dagger}PU)^3 \in Z$$