Entanglement & the geometry of two qubits

With Oded Kenneth

qubit

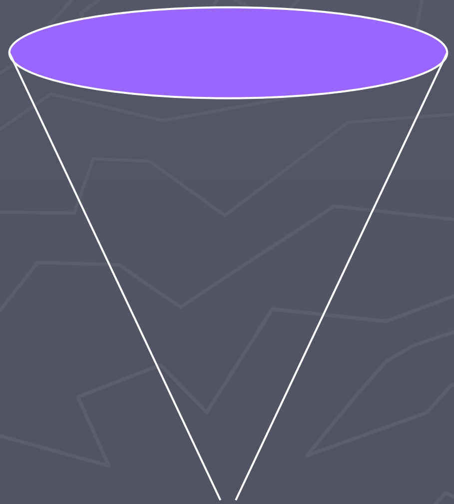
Hilbert space : $\mathbb{C}^2$

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{\mu=0}^{3} n_\mu \sigma_\mu,$$

Observables: 4 dim linear space

States: 4 dim cone

Pure states: Boundary



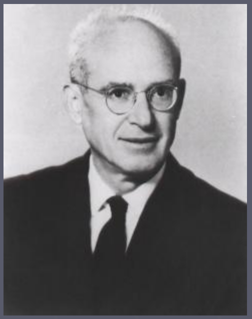
Alice & Bob

$$|A\rangle \otimes |B\rangle$$

$$\rho = \sum_{\mu\nu=0}^{3} \rho_{\mu\nu} \sigma_\mu \otimes \sigma_\nu = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

16 dimensional

Separable states



$$\rho_S = \sum p_j \, \rho_j^A \otimes \rho_j^B , \qquad p_j > 0$$

Entangled states= Unseparable

Example: Bell or EPR states

$$\rho_B = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Separable states: (in) significance

$$Prob(\rho, A) = Tr(\rho A)$$

$$\rho = \rho^A \otimes \rho^B$$

$$P(A \otimes B; \rho_A \otimes \rho_B) = P_A(\rho_A) P_B(\rho_B)$$

Statistical independence

Local operations and classical communication

Witnesses

$$Tr(\rho_s W) > 0$$

Example: or why Bell is entangled

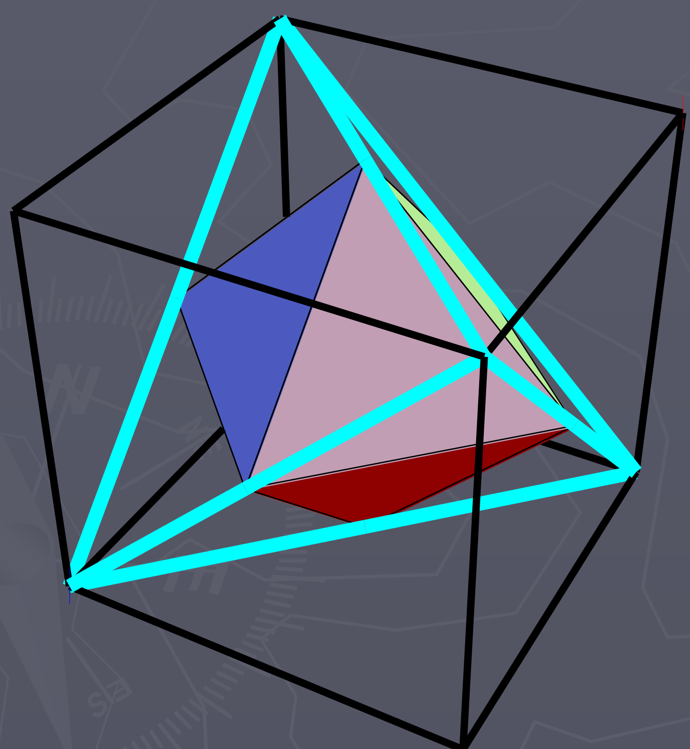
$$E\,|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$$

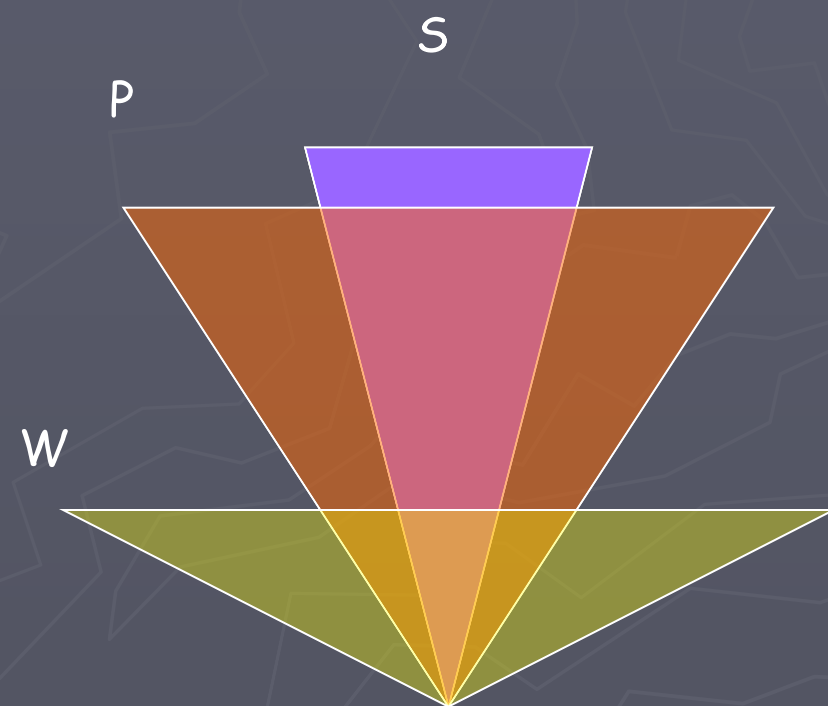
$$Tr(\rho_s E) = |\langle\phi|\psi\rangle|^2 \geq 0 \qquad \rho_s \leftrightarrow |\psi\rangle \otimes |\phi\rangle$$

But

$$E\,|Bell\rangle = -\,|Bell\rangle\,, \qquad |Bell\rangle = |1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle$$

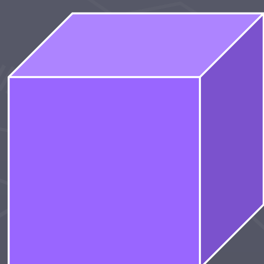
Geometric characterization





Local equivalence

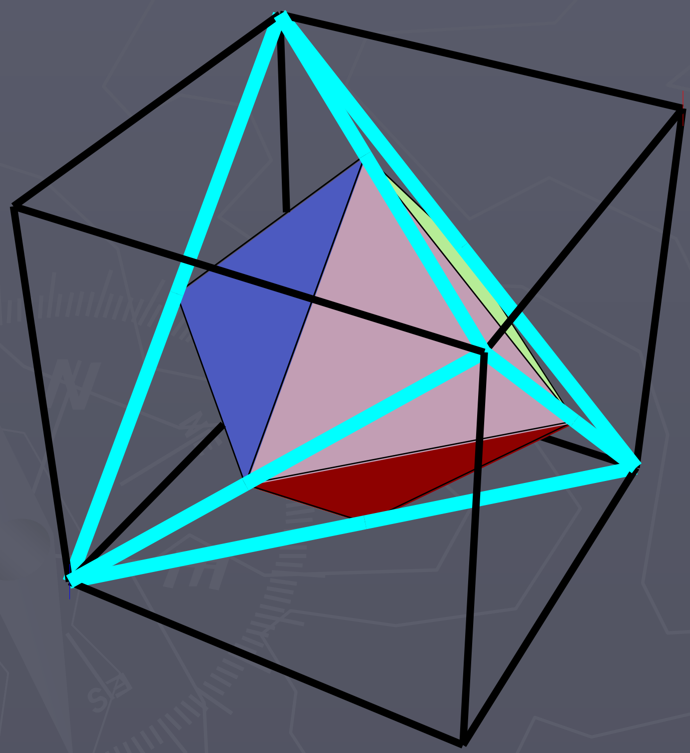
$$A \otimes B \sim (UAU^*) \otimes (VBV^*), \qquad U, V \in SU(2)$$



15- (3+3)= 9

Horodeckies

States with maximally mixed subsystems



$$Tr_a\rho = Tr_b\rho = \mathbb{I}$$

15-(3+3)-(3+3)=3

Local equivalence

$$A \otimes B \sim (UAU^*) \otimes (VBV^*), \qquad U, V \in SL(2, \mathbb{C})$$

15- (6+6)= 3

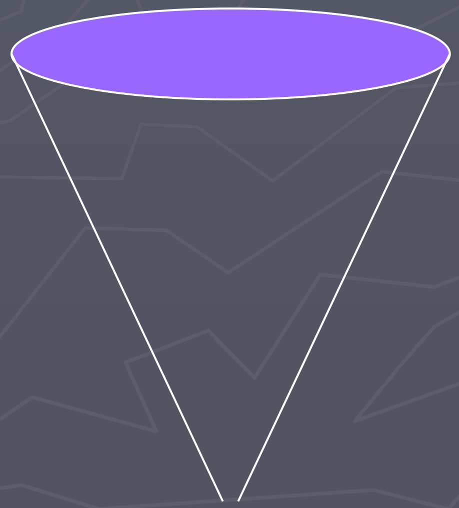
Lorentz geometry

$$\det(a \cdot \sigma) = a_\mu a^\mu$$

$$G(a \cdot \sigma)G^* = (\Lambda_G a) \cdot \sigma, \quad G \in SL(2, \mathbb{C})$$

Mixed state are time-like

Pure states are light-like



CANONICAL FORM

Suppose A maps the forward light-cone into itself then there are a pair of Lorentz transformations that bring A into a diagonal form.

$$w_{\mu\nu} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \quad a \geq b \geq c \geq |d|$$

$$Spec(W) = a \pm b \pm c \pm d,$$

Peres test is iff

$$\rho = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \qquad \rho^P = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$$

If transform has negative eigenvalue state is entangled

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \rho^P = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$