Hofstadter models

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outline

- Comparing infinite projections
- Bellissard formula
- Lattice gauge theories
- Hofstadter model
- Hofstadter butterfly

Orthogonal projections

Orthogonal projections:

$$P^2 = P, \qquad P = P^\dagger$$

Pair of projections

$$P \quad Q, \qquad Q_\perp = 1 - Q$$

Non-commuting trigonometry

Let: $$S = P - Q, \qquad C = P - Q_\perp$$

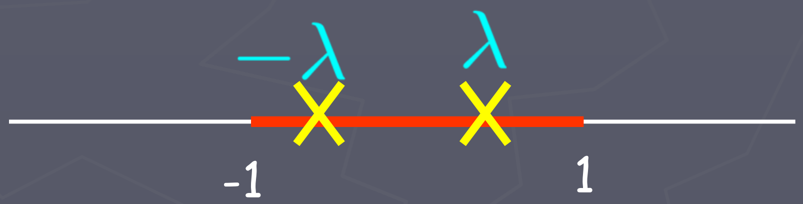
Then $$S^2 + C^2 = 1$$

$$CS + SC = 0$$

Corrolaries

The spectrum of P-Q is

1. Contained in [-1,1]
2. Balanced in (-1,1)



Theorem: Suppose S is compact. Then

$$Tr(P-Q)^{2n+1} = dimKer(P-Q-1) - dimKer(P-Q+1)$$

An integer independent of n

Allows for comparing infinite dimensions

Spectrum of P-Q

Recall $$S = P - Q$$

From $$S^2 + C^2 = 1$$

Follows $$Spec(S) \subseteq [-1, 1]$$

P-Q: Symmetry

Suppose S compact. If

$$S\,|\psi\rangle = \lambda\,|\psi\rangle\,, \qquad |\lambda| < 1$$

From the non-commutatively of C and S

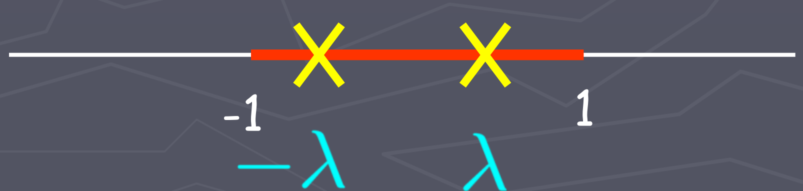
$$-CS\,|\psi\rangle = SC\,|\psi\rangle = -\lambda C\,|\psi\rangle$$

$$S\,|\phi\rangle = -\lambda\,|\phi\rangle\,, \qquad |\phi\rangle = C\,|\psi\rangle$$

Where

$$\langle\phi|\phi\rangle = \langle\psi|\,C^2\,|\psi\rangle = 1 - \lambda^2 \neq 0$$

Spectrum is balanced except possibly at the end points



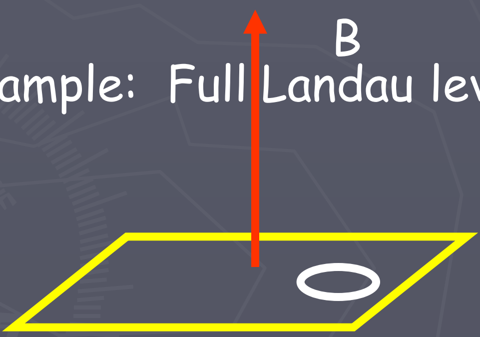
Comparing infinite dimensions

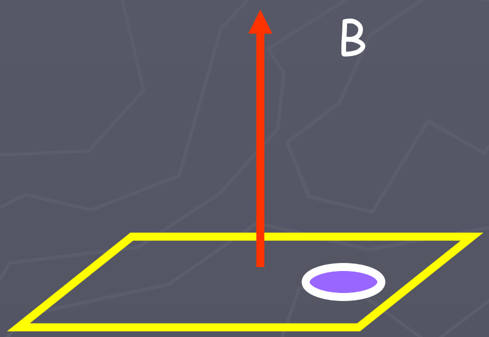
When P and Q are finite dimensional then

$$Tr(P-Q) = Tr(P) - Tr(Q)$$

$$= Tr(P-Q)^{2n+1} = \sum \lambda_j^{2n+1} \in Z$$

Example: Full Landau level





$TrP = \infty$ # electrons in Landau level

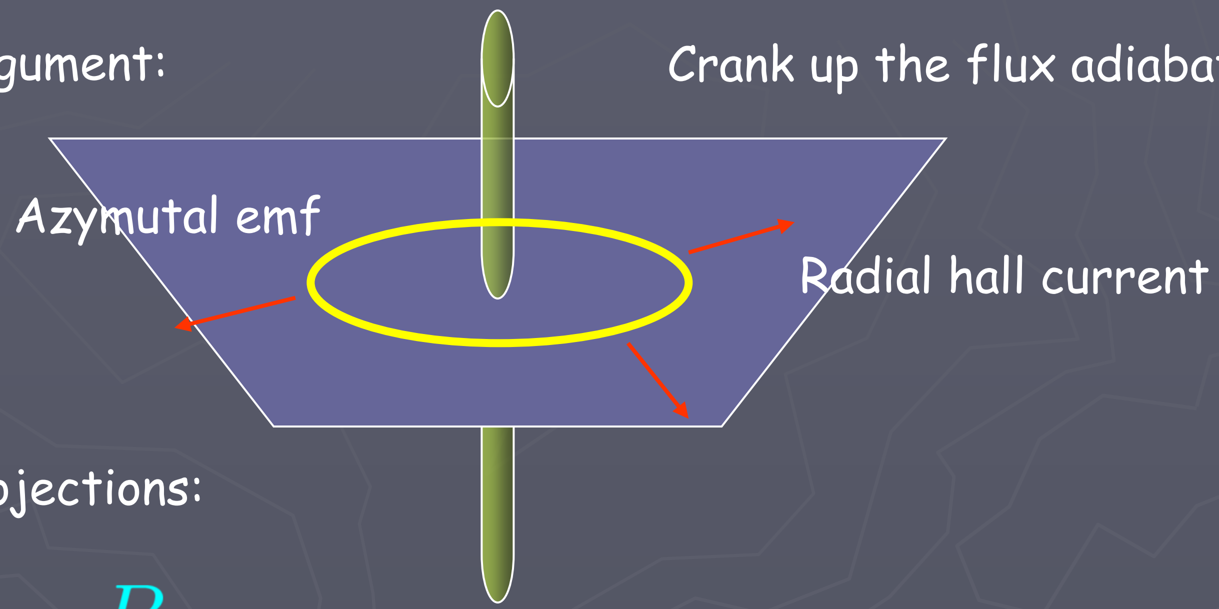
$TrQ = \infty$ # electrons punctured Landau level

Expect: # electrons in puncture = number of flux quanta

Application to QHE

Laughlin argument:

Crank up the flux adiabatically



The two projections:

Initial state P

Final state $Q = UPU^\dagger$ $\quad U = \frac{z}{|z|}$

Hall conductance$= Tr(P - Q) \in Z$

Bellissard formula: Hall conductance is an index

- P a spectral projection in two dimensions.
- $\langle x| P |y\rangle$ decays sufficiently fast in $x - y$
- U AB flux tube
- Mild conditions about translation invariance

The Hall conductance is

$$Index(PUP) = Tr(P - U^{\dagger} PU)^3 \in Z$$

Projections and integers

Chern integer for smooth bi-periodic spectral projections

$$P(\Phi_1, \Phi_2) \to Ch(P) \in Z$$

Fredholm integer for infinite dimensional spectral
Projections associated with quantum system in the plane

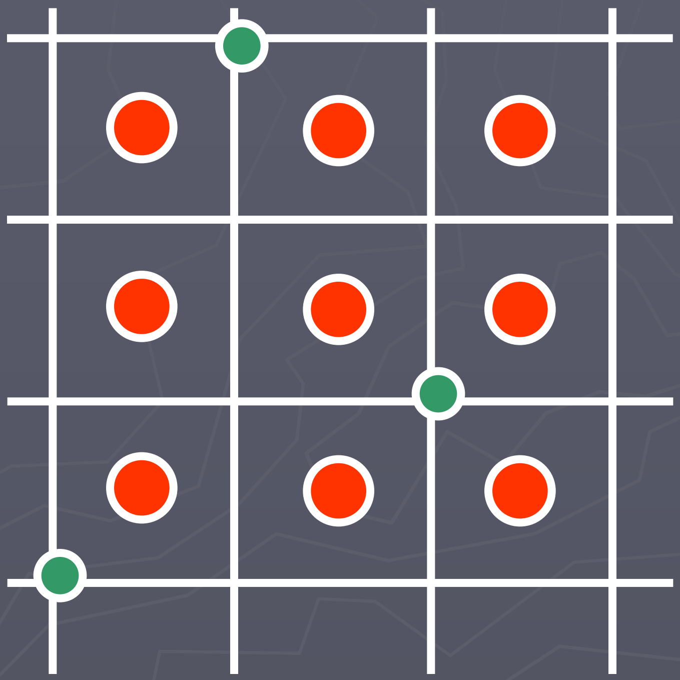
$$\left\{ \dim P = \infty, \quad U = \frac{z}{|z|} \right\} \to \in Index(PUP)$$

Hofstadteer models: Rich family of examples

$$P(B, \mu)$$

Hofstadter models

- 2 dimensional lattice
- Uniform magnetic field Φ
- Finite density of electrons fixed by chemical potential μ



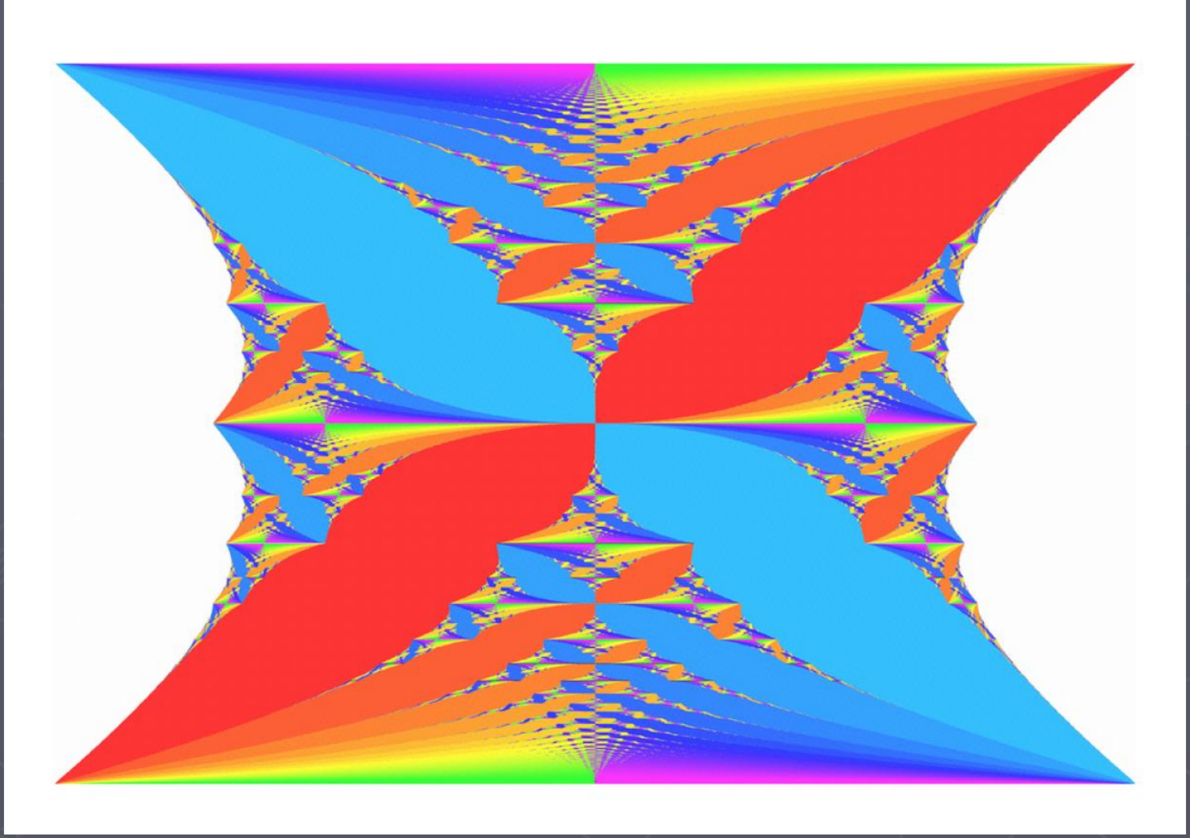
$$H(\Phi, \mu)$$

We will be interested in the projection on the ground state

$$P(\Phi, \mu)$$

Colored butterfly

Φ



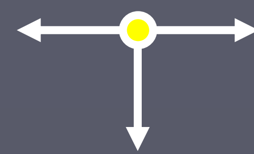
Chemical potential

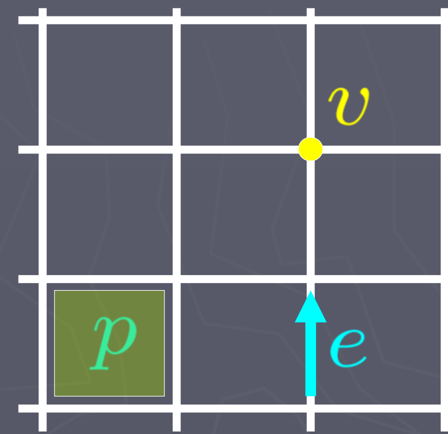
Phase diagram for the QHE

(Abelian) Lattice gauge theories

Wave functions: on vertices

$$\Psi(v) \in C$$



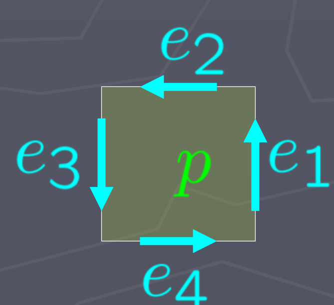


Gauge potentials on (oriented) edges

$$e^{i\gamma(e)} = V(e) = V^{-1}(-e) \in U(1)$$

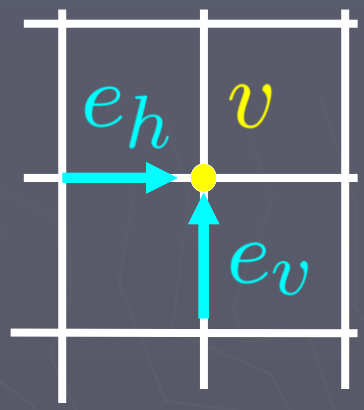
Gauge fields: on plaquetts

$$e^{i\Phi(p)} = F(p) = F^{-1}(p) = \prod_{\partial p} U(e_j)$$



$$\partial p = e_1 + e_2 + e_3 + e_4$$

Generalities



$$H = V_h + V_h^* + V_v + V_v^*$$

$$(V_h\Psi)(m,n) \;=\; e^{i\gamma(e_h)}\Psi(m-1,n)$$

$$(V_v\Psi)(m,n) \;=\; e^{i\gamma(e_v)}\Psi(m,n-1)$$

$$v = (m,n)$$

In Landau gauge, U is the ordinary shift

$$H = H^*, \qquad \|H\| \le 4$$

It follows

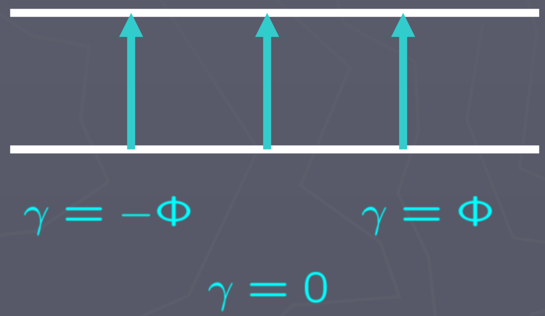
$$Spec(H) \subseteq [-4,4]$$

By gauge invariance Spec(H) is a function of $\{\Phi(p)\}$

Constant flux

Landau gauge: Gauge field on vertical bonds only

$$\gamma(e_v) = m\Phi$$



$$\begin{aligned}(H\Psi)(m,n) &= \Psi(m+1,n) + \Psi(m-1,n) \\ &+ e^{im\Phi}\Psi(m,n+1) + e^{-im\Phi}\Psi(m,n-1)\end{aligned}$$

Translation invariant in horizontal direction $\quad UH = HU$

Hofstadter model

$$
\begin{aligned}
(H\Psi)(m,n) &= \Psi(m+1,n)+\Psi(m-1,n) \\
&+ e^{im\Phi}\Psi(m,n+1)+e^{-im\Phi}\Psi(m,n-1)
\end{aligned}
$$

Separation of variables

$$\Psi(n,m)=e^{i\theta n}\psi(m), \quad \theta\in[-\pi,\pi)$$

$$
\begin{aligned}
(H_\theta\psi)(m) &= \psi(m+1)+\psi(m-1) \\
&+ e^{i(m\Phi+\theta)}\psi(m)+e^{-i(m\Phi+\theta)}\psi(m)
\end{aligned}
$$

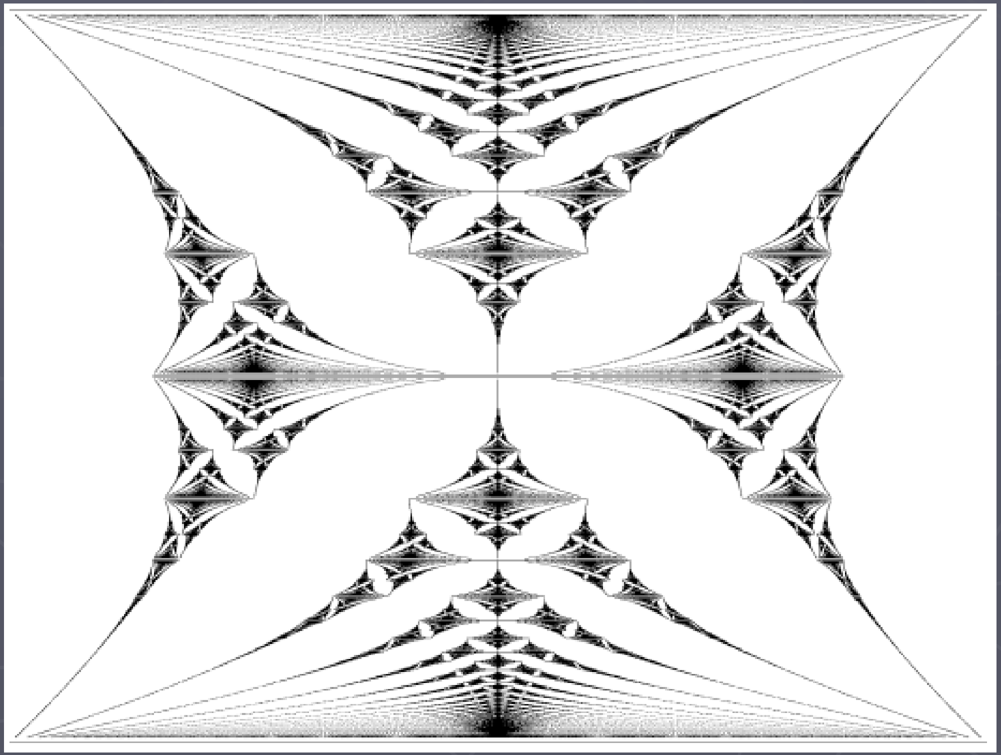
Wannier-Hofstadter model

$$(H\psi)(m)=\psi(m+1)+\psi(m-1)+2\cos(m\Phi+\theta)\psi(m)$$

A rich spectral problem in one dimension

Spectrum

Φ



Spec(H)

Hofstadter model: Sensitivity

$$(H\psi)(m) = \psi(m+1) + \psi(m-1) + 2\cos(m\Phi + \theta)\psi(m)$$

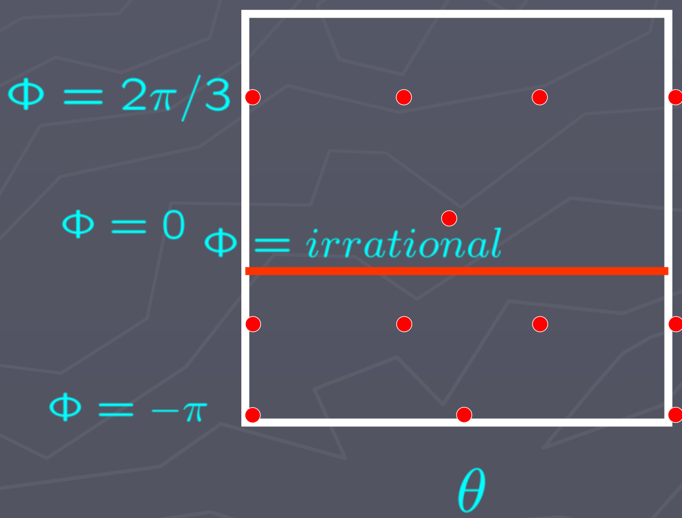
Depends on two angles sensitively

$$H(\Phi, \theta), \qquad \Phi,\ \theta \in [\pi, \pi)$$

A rich spectral problem in one dimension

$$H(\Phi, \theta) = U H(\Phi, \theta + \Phi) U^*$$

Independent of theta
For irrational fluxes



Rational flux

$$(H\psi)(m) = \psi(m+1) + \psi(m-1) + 2\cos(m\Phi + \theta)\psi(m)$$

Depends on two angles sensitively

$$\Phi = \frac{2\pi p}{q}, \quad HT = TH, \quad (T\psi)(m) = \psi(m-q)$$

Conservation of (quasi) momentum

$$\psi(m-q) = e^{-i\phi}\psi(m), \quad \phi \in [-\pi, \pi)$$

Reduction to pXp matrices

pXp hermitian matrix on the two torus:

$$H(\theta,\phi) = \begin{pmatrix} 2\cos(\theta) & 1 & 0 & 0 & e^{i\phi} \\ 1 & 2\cos(\Phi+\theta) & 1 & 0 & 0 \\ 0 & 1 & 2\cos(2\Phi+\theta) & 1 & 0 \\ & \dots & \dots & \dots & \\ e^{-i\phi} & 0 & 0 & 0 & 2\cos(-\Phi+\theta) \end{pmatrix}$$

$$\Phi = \frac{2\pi p}{q}$$

Spect(H)= p bands

Each band has a Chern integer

Credits

Atiyah (Index)
Bellissard (QHE NC geometry)
Berry (Adiabatic curvature, Berry's phase, levitron)
Herbst-Simon, Zak (magnetic translations)
Kato (Adiabatic theorem)
Last, Gat, Panati, Teufel (Hofstadter)
Seiler (Pauli, Adiabatic theorem, qhe)
Simon (Hofstadter, Adiabatic holonomy, Homotopy)
Seiler-Simon (Relative index, pairs of projections)
Segert-Sadun (Homotopy of simple matrices)
Osadchy (Colored butterfly)
Thouless (QHE, TKNN integers, Hofstadter)