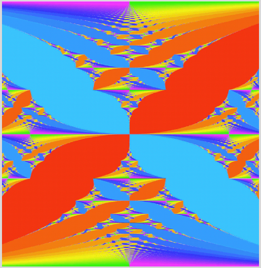
Geometry of Quantum Transport

Yosi Avron, Martin Fraas, Gian Michele Graf, Oded Kenneth

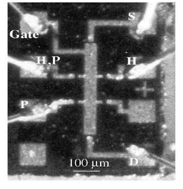
November 28, 2010



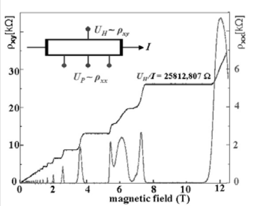
Outline

1. Motivation: QHE
   - Control
   - Reponse
   - Geometry
2. Transport is geometric
3. Open systems
   - Lindbladians
   - Spectral properties
   - Adiabatic evolution
   - Splitting principle
4. Kähler
   - Complex structure
   - Qubit and QHE

Motivation: Quantum Hall effect

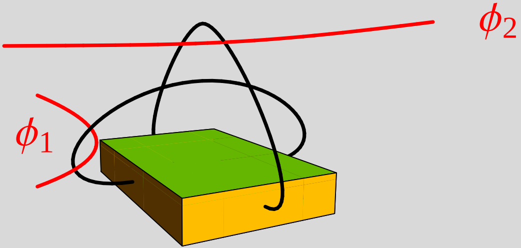


- Micro ill characterized; coupled to environment
- Quantized Hall resistivity $\frac{h}{e^2}\frac{1}{\mathbb{Z}}$
- Accuracy:12 significant digits
- Chern number of (spectral) bundle P
- Chern numbers in open q-system?



Controls in QHE

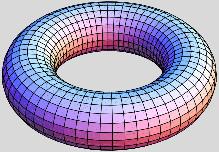
- Controls: Magnetic flux tubes $\mathcal{M} = \mathbb{T}^2$
- Aharonov-Bohm periodicity: $H(\phi) \equiv H(\phi + 2\pi)$

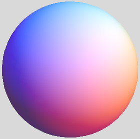


Topology of QHE in physical space

Controlled Hamiltonians

- Space of controls: $\phi \in \mathcal{M}$ two dimensional
- Controlled Hamiltonian: $H(\phi)$
- $\mathcal{M}$ has no a-priori metric
- $\mathcal{M}$ has a-priori topology, e.g. $\mathbb{T}^2, \mathbb{S}^2, \mathbb{R}^2$

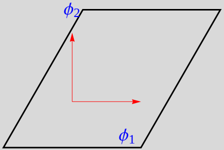




$\mathcal{M}$: Control spaces

Response and transport coefficients

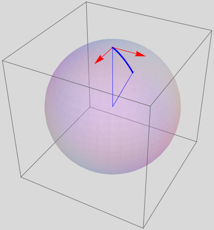
- Controls $= \phi_\mu$ Fluxes in the QHE
- Driving= control rates $= \dot{\phi}_\mu$ emf
- Response: $\frac{-\partial H}{\partial \phi_\nu}$ Loop currents
- Transport coefficients: $\mathrm{Tr}(\rho \partial_\mu H) = \dots + f_{\mu\nu}(\phi)\dot{\phi}_\nu + \dots$
- Dissipative and reactive response $f = f^S + f^A$
- $f^A \neq 0$: symplectic structure
- $f^S \geq 0$: metric



Symplectic structure

Geometry in Hilbert space

- Suppose $P(\phi)$ smooth family of projections
- Example: spin 1/2 $P(\theta,\phi) = \frac{1}{2}\begin{pmatrix} 1-\cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & 1+\cos\theta \end{pmatrix}$, $\mathcal{M} = \mathbb{S}^2$
- Fubini-Study metric $g_{\mu\nu}(\phi) = \mathrm{Tr}\ P_\perp\{\partial_\nu P,\, \partial_\mu P\}$
- Symplectic structure $\omega_{\mu\nu}(\phi) = i\,\mathrm{Tr}\ P_\perp[\partial_\nu P,\, \partial_\mu P]$
- Endows control space with geometry



Metric and symplectic structures

Main result

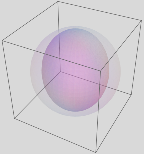
- Open systems, governed by dephasing Lindbladians
- Dephasing rate γ
- Transport coefficients: $f^S = \frac{\gamma}{1+\gamma^2}\, g, \quad f^A = \frac{1}{1+\gamma^2}\, \omega$
- Good news: Transport is geometric
- Bad news: Hall conductance $\neq$ Chern number
- Kähler Control space

$$(f^{-1})^S = \gamma\, g^{-1}, \quad (f^{-1})^A = \omega^{-1}$$

- Good news: Hall resistence $=$ Chern number

Open systems

- Hamiltonians: Generate unitary evolutions
- Lindbladians: Generate (completely) positive maps, $\rho \geq 0$
- $\rho \to \sum_j A_j \rho A_j^*, \quad \sum A_j^* A_j = 1$; Evolves ρ, not $|\psi\rangle$
- Contracting
- Interpretation: Measurement, Coupling to a Markovian bath;
  Stochastic unitary evoltion



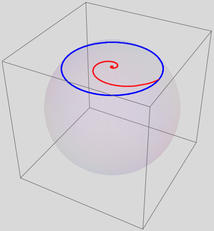
Contraction of Bloch sphere

Lindbladians

- Unitary $\mathcal{L}(\rho) = -i[H, \rho]$
- Dephasing Lindbladiand:

$$\mathcal{L}(\rho) = -i[H, \rho] + [f(H), [\rho, f(H)]]$$

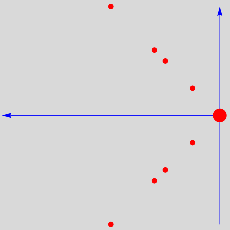
- P Spectral projections of H stationary, $\mathcal{L}(P) = 0$; Energy conserved
- Interpretation: Measurement of H; Stochastic evolution; Shallow pockets bath



Unitary vs dephasing orbits

Spectral properties

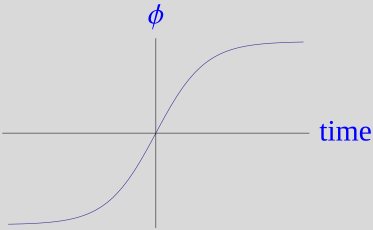
- $\mathcal{L}(|j\rangle\langle k|) = \lambda_{jk}\ |e_j\rangle\langle e_k|$
  $\lambda_{jk} = i(e_j - e_k) - \big(f(e_j) - f(e_k)\big)^2$
- Since $\mathcal{L}$ is contracting the spectrum is on the half plane
- Dephasing Lindblad 0 is multiply degenerate
- When the spectrum lies on rays one gets the simple formulas



The spectrum of Dephasing Lindblad

Adiabatic evolutions

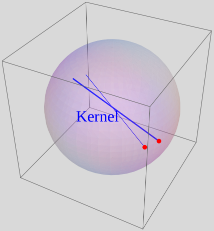
- Controls $\phi(s)$ and $H(\phi)$ change adiabatically,
- H determines dephasing Lindbladian
- Initial data: $P(\phi)$, instantaneous stationary state
- Adiabatic evolutions $\epsilon\dot{\rho} = \mathcal{L}(\rho), \quad \dot{\phi} = O(\epsilon)$



Adiabatic switching of controls

Adiabatic splitting principle

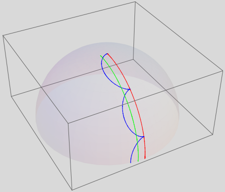
- Distinct evolutions in $Ker\,\mathcal{L}$ and $Range\,\mathcal{L}$
- $P_j \in Ker\,\mathcal{L}$
- Basic identity $P^2 = P \Longrightarrow P\dot{P}P = 0$
- $Ker\,\mathcal{L}$ evolves like a rigid body
- Motion in $Ker\,\mathcal{L}$ is tunneling
- $\rho \approx P$ to lowest order in ϵ: Frozen in kernel



Motion of Kernel

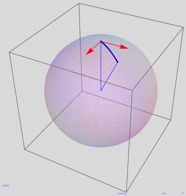
Linear response

- Linear response–first order in adiabaticity
- Motion in $Range\,\mathcal{L}$ to leading order $\mathcal{L}^{-1}(\dot{P})$
- Suppose (4 simplicity) $H(\phi) = \sum e_j P_j(\phi) \to \partial_\phi H = \sum e_j \partial_\phi P_j$
- Since $Tr(P_j \partial_\phi P_k) = 0$
- $tr(\rho \partial H) = \sum_j e_j tr(\mathcal{L}^{-1}(\dot{P}) \partial_\phi P_j), \quad P = P_0$

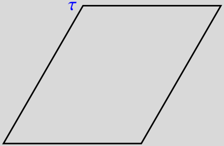

,
Orbits and approximants

Kähler structures

- The symplectic structure ω defines an area form
- The metric g defines length
- Compatible: Coinciding areas $\det g = \det \omega$
- Basic example: $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- Transport matrix $\quad f = \frac{1}{1+\gamma^2} \begin{pmatrix} \gamma & 1 \\ -1 & \gamma \end{pmatrix}$
- The inverse transport matrix $f^{-1} = \begin{pmatrix} \gamma & 1 \\ -1 & \gamma \end{pmatrix}$
- Anti-Symmetric part immune to dephasing



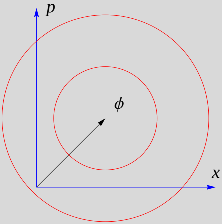
Kähler is complex structure



- $P_\perp(\partial_1 P + \tau \partial_2 P) = 0, Im\ \tau > 0 \Longrightarrow$ Kähler
- Harmonic oscillator, $\mathcal{M} = \mathbb{R}^2$

$$H(\phi) = \frac{1}{2}(p - \phi_p)^2 + \frac{1}{2}(x - \phi_x)^2$$

$$P(\phi) = U(\phi) P U^*(\phi), \quad U(\phi) = e^{i(\phi_p x + \phi_x p)}$$



- Coherent states $\Longleftrightarrow P_\perp(ip + x)P = 0 \Longleftrightarrow$ Kähler
- Ground state bundle is Kähler

Qubit and QHE

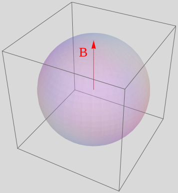
- Spin 1/2 $\mathcal{M} = \mathbb{S}^2$

$$P(\theta,\phi) = U(\theta,\phi)PU^*(\theta,\phi), \quad U(\theta,\phi) = e^{i\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)}$$

- By symmetry, enough to check at north pole

$$2P = 1 + \sigma_z, \quad 2P_\perp = 1 - \sigma_z$$

$$(1-\sigma_z)(\sigma_x + i\sigma_y)(1+\sigma_z) = 2(\sigma_x + i\sigma_y)(1+\sigma_z) = 0$$



- QHE

Summary

- Transport in open quantum systems geometric
- Dephasing dynamics induces geometry on control space
- Dissipation $\propto$ Fubini-Study metric
- Non-dissipative transport $\propto$ adiabatic curvature
- Kähler structure $\Longrightarrow$ immunity of Chern numbers