# Optimal parametrization of adiabatic paths

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# Outline

- The problem
- The unitary case
- Dephasing Lindblad
- Optimal paths
- Application to search algorithms

# Simple Hamiltonians

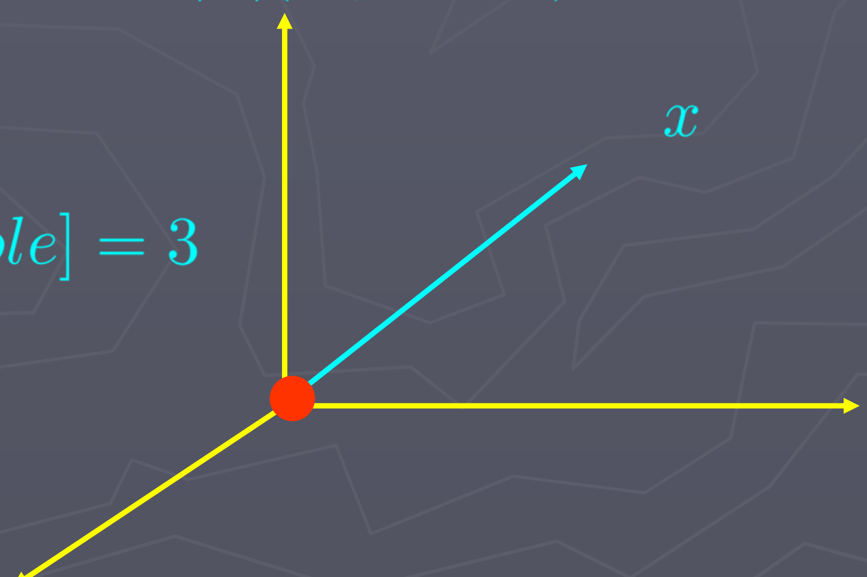
Simple, Hermitian, matrices

$$H = \sum e_j P_j, \quad P_j = |j\rangle \langle j|, \quad e_0 < e_1 \ldots < e_{N-1}$$

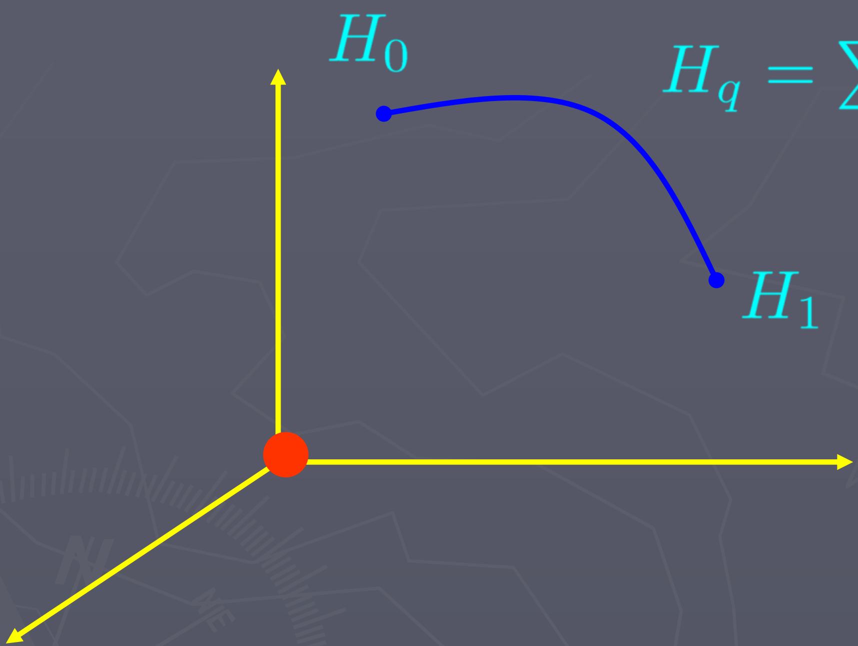
Example:

$$H = x \cdot \sigma = |x|(P_+ - P_-), \quad P_\pm = \frac{1 \pm \hat{x} \cdot \sigma}{2}$$

$$codim\ [not\ simple] = 3$$



# Interpolation




$$H_q = \sum e_j(q) P_j(q)$$

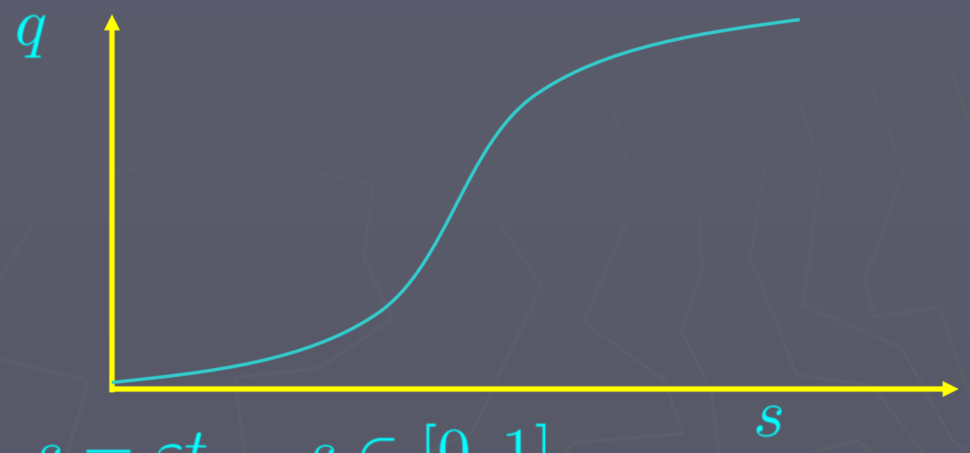
Example: Linear interpolation

$$H_q = (1-q)H_0 + qH_1, \quad q \in [0,1]$$

Interesting if $[H_0, H_1] \neq 0$

# Adiabatic parametrization

Parametrization




Slow time

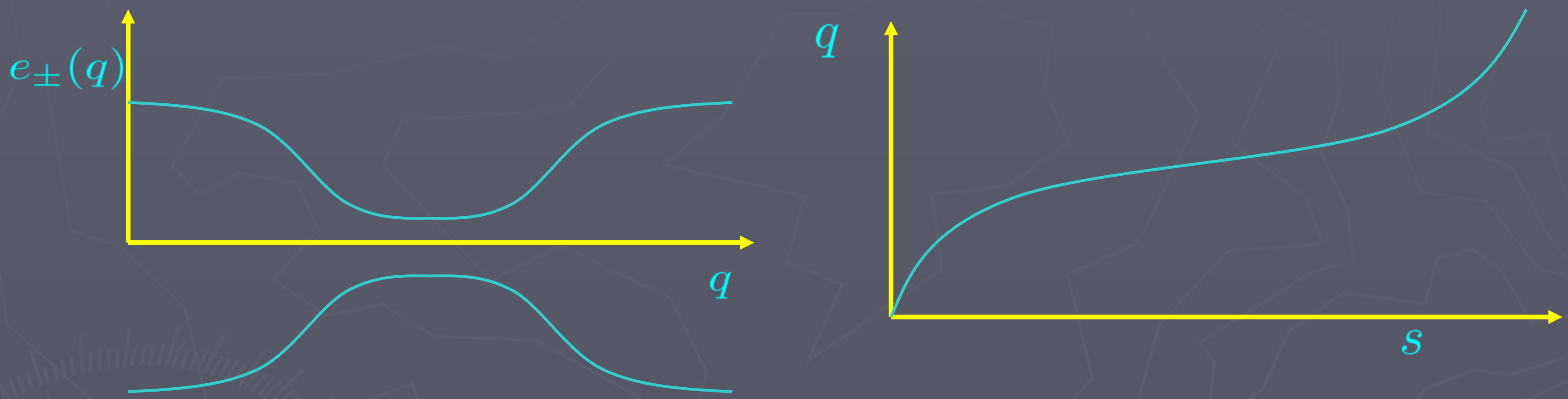
$$s = \varepsilon t, \quad s \in [0,1]$$

Adiabatic evolutions

$$\varepsilon \dot{\rho} = -i[H_q, \rho], \quad q = q(s), \quad \rho(0) = P_0(0)$$

H changes by O(1)

# The problem

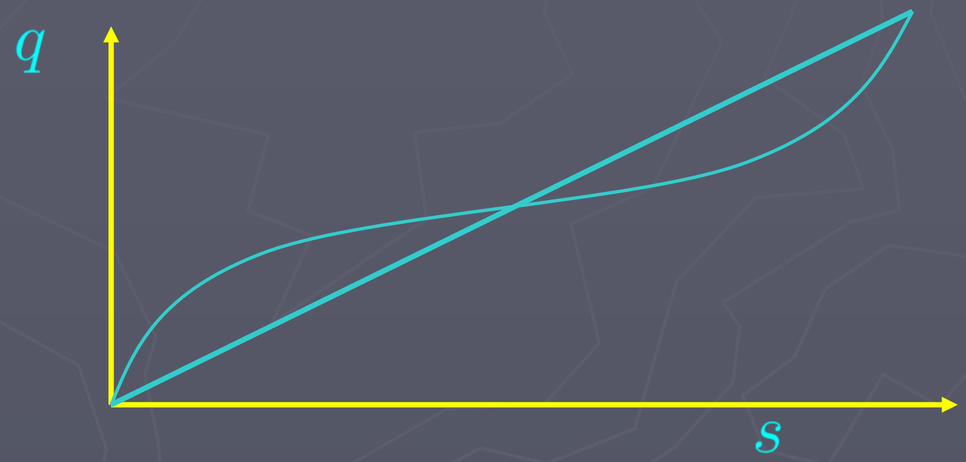



Optimize upper bounds on tunneling: Cerf, Lidar, Regev, Seiler, Vaziarni

Expectation: Go slow when gap small

# The controls:

The parametrization $q(s)$




The allotted time

$$\mathcal{T} = \frac{1}{\varepsilon}$$

# The cost function

Maximize fidelity ; minimize tunneling

$$F[q(s)] = \mathrm{tr}\big(P_0(q)\rho(s)\big)\Big|_{q=s=1}$$

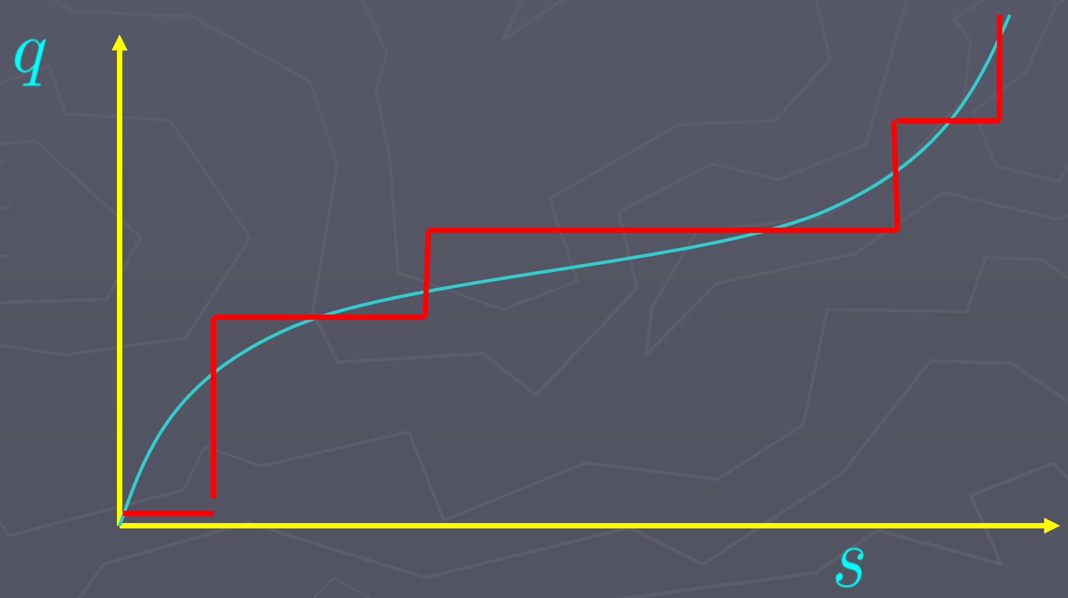
The problem:

Given $\varepsilon$ find the optimal parametrization $q(s)$

# The surprise

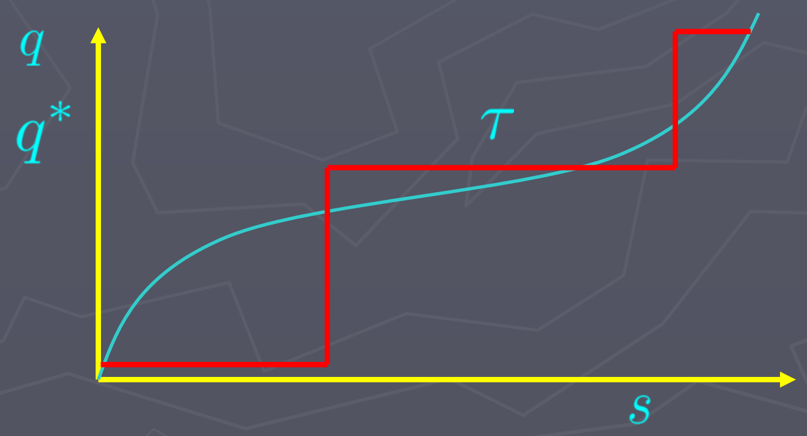
In nbhd of ANY interpolation;

- Infinitely many interpolations with NO tunneling
- Infinitely many smooth int. with arbitrarily small tunneling



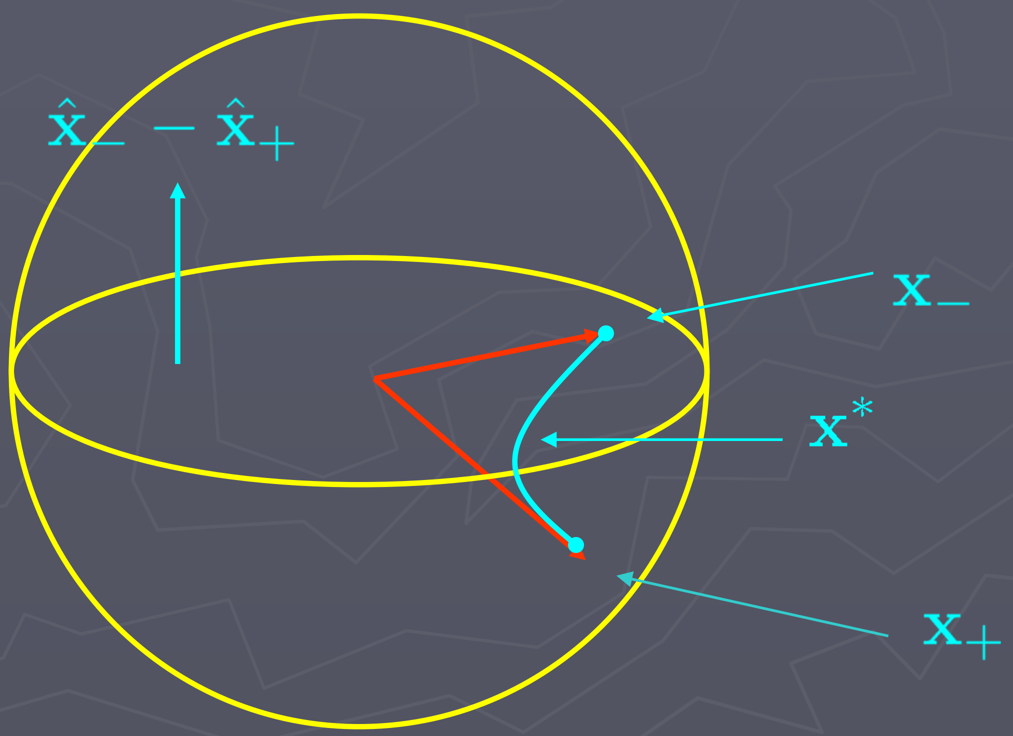
# Sketch of proof


$$e^{-iH(q^*)\tau/\varepsilon} P(q_0) e^{iH(q^*)\tau/\varepsilon} = P(q_1)$$



# How to find x*

Equatorial plane perp to



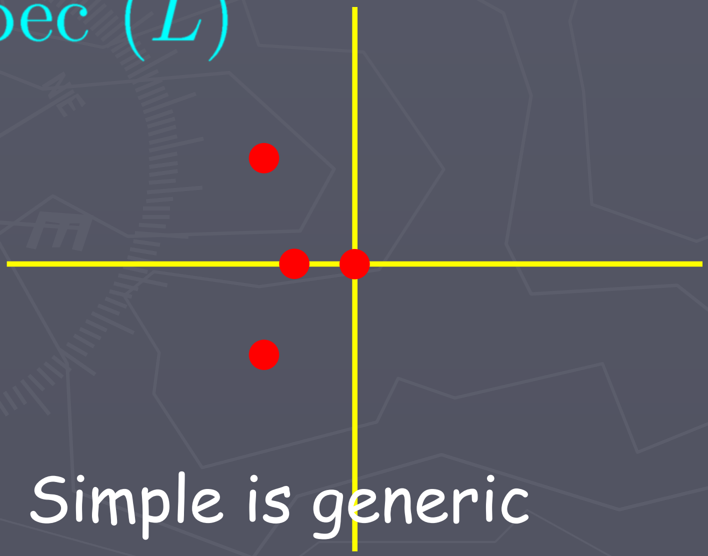
# Lindblad: System coupled to Markovian bath

$$L(\rho) = -i[H,\rho] + \sum_{a=1}^{M} (2\Gamma_a \rho \Gamma_a^* - \Gamma_a^* \Gamma_a \rho - \rho \Gamma_a^* \Gamma_a)$$

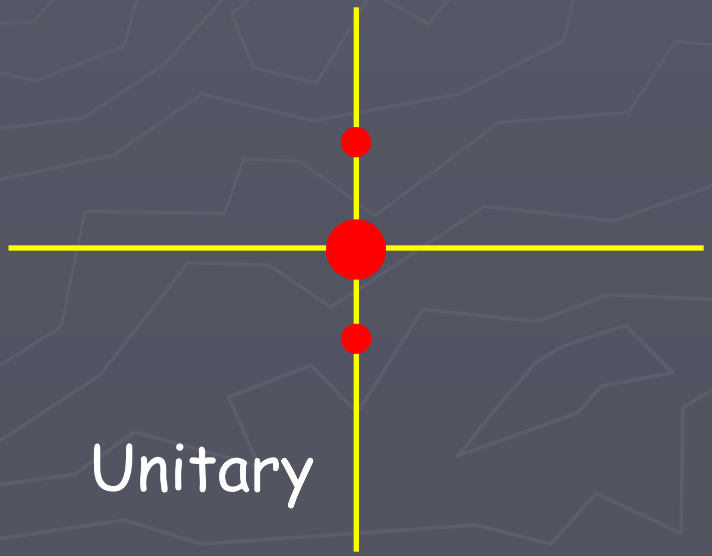
Unitary $\Gamma_a = 0$

$\text{Spec } (L) = \{i(e_j - e_k)\}$

$\text{Spec } (L)$

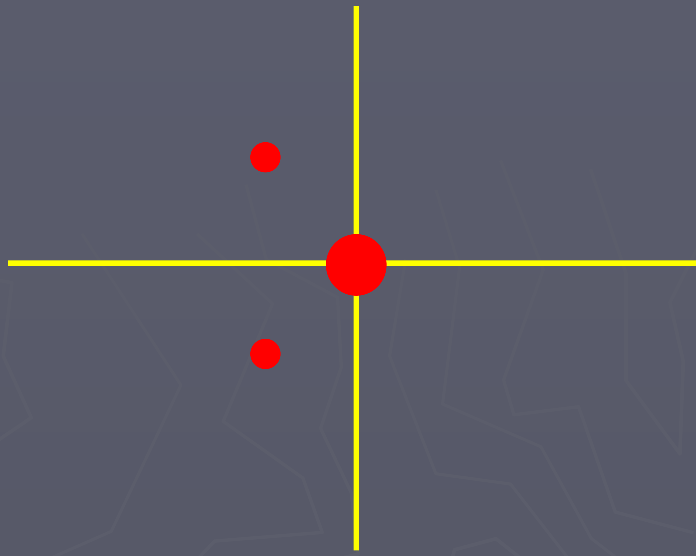


Simple is generic



Unitary

# Dephasing Lindblad



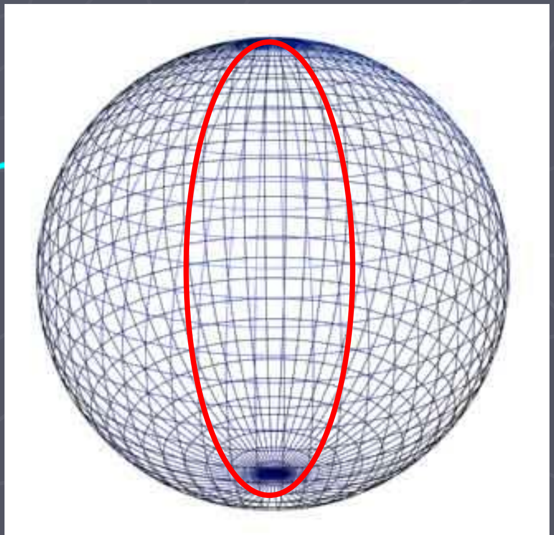
$$H = \sum e_j P_j, \quad L(P_j) = 0$$

Equivalently: $$\Gamma_a = \sum \gamma_{aj} P_j$$

$\Gamma_a$ a slave of $H$

Example:

$$L(\rho) = -i[H,\rho] - \gamma \sum_{j \neq k} P_j \rho P_k,$$



# Adiabatic Dephasing: New model

$$\varepsilon \dot{\rho} = L_q(\rho), \quad q = q(s)$$

In particular

$$H_q = \sum e_j(q) P_j(q), \quad \Gamma_j(q) = \sqrt{\gamma} P_j(q)$$

$$\Gamma_a \text{ a slave of } H$$

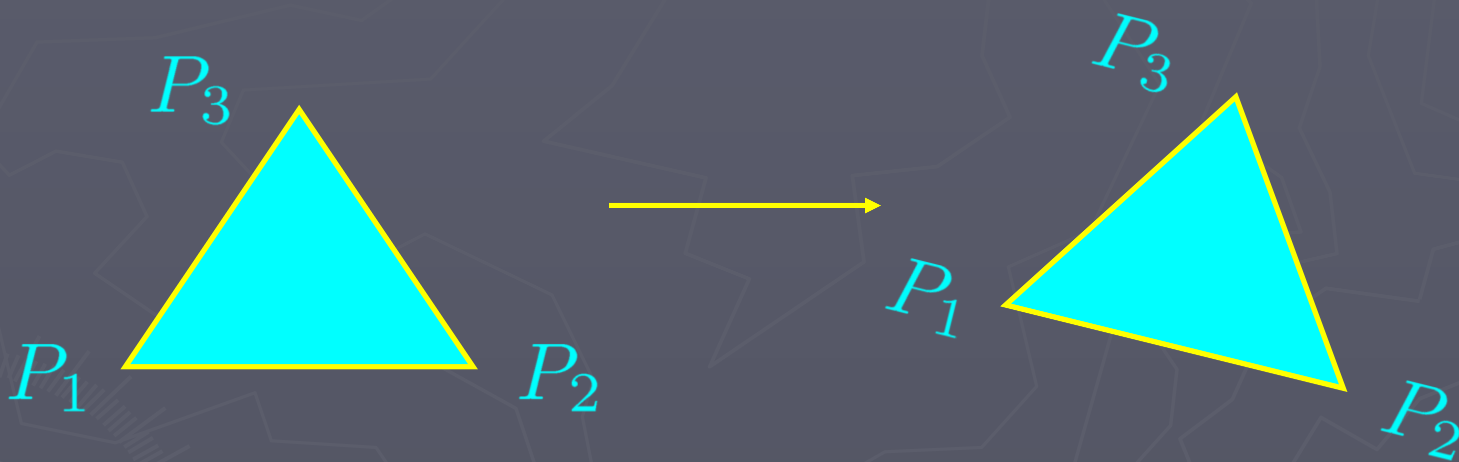
Example:

$$L_q(\rho) = -i[H_q, \rho] - \gamma \sum_{j \neq k} P_j(q) \rho P_k(q), \quad \gamma \geq 0$$

# Parametric evolution

$$\mathrm{Ker}\ L_q \longrightarrow \sum p_j P_j(q)$$

Simplex




Kernel rotates like a rigid body

$$P_j(q) P_k(q) = \delta_{jk} P_j(q) \longrightarrow \mathrm{tr}(\dot{P}_j P_k) = 0$$

$$\mathrm{Ker}\ L_q = \mathrm{Ker}\ L_q^* \perp \mathrm{Range}\ L_q \longrightarrow \sum_{j \neq k} P_j A P_k$$

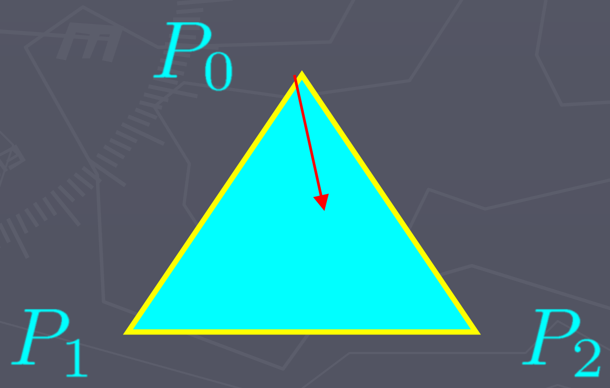
# Adiabatic expansion

$$\rho = \sum \varepsilon^n (K_n + R_n), \quad K_n \in \mathrm{Ker}\ L, \quad R_n \in \mathrm{Range}\ L$$

$$\varepsilon \dot{\rho} = L(\rho)$$

Adiabatic expansion

$$\dot{K}_n + \dot{R}_n = L(K_{n+1} + R_{n+1}) = L(R_{n+1})$$




# Tunneling rate

Use the adiabatic expansion

$$\frac{d}{ds}\mathrm{tr}(P_0\rho) = \varepsilon\, M(q)\,\dot{q}^2(s) + O(\varepsilon^2)$$

$$M(q) = \sum_{a\neq 0} \frac{\gamma\,\mathrm{tr}(P_a P_0'^2)}{(e_0(q)-e_a(q))^2+\gamma^2} \geq 0$$

# Irreversible tunneling

$$T_{q,\varepsilon}(1) = 2\varepsilon \int_0^1 M(q)\,\dot{q}^2\,ds + O(\varepsilon^2)$$

# Optimal paths

- Standard variational principle
- Minimizer: Unique, Euler Lagrange
- Const tunneling rate for optimal parametrization

$$M(q)\,\dot{q}^2 = const$$

$$M(q) = \sum_{a\neq 0} \frac{\gamma\,\mathrm{tr}(P_a P_0'^2)}{(e_0(q)-e_a(q))^2+\gamma^2} \geq 0$$

# Application to Grover

$$H_0 = 1 - |\psi\rangle\langle\psi|$$

$$H_1 = 1 - |k\rangle\langle k|$$

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum|j\rangle$$

Spec H(q)

$$gap \sim \frac{1}{\sqrt{N}}$$

Optimal tunneling

$$\varepsilon \frac{\gamma N}{1+N\gamma^2}$$

Model does not fix how does $\gamma$ scale with N

# Three scenarios

- Weak dephasing $\gamma \ll gap$

  If $\gamma \ll \varepsilon$ back to unitary case

  If $\gamma \gg \varepsilon$ Slow search: $T = \frac{1}{\varepsilon} \gg \frac{1}{\gamma} \gg \sqrt{N}$

- Single scale $\gamma \sim gap \sim \frac{1}{\sqrt{N}}$

  Optimal tunneling $\varepsilon \frac{\gamma N}{1 + N\gamma^2} \sim \varepsilon\sqrt{N}$ Grover

- Strong dephasing $\gamma \gg gap$

  Optimal tunneling $\varepsilon \frac{\gamma N}{1 + N\gamma^2} \sim \frac{\varepsilon}{\gamma}$ Beats Grover

Boixo et. al.

# Hidden resources

- Bath is system specific and not universal
- Bath has a premonition of answer
- Zeno, quick monitoring, time-energy uncertainty

$$\text{C}\hbar\text{urc}\hbar\text{-Turing thesis}$$

Universal, physical markovian baths can not dephase too quickly

$$\gamma = O(gap)$$