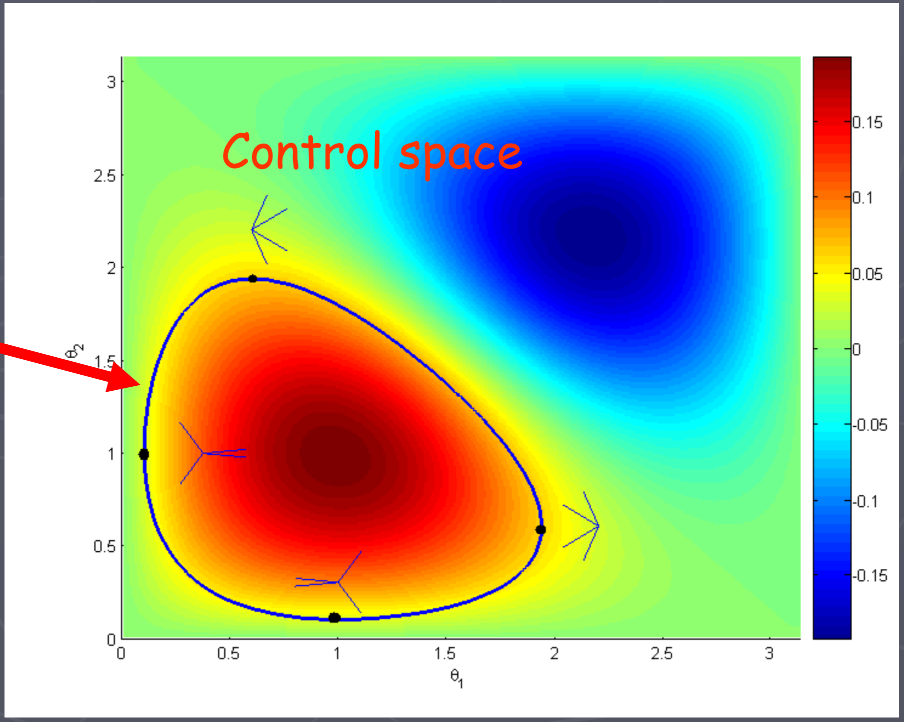
Swimming as a gauge theory

With Oren Raz

Message

- How to analyze control problem with pictures
- Solving ODE on path space with landscape diagrams
- Non abelain gauge fields without QCD

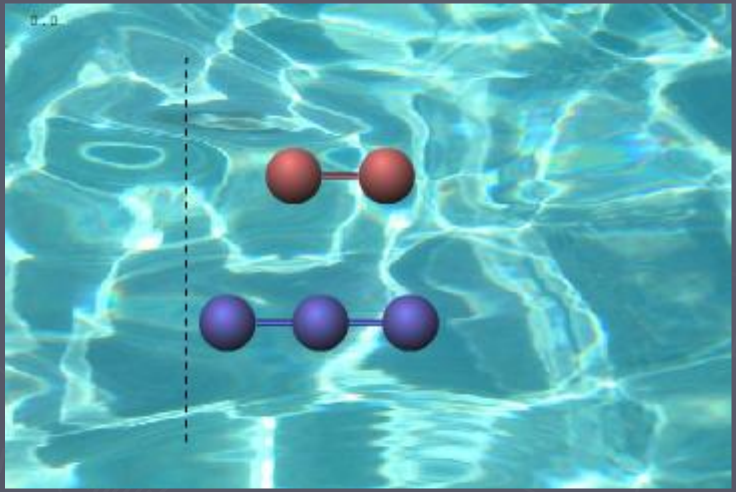
stroke



Outline

- Adiabatic swimming is statics
- Swimming and linear response
- Slender is beautiful: Cox for pedestrians
- An Abelian swimmer
- Metric and curvature in the space of controls
- Purcell non Abelian swimmer

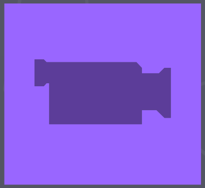
Showtime: Robot race



Robot race

Najafi and Golestanian, 2004

Euglena



http://lifesci.rutgers.edu/~triemer/movies.htm#Metaboly

Adiabatic swimming is F=0

Stokes: $F = 6\pi\mu RV$

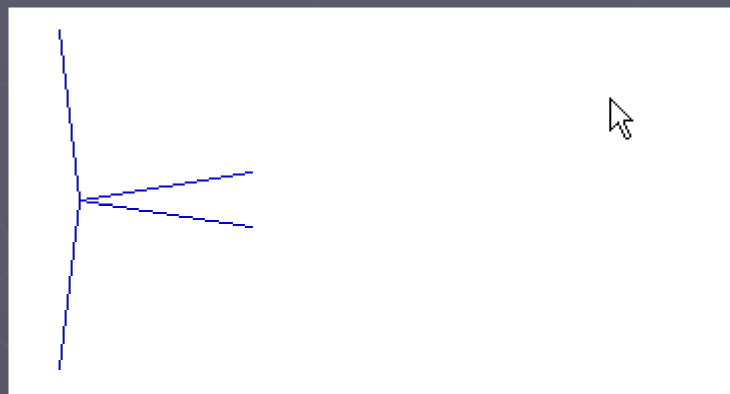
Newton

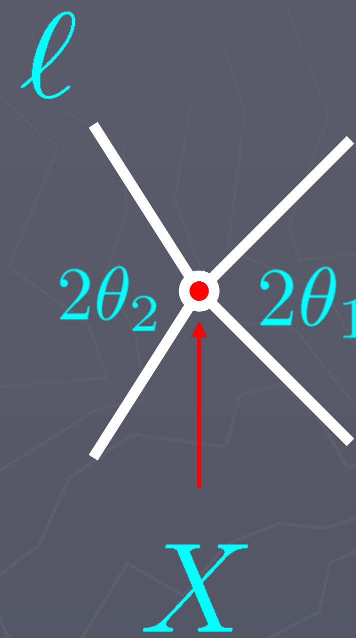
$$\overbrace{F}^{O\left(\frac{R}{T}\right)} = \overbrace{m\dot{V}}^{O\left(\frac{R^3}{T^2}\right)}$$

In adiabatic limit, for small objects, equation of motion is

$$F_{total} \approx 0$$

Abelian Sysiphian swimmer





Linear reposne

$$F_x = \mu_x(\theta)\dot{X} + f_1(\theta)\dot{\theta}_1 + f_2(\theta)\dot{\theta}_2$$

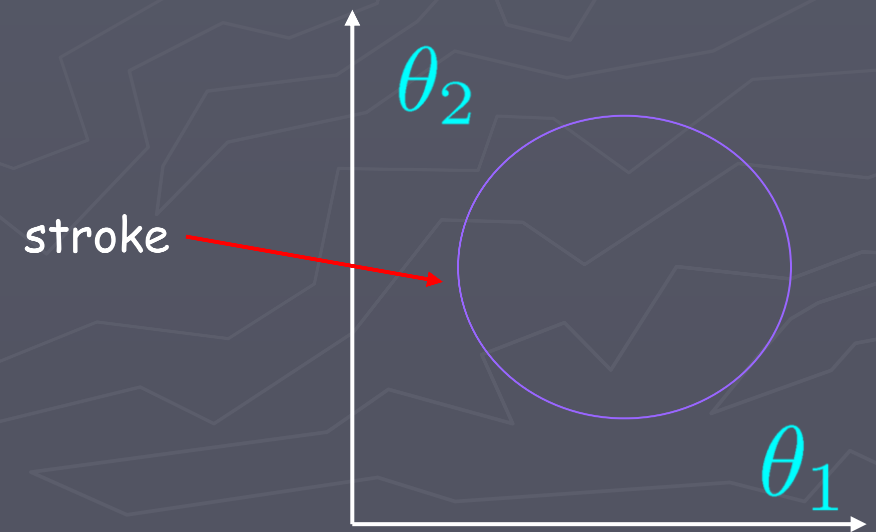
Swimming equation:

$$đX = A_1(\theta)\ d\theta_1 + A_2(\theta)\ d\theta_2$$

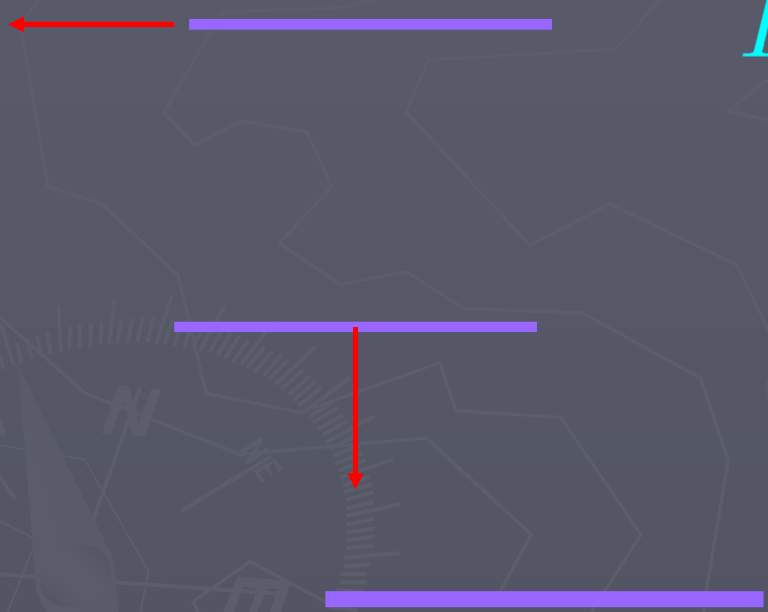
Three problems

$$\text{đ}X = A_1(\theta)\ d\theta_1 + A_2(\theta)\ d\theta_2$$

1. Generalization to Euclidean motions (non-Abelian).
2. Compute the Gauge potential A from hydrodynamics
3. Analyze the ODE on control space?



Weirdness of Low Reynolds: Dragging needles



$$F = k\ell V, \quad k = \frac{2\pi\mu}{\log \ell/R}$$

slenderness

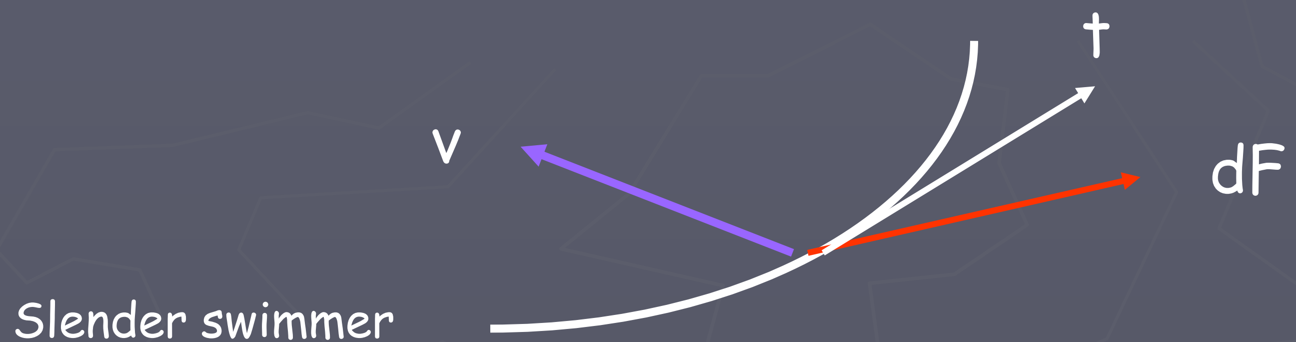
$$F = 2k\ell V$$

$$k\ell < (6\pi\mu R)\frac{\ell}{2R}$$

Friction of one ball

Number of balls

Cox for pedestrians

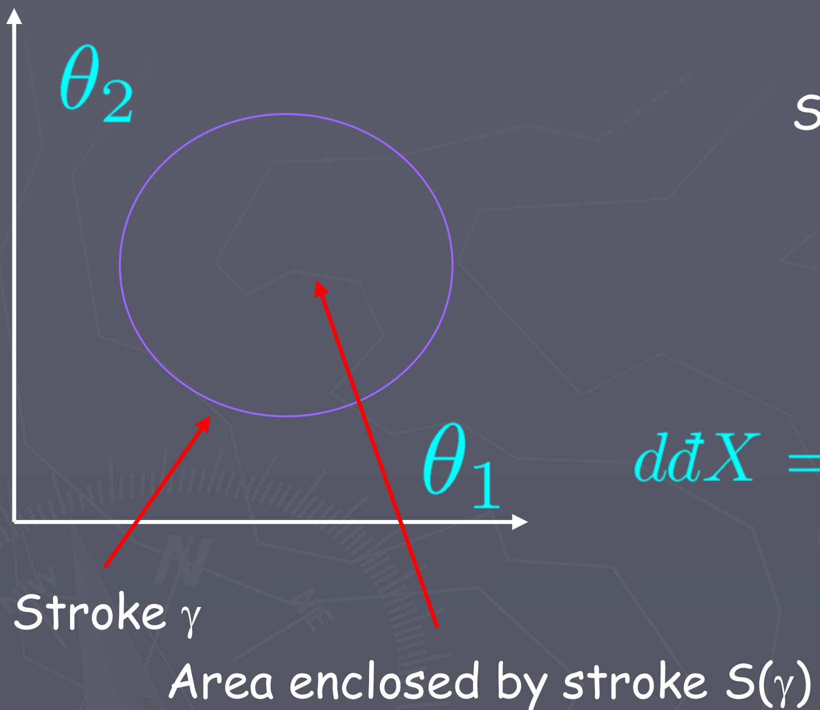


$$dF(x) = k(\mathbf{t}(\mathbf{t}\cdot\mathbf{v}) - 2\mathbf{v})d\ell,$$

$$F_x = f_\theta\,\dot{\theta} + f_X\,\dot{X}$$

$$f_X = k(\cos^2\theta - 2)\int_0^\ell ds\,, \qquad f_\theta = 2k\sin\theta\int_0^\ell s\,ds$$

Connection and curvature



Stroke: closed path in control space

$$\text{đ}X = A_1 d\theta_1 + A_2 d\theta_2$$

$$d\text{đ}X = \mathcal{F} d\theta_1 \wedge d\theta_2, \quad \mathcal{F} = \partial_1 A_2 - \partial_2 A_1$$

- No loss of information (abelian)

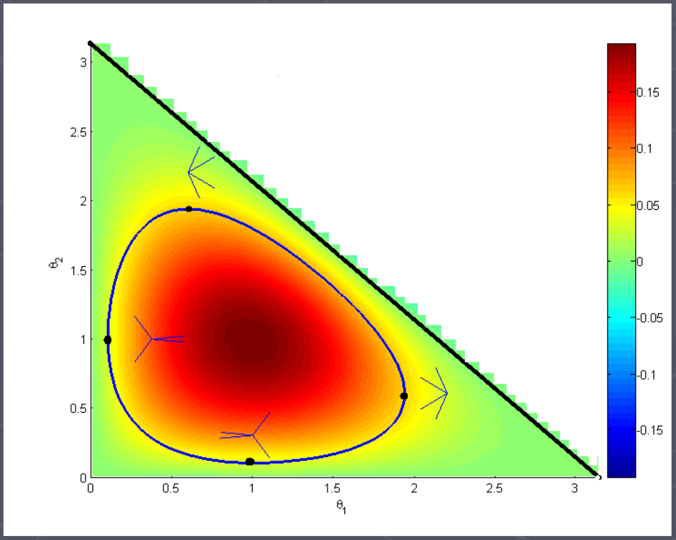
$$\oint_\gamma \text{đ}X = \int_{S(\gamma)} d\text{đ}X$$

Example: Aebelian swimmer

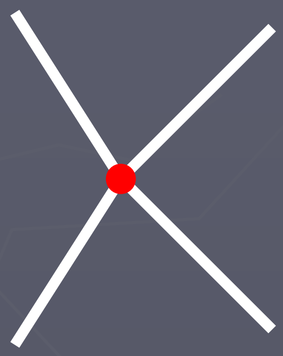
$$d\text{đ}x = a^2(\theta)(\cos\theta_2 - \cos\theta_1)\left(\sin\theta_1 \sin\theta_2\right) d\theta_1 \wedge d\theta_2$$

Rational trig function

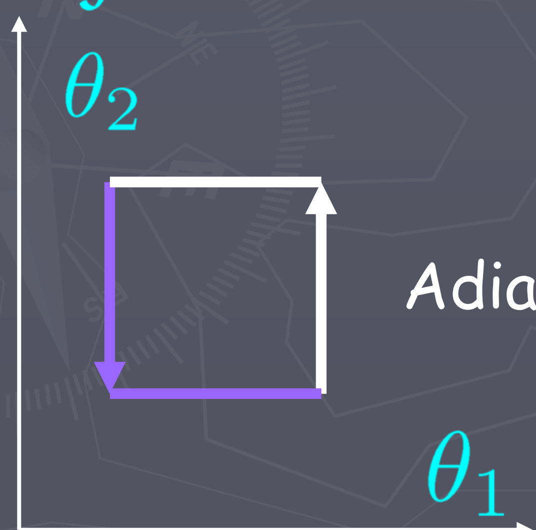
Vanishes on boundary of triangle



Gauge issues: What's good about curvature?



$đX$	Depends on fiducial point	sysiphian
$\mathcal{F}$	independent	Takes account of cancellations



Adiabatic swimming: a failure of commutation

How to plot 2 forms?

$$d\,\text{đ}X = \mathcal{F} d\theta_1 \wedge d\theta_2, \quad \mathcal{F} = \partial_1 A_2 - \partial_2 A_1$$

Need a metric in control space

$$dS = \sqrt{\det g}\; d\theta_1 \wedge d\theta_2$$

Can plot function

$$\frac{\mathcal{F}}{\sqrt{\det g}}$$

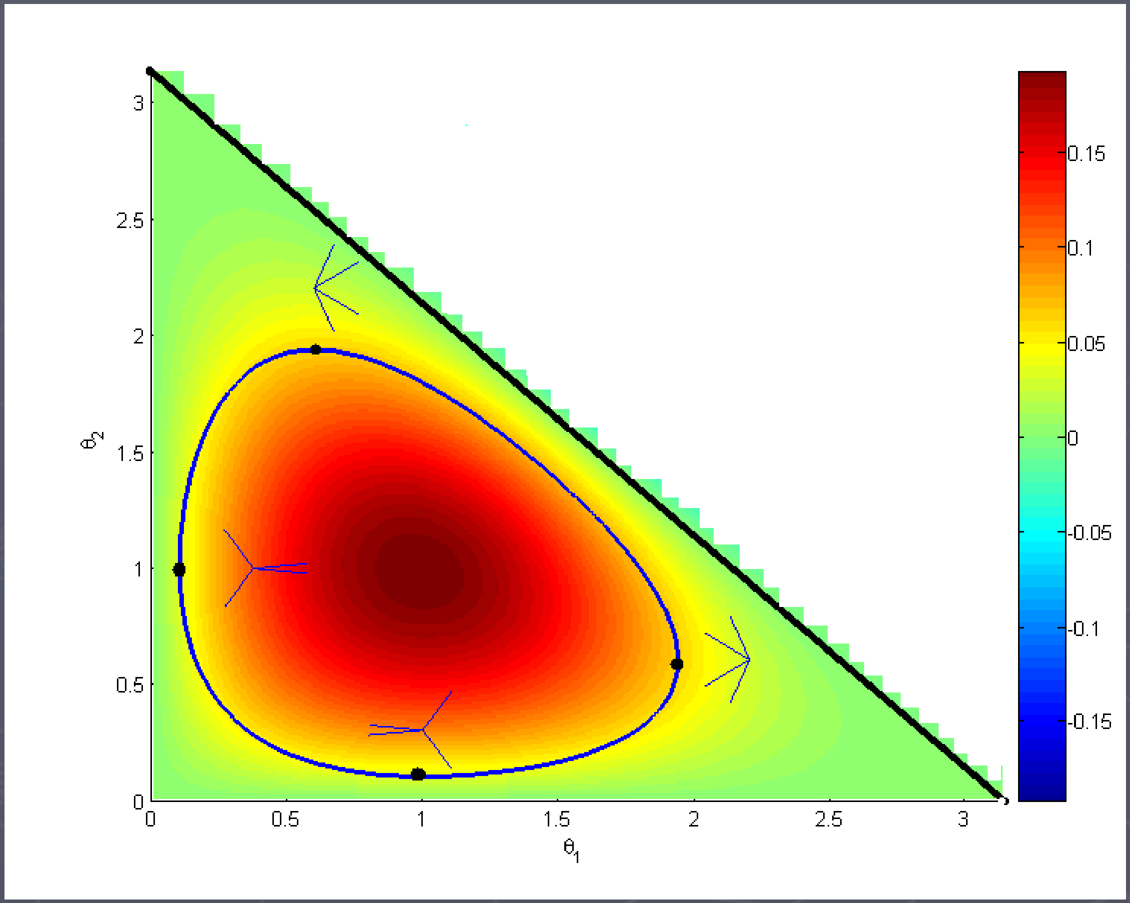
Dissipation: Metric in control space

$$Power = \sum g_{ij}(\theta)\ \dot{\theta}_i \dot{\theta}_j$$

Example: the abelian simple swimmer

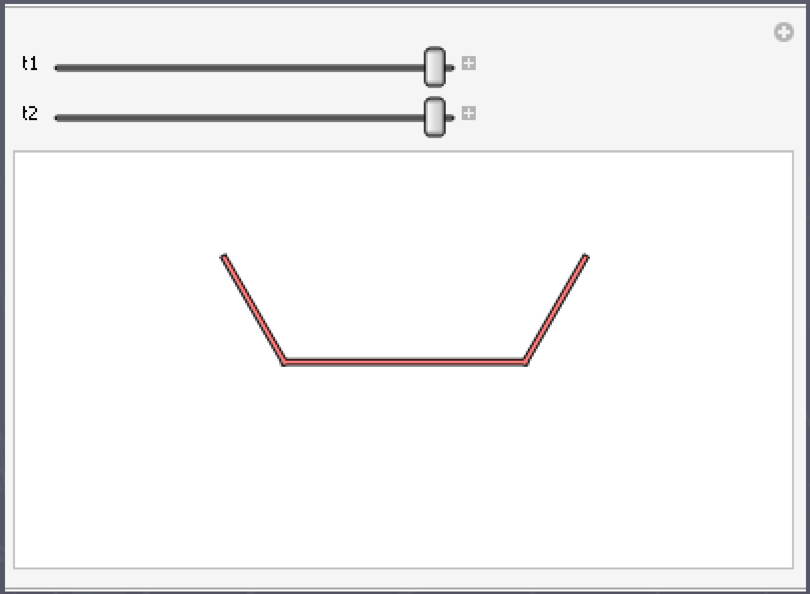
$$6\dot{x}^2 + 2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - 6(\dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2)\dot{x}$$

Pictures for ode on control space



Optimizer exists because the curvature vanishes on bdry

Purcell non-Abelian swimmer



$$E(\theta, x, y) = \begin{pmatrix} \cos\phi & \sin\phi & x \\ -\sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}$$

Euclidean motions: Non commutative group

Lab frame

Lab frame

$$đx^{\alpha} = \mathcal{A}_j^{\alpha}\, d\theta_j \qquad x^{\alpha} = (\phi, x, y)$$

Gauge fields are not function of controls

$$\mathcal{A}_i^{\alpha}(\phi,\theta) = R^{\alpha\beta}(\phi) A_i^{\beta}(\theta)\,; \qquad R^{\alpha\beta}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Body frame

$$\text{đ}y = A_j \, d\theta_j$$

Lie algebra valued objects

Composition: matrix multiplication

$$E(\gamma) = \prod_{\theta \in \gamma} E\big(dy^\alpha(\theta)\big)$$

Gauge fields A are matrix valued function of the controls

Non Abelian curvature

dy Lie algebra valued

$$\delta y = \mathcal{F} d\theta_1 \wedge d\theta_2$$

Curvature: Matrix valued

Matrix form:

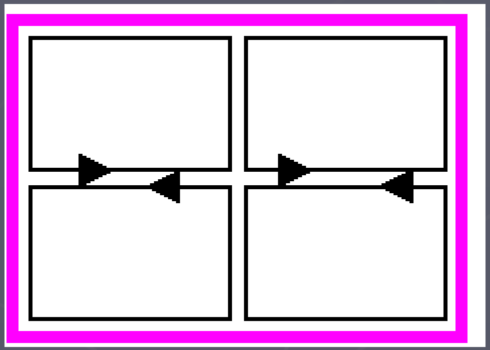
$$\mathcal{F} = \partial_1 A_2 - \partial_2 A_1 - [A_1, A_2]$$

In components:

$$\mathcal{F}^\alpha = \partial_1 A_2^\alpha - \partial_2 A_1^\alpha + \varepsilon^{0\alpha\beta} \left(A_1^0 A_2^\beta - A_2^0 A_1^\beta \right)$$

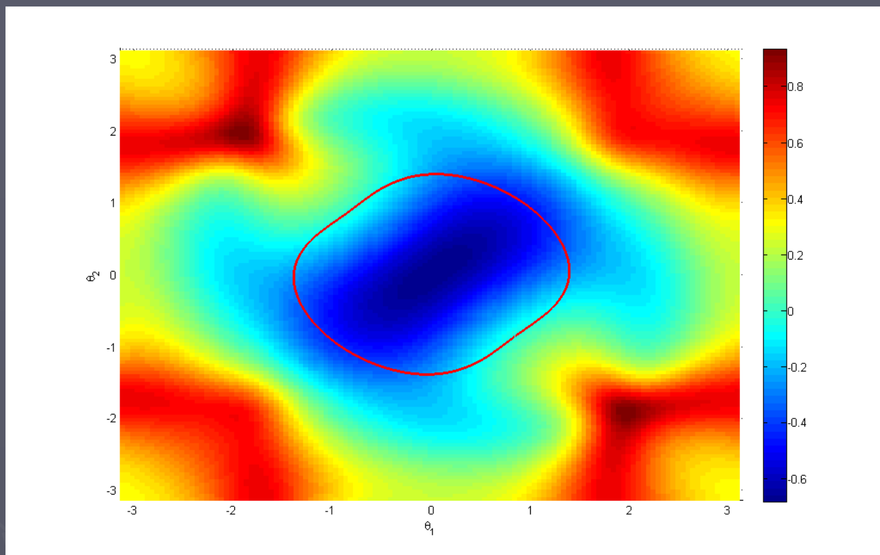
Note that the rotational curvature remains abelian

Stokes failure



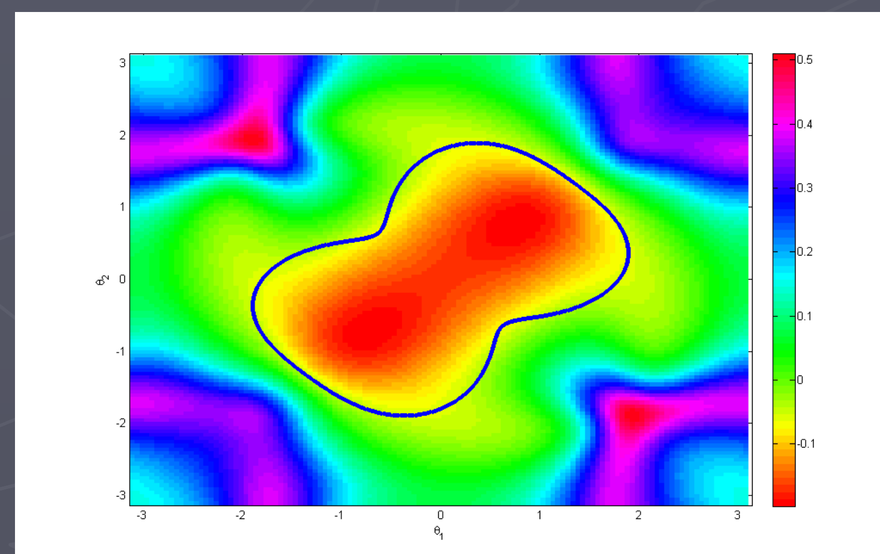
Precise information on small strokes
Qualitative information for large strokes

Curvatures



Optimal swimming efficiency,
Hosoi and Tam

Optimal distance stroke
Tam and Hososi



Conclusions

- Control problem with pictures
- Solving ODE on path space with landscape diagrams
- Introduction to Non abelain gauge fields

stroke